In this Maple sheet, we compute the Lax matrices using the asymptotics of the wave functions and the local diagonalization for the Painlevé 4 equation.

For convenience the irregular times and monodromies are denoted t_{i,j} at lambda=infinity and s_{i,j} at lambda=t

We first use the expression of the coefficients of the spectral curve in terms of the irregular times and monodromies.

```
> restart:
P2:=unapply(s10*s20/(lambda-t)^2+P012/(lambda-t)-t12*s10-t12*s20-t12*t10+t10*t22+t11*t21+(t11*t22+t12*t21)*lambda+t12*t22*lambda^2,lambda);
tdP2:=unapply(s10*s20/(lambda-t)^2-t12*s10-t12*s20-t12*t10+t10*t22+t11*t21+(t11*t22+t12*t21)*lambda+t12*t22*lambda^2,lambda);
P1:=unapply((s10+s20)/(lambda-t)-t11-t21+(-t12-t22)*lambda,lambda);
```

$$P2 := \lambda \rightarrow \frac{s10\, s20}{(\lambda - t)^2} + \frac{P012}{\lambda - t} - t12\, s10 - t12\, s20 - t12\, t10 + t10\, t22 + t11\, t21 + (t11\, t22 + t12\, t21)\, \lambda + t12\, t22\, \lambda^2 \quad (1)$$

$$tdP2 := \lambda \rightarrow \frac{s10\, s20}{(\lambda - t)^2} - t12\, s10 - t12\, s20 - t12\, t10 + t10\, t22 + t11\, t21 + (t11\, t22 + t12\, t21)\, \lambda + t12\, t22\, \lambda^2$$

$$P1 := \lambda \rightarrow \frac{s10 + s20}{\lambda - t} - t11 - t21 + (-t12 - t22)\, \lambda$$

Expression of the Lax matrix L

Study at infinity

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> logPsi1Infty:=-t12/2/h*lambda^2-t11/h*lambda-t10/h*ln(lambda)+A10-A12/(2-1)/lambda^(2-1)-A13/(3-1)/lambda^(3-1)-A14/(4-1)/lambda^(4-1)-A15/(5-1)/lambda^(5-1)-A16/(6-1)/lambda^(6-1)-A17/(7-1)/lambda^(7-1)   ;
logPsi2Infty:=-t22/2/h*lambda^2-t21*lambda/h-t20/h*ln(lambda)-1*ln(lambda)+A20-A22/(2-1)/lambda^(2-1)-A23/(3-1)/lambda^(3-1)-A24/(4-1)/lambda^(4-1)-A25/(5-1)/lambda^(5-1)-A26/(6-1)/lambda^(6-1)-A27/(7-1)/lambda^(7-1)   ;
Llogpsi1Infty:=-Lt12/2/h*lambda^2-Lt11/h*lambda-Lt10/h*ln(lambda)+LA10-LA12/(2-1)/lambda^(2-1)-LA13/(3-1)/lambda^(3-1)-LA14/(4-1)/lambda^(4-1)-LA15/(5-1)/lambda^(5-1)-LA16/(6-1)/lambda^(6-1)-LA17/(7-1)/lambda^(7-1)   ;
Llogpsi2Infty:=-Lt22/2/h*lambda^2-Lt21*lambda/h-Lt20/h*ln(lambda)+LA20-LA22/(2-1)/lambda^(2-1)-LA23/(3-1)/lambda^(3-1)-LA24/(4-1)/lambda^(4-1)-LA25/(5-1)/lambda^(5-1)-LA26/(6-1)/lambda^(6-1)-LA27/(7-1)/lambda^(7-1)   ;
```

```
Lpsi1Infty := exp(1/h*(-1/2*t12*lambda^2-t11*lambda-t10*ln
(lambda)+h*A10-h*A12/lambda-1/2*h*A13/lambda^2-1/3*h*
A14/lambda^3-1/4*h*A15/lambda^4-1/5*h*A16/lambda^5-1/6*h*
A17/lambda^6))*1/h*(-1/2*Lt12*lambda^2-Lt11*lambda-Lt10*ln
(lambda)+h*LA10-h*LA12/lambda-1/2*h*LA13/lambda^2-1/3*h*
LA14/lambda^3-1/4*h*LA15/lambda^4-1/5*h*LA16/lambda^5-1/6*h*
LA17/lambda^6);
Lpsi2Infty := exp(1/h*(-1/2*t22*lambda^2-t21*lambda-t20*ln
(lambda)-h*ln(lambda)+h*A20-h*A22/lambda-1/2*h*A23/lambda^2
-1/3*h*A24/lambda^3-1/4*h*A25/lambda^4-1/5*h*A26/lambda^5-1/6*
h*A27/lambda^6))*1/h*(-1/2*Lt22*lambda^2-Lt21*lambda-Lt20*ln
(lambda)+h*LA20-h*LA22/lambda-1/2*h*LA23/lambda^2-1/3*h*
LA24/lambda^3-1/4*h*LA25/lambda^4-1/5*h*LA26/lambda^5-1/6*h*
LA27/lambda^6);
psi1Infty:=exp(logPsi1Infty);
psi2Infty:=exp(logPsi2Infty);
dpsi1dlambdaInfty:=diff(psi1Infty,lambda):
dpsi2dlambdaInfty:=diff(psi2Infty,lambda):
d2psi1dlambda2Infty:=diff(psi1Infty,lambda$2):
d2psi2dlambda2Infty:=diff(psi2Infty,lambda$2):
Vinfty1:=-t12/2/lambda^2-t11/lambda+t10*ln(lambda);
Vinfty2:=-t22/2/lambda^2-t21/lambda+t20*ln(lambda);


WronskianLambdaInfty:=h*factor(psi1Infty*dpsi2dlambdaInfty-
psi2Infty*dpsi1dlambdaInfty):
WronskianLambdabisInfty:=h*simplify(factor( (diff(logPsi2Infty,
lambda)-diff(logPsi1Infty,lambda))*exp(logPsi1Infty+
logPsi2Infty))):

WronskianTildeLambdaInfty:=h^3*factor(dpsi2dlambdaInfty*
d2psi1dlambda2Infty-dpsi1dlambdaInfty*d2psi2dlambda2Infty):
```

$$logPsi1Infty := -\frac{1}{2}\frac{t12\,\lambda^2}{h} - \frac{t11\,\lambda}{h} - \frac{t10\ln(\lambda)}{h} + A10 - \frac{A12}{\lambda} - \frac{1}{2}\frac{A13}{\lambda^2} - \frac{1}{3}\frac{A14}{\lambda^3} - \frac{1}{4}\frac{A15}{\lambda^4} - \frac{1}{5}\frac{A16}{\lambda^5} - \frac{1}{6}\frac{A17}{\lambda^6} \quad \textbf{(1.1)}$$

$$logPsi2Infty := -\frac{1}{2}\frac{t22\,\lambda^2}{h} - \frac{t21\,\lambda}{h} - \frac{t20\ln(\lambda)}{h} - \ln(\lambda) + A20 - \frac{A22}{\lambda} - \frac{1}{2}\frac{A23}{\lambda^2} - \frac{1}{3}\frac{A24}{\lambda^3} - \frac{1}{4}\frac{A25}{\lambda^4} - \frac{1}{5}\frac{A26}{\lambda^5} - \frac{1}{6}\frac{A27}{\lambda^6}$$

$$Llogpsi1Infty := -\frac{1}{2}\frac{Lt12\,\lambda^2}{h} - \frac{Lt11\,\lambda}{h} - \frac{Lt10\ln(\lambda)}{h} + LA10 - \frac{LA12}{\lambda} - \frac{1}{2}\frac{LA13}{\lambda^2} - \frac{1}{3}\frac{LA14}{\lambda^3} - \frac{1}{4}\frac{LA15}{\lambda^4} - \frac{1}{5}\frac{LA16}{\lambda^5} - \frac{1}{6}\frac{LA17}{\lambda^6}$$

$$Llogpsi2Infty := -\frac{1}{2}\frac{Lt22\,\lambda^2}{h} - \frac{Lt21\,\lambda}{h} - \frac{Lt20\ln(\lambda)}{h} + LA20 - \frac{LA22}{\lambda} - \frac{1}{2}\frac{LA23}{\lambda^2} - \frac{1}{3}\frac{LA24}{\lambda^3} - \frac{1}{4}\frac{LA25}{\lambda^4} - \frac{1}{5}\frac{LA26}{\lambda^5} - \frac{1}{6}\frac{LA27}{\lambda^6}$$

$$Lpsi1Infty := 1 \Bigg/ \left(h\, e^{\frac{1}{h}\left(-\frac{1}{2}t12\,\lambda^2 - t11\,\lambda - t10\ln(\lambda) + h\,A10 - \frac{h\,A12}{\lambda} - \frac{1}{2}\frac{h\,A13}{\lambda^2} - \frac{1}{3}\frac{h\,A14}{\lambda^3} - \frac{1}{4}\frac{h\,A15}{\lambda^4} - \frac{1}{5}\frac{h\,A16}{\lambda^5} - \frac{1}{6}\frac{h\,A17}{\lambda^6}\right)} \left(-\frac{1}{2}Lt12\,\lambda^2 - Lt11\,\lambda - Lt10\ln(\lambda) + h\,LA10 - \frac{h\,LA12}{\lambda} - \frac{1}{2}\frac{h\,LA13}{\lambda^2} - \frac{1}{3}\frac{h\,LA14}{\lambda^3} - \frac{1}{4}\frac{h\,LA15}{\lambda^4} - \frac{1}{5}\frac{h\,LA16}{\lambda^5} - \frac{1}{6}\frac{h\,LA17}{\lambda^6}\right)\right)$$

$$Lpsi2Infty := 1 \Bigg/ \left(h\, e^{\frac{1}{h}\left(-\frac{1}{2}t22\,\lambda^2 - t21\,\lambda - t20\ln(\lambda) - h\ln(\lambda) + h\,A20 - \frac{h\,A22}{\lambda} - \frac{1}{2}\frac{h\,A23}{\lambda^2} - \frac{1}{3}\frac{h\,A24}{\lambda^3} - \frac{1}{4}\frac{h\,A25}{\lambda^4} - \frac{1}{5}\frac{h\,A26}{\lambda^5} - \frac{1}{6}\frac{h\,A27}{\lambda^6}\right)} \left(-\frac{1}{2}Lt22\,\lambda^2 - Lt21\,\lambda - Lt20\ln(\lambda) + h\,LA20 - \frac{h\,LA22}{\lambda} - \frac{1}{2}\frac{h\,LA23}{\lambda^2} - \frac{1}{3}\frac{h\,LA24}{\lambda^3} - \frac{1}{4}\frac{h\,LA25}{\lambda^4} - \frac{1}{5}\frac{h\,LA26}{\lambda^5} - \frac{1}{6}\frac{h\,LA27}{\lambda^6}\right)\right)$$

$$psi1Infty := e^{-\frac{1}{2}\frac{t12\,\lambda^2}{h} - \frac{t11\,\lambda}{h} - \frac{t10\ln(\lambda)}{h} + A10 - \frac{A12}{\lambda} - \frac{1}{2}\frac{A13}{\lambda^2} - \frac{1}{3}\frac{A14}{\lambda^3} - \frac{1}{4}\frac{A15}{\lambda^4} - \frac{1}{5}\frac{A16}{\lambda^5} - \frac{1}{6}\frac{A17}{\lambda^6}}$$

$$psi2Infty := e^{-\frac{1}{2}\frac{t22\,\lambda^2}{h} - \frac{t21\,\lambda}{h} - \frac{t20\ln(\lambda)}{h} - \ln(\lambda) + A20 - \frac{A22}{\lambda} - \frac{1}{2}\frac{A23}{\lambda^2} - \frac{1}{3}\frac{A24}{\lambda^3} - \frac{1}{4}\frac{A25}{\lambda^4} - \frac{1}{5}\frac{A26}{\lambda^5} - \frac{1}{6}\frac{A27}{\lambda^6}}$$

$$Vinfty1 := -\frac{1}{2}\frac{t12}{\lambda^2} - \frac{t11}{\lambda} + t10\ln(\lambda)$$

$$Vinfty2 := -\frac{1}{2}\frac{t22}{\lambda^2} - \frac{t21}{\lambda} + t20\ln(\lambda)$$

```
> L21Infty:=factor(simplify
  (WronskianTildeLambdaInfty/WronskianLambdabisInfty)):
  L21InftyOrderlambda3:=factor(-residue(L21Infty/lambda^4,lambda=
  infinity));
  L21InftyOrderlambda2:=factor(-residue(L21Infty/lambda^3,lambda=
  infinity));
  L21InftyOrderlambda1:=factor(-residue(L21Infty/lambda^2,lambda=
  infinity));
  L21InftyOrderlambda0:=factor(-residue(L21Infty/lambda^1,lambda=
  infinity));
  L21InftyOrderlambdaMinus1:=factor(-residue(L21Infty/lambda^0,
  lambda=infinity)):
  factor(series(L21InftyOrderlambdaMinus1,h=0));
```

$$L21InftyOrderlambda3 := 0 \quad (1.2)$$

$$L21InftyOrderlambda2 := -t22\, t12$$

$$L21InftyOrderlambda1 := -t11\, t22 - t12\, t21$$

$$L21InftyOrderlambda0 := -h\, t12 - t10\, t22 - t11\, t21 - t12\, t20 - t10\, t21 - t11\, t20 + \frac{A12\, t12\, t22 - A12\, t22^2 + A22\, t12^2 - A22\, t12\, t22 - t11\, t12 + t12\, t21}{-t22 + t12}\, h$$

We get that L_{2,1} behaves at infinity like -t22*t11*lambda^2-(t21*t12+t22*t11)*lambda-(t12*t20+t22*t10+t11*t21+h*t12) +O(1/lambda)

```
> L22Infty:=factor(h*simplify(diff(WronskianLambdabisInfty,
  lambda)/WronskianLambdabisInfty)):
  L22InftyOrderlambda3:=factor(-residue(L22Infty/lambda^4,lambda=
  infinity));
  L22InftyOrderlambda2:=factor(-residue(L22Infty/lambda^3,lambda=
  infinity));
  L22InftyOrderlambda1:=factor(-residue(L22Infty/lambda^2,lambda=
  infinity));
  L22InftyOrderlambda0:=factor(-residue(L22Infty/lambda^1,lambda=
  infinity));
  L22InftyOrderlambdaMinus1:=factor(-residue(L22Infty/lambda^0,
  lambda=infinity));
  L22InftyOrderlambdaMinus2:=factor(-residue(L22Infty/lambda^
  (-1),lambda=infinity));
  L22InftyOrderlambdaMinus3:=factor(-residue(L22Infty/lambda^
  (-2),lambda=infinity)):
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$$L22InftyOrderlambda3 := 0 \quad (1.3)$$

$$L22InftyOrderlambda2 := 0$$
$$L22InftyOrderlambda1 := -t12 - t22$$
$$L22InftyOrderlambda0 := -t11 - t21$$
$$L22InftyOrderlambdaMinus1 := -t10 - t20$$
$$L22InftyOrderlambdaMinus2 := \frac{h\,(A12\,t12 - A12\,t22 + A22\,t12 - A22\,t22 - t11 + t21)}{-t22 + t12}$$

We get that L_{2,2} behaves at infinity like -(t12+t22)*lambda-(t11+t21)-(t10+t20)/lambda+h*O(1/lambda^2)

Study at lambda=t

```
> logPsi1T:=s10/h*ln(lambda-t)+B10+B12/(2-1)*(lambda-t)^(2-1)+
B13/(3-1)*(lambda-t)^(3-1)+B14/(4-1)*(lambda-t)^(4-1)+B15/(5-1)
*(lambda-t)^(5-1)+B16/(6-1)*(lambda-t)^(6-1)+B17/(7-1)*(lambda-
t)^(7-1)  ;
logPsi2T:=s20/h*ln(lambda-t)+B20+B22/(2-1)*(lambda-t)^(2-1)+
B23/(3-1)*(lambda-t)^(3-1)+B24/(4-1)*(lambda-t)^(4-1)+B25/(5-1)
*(lambda-t)^(5-1)+B26/(6-1)*(lambda-t)^(6-1)+B27/(7-1)*(lambda-
t)^(7-1)  ;
LlogpsilT:=Ls10/h*ln(lambda-t)+LB10+LB12/(2-1)*(lambda-t)^(2-1)
+LB13/(3-1)*(lambda-t)^(3-1)+LB14/(4-1)*(lambda-t)^(4-1)+LB15/
(5-1)*(lambda-t)^(5-1)+LB16*(6-1)/(lambda-t)^(6-1)+LB17/(7-1)
*(lambda-t)^(7-1)
-s10/h/(lambda-t)*Lt-B12*(lambda-t)^(2-1-1)*Lt-B13*(lambda-t)^
(3-1-1)*Lt-B14*(lambda-t)^(4-1-1)*Lt ;
Llogpsi2T:=Ls20/h*ln(lambda-t)+LB20+LB22/(2-1)*(lambda-t)^(2-1)
+LB23/(3-1)*(lambda-t)^(3-1)+LB24/(4-1)*(lambda-t)^(4-1)+LB25/
(5-1)*(lambda-t)^(5-1)+LB26/(6-1)*(lambda-t)^(6-1)+LB27/(7-1)
*(lambda-t)^(7-1)
-s20/h/(lambda-t)*Lt-B22*(lambda-t)^(2-1-1)*Lt-B23*(lambda-t)^
(3-1-1)*Lt-B24*(lambda-t)^(4-1-1)*Lt ;  ;
Lpsi1T := exp((s10/h*ln(lambda-t)+B10+B12/(2-1)*(lambda-t)^(2
-1)+B13/(3-1)*(lambda-t)^(3-1)+B14/(4-1)*(lambda-t)^(4-1)+B15/
(5-1)*(lambda-t)^(5-1)+B16/(6-1)*(lambda-t)^(6-1)+B17/(7-1)
*(lambda-t)^(7-1)))
*(Ls10/h*ln(lambda-t)+LB10+LB12/(2-1)*(lambda-t)^(2-1)+LB13/(3
-1)*(lambda-t)^(3-1)+LB14/(4-1)*(lambda-t)^(4-1)+LB15/(5-1)
*(lambda-t)^(5-1)+LB16/(6-1)*(lambda-t)^(6-1)+LB17/(7-1)
*(lambda-t)^(7-1)-s10/h/(lambda-t)*Lt-B12*(lambda-t)^(2-1-1)*
Lt-B13*(lambda-t)^(3-1-1)*Lt-B14*(lambda-t)^(4-1-1)*Lt);
Lpsi2T := exp((s20/h*ln(lambda-t)+B20+B22/(2-1)*(lambda-t)^(2
-1)+B23/(3-1)*(lambda-t)^(3-1)+B24/(4-1)*(lambda-t)^(4-1)+B25/
(5-1)*(lambda-t)^(5-1)+B26/(6-1)*(lambda-t)^(6-1)+B27/(7-1)
*(lambda-t)^(7-1)))
*(Ls20/h*ln(lambda-t)+LB20+LB22/(2-1)*(lambda-t)^(2-1)+LB23/(3
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-1)*(lambda-t)^(3-1)+LB24/(4-1)*(lambda-t)^(4-1)+LB25/(5-1)
*(lambda-t)^(5-1)+LB26/(6-1)*(lambda-t)^(6-1)+LB27/(7-1)
*(lambda-t)^(7-1)-s20/h/(lambda-t)*Lt-B22*(lambda-t)^(2-1-1)*
Lt-B23*(lambda-t)^(3-1-1)*Lt-B24*(lambda-t)^(4-1-1)*Lt);
psi1T:=exp(logPsi1T);
psi2T:=exp(logPsi2T);
dpsi1dlambdaT:=diff(psi1T,lambda):
dpsi2dlambdaT:=diff(psi2T,lambda):
d2psi1dlambda2T:=diff(psi1T,lambda$2):
d2psi2dlambda2T:=diff(psi2T,lambda$2):
VT1:=s10*ln(lambda-t);
VT2:=s20*ln(lambda-t);

WronskianLambdaT:=h*factor(psi1T*dpsi2dlambdaT-psi2T*
dpsi1dlambdaT):
WronskianLambdabisT:=h*simplify(factor( (diff(logPsi2T,lambda)-
diff(logPsi1T,lambda))*exp(logPsi1T+logPsi2T))):

WronskianTildeLambdaT:=h^3*factor(dpsi2dlambdaT*
d2psi1dlambda2T-dpsi1dlambdaT*d2psi2dlambda2T):
```

$$logPsi1T := \frac{s10\ln(\lambda-t)}{h} + B10 + B12\,(\lambda-t) + \frac{1}{2}\,B13\,(\lambda-t)^2 + \frac{1}{3}\,B14\,(\lambda-t)^3 + \frac{1}{4}\,B15\,(\lambda-t)^4 + \frac{1}{5}\,B16\,(\lambda-t)^5 + \frac{1}{6}\,B17\,(\lambda-t)^6 \tag{1.4}$$

$$logPsi2T := \frac{s20\ln(\lambda-t)}{h} + B20 + B22\,(\lambda-t) + \frac{1}{2}\,B23\,(\lambda-t)^2 + \frac{1}{3}\,B24\,(\lambda-t)^3 + \frac{1}{4}\,B25\,(\lambda-t)^4 + \frac{1}{5}\,B26\,(\lambda-t)^5 + \frac{1}{6}\,B27\,(\lambda-t)^6$$

$$Llogpsi1T := \frac{Ls10\ln(\lambda-t)}{h} + LB10 + LB12\,(\lambda-t) + \frac{1}{2}\,LB13\,(\lambda-t)^2 + \frac{1}{3}\,LB14\,(\lambda-t)^3 + \frac{1}{4}\,LB15\,(\lambda-t)^4 + \frac{5\,LB16}{(\lambda-t)^5} + \frac{1}{6}\,LB17\,(\lambda-t)^6 - \frac{s10\,Lt}{h\,(\lambda-t)} - B12\,Lt - B13\,(\lambda-t)\,Lt - B14\,(\lambda-t)^2\,Lt$$

$$Llogpsi2T := \frac{Ls20\ln(\lambda-t)}{h} + LB20 + LB22\,(\lambda-t) + \frac{1}{2}\,LB23\,(\lambda-t)^2 + \frac{1}{3}\,LB24\,(\lambda-t)^3 + \frac{1}{4}\,LB25\,(\lambda-t)^4 + \frac{1}{5}\,LB26\,(\lambda-t)^5 + \frac{1}{6}\,LB27\,(\lambda-t)^6 - \frac{s20\,Lt}{h\,(\lambda-t)} - B22\,Lt - B23\,(\lambda-t)\,Lt - B24\,(\lambda-t)^2\,Lt$$

$$Lpsi1T :=$$

$$e^{\frac{s10\ln(\lambda-t)}{h}+B10+B12(\lambda-t)+\frac{1}{2}B13(\lambda-t)^2+\frac{1}{3}B14(\lambda-t)^3+\frac{1}{4}B15(\lambda-t)^4+\frac{1}{5}B16(\lambda-t)^5+\frac{1}{6}B17(\lambda-t)^6}\left(\frac{Ls10\ln(\lambda-t)}{h}+LB10+LB12(\lambda-t)+\frac{1}{2}LB13(\lambda-t)^2+\frac{1}{3}LB14(\lambda-t)^3+\frac{1}{4}LB15(\lambda-t)^4+\frac{1}{5}LB16(\lambda-t)^5+\frac{1}{6}LB17(\lambda-t)^6-\frac{s10\,Lt}{h(\lambda-t)}-B12\,Lt-B13(\lambda-t)\,Lt-B14(\lambda-t)^2\,Lt\right)$$

$$Lpsi2T := e^{\frac{s20\ln(\lambda-t)}{h}+B20+B22(\lambda-t)+\frac{1}{2}B23(\lambda-t)^2+\frac{1}{3}B24(\lambda-t)^3+\frac{1}{4}B25(\lambda-t)^4+\frac{1}{5}B26(\lambda-t)^5+\frac{1}{6}B27(\lambda-t)^6}\left(\frac{Ls20\ln(\lambda-t)}{h}+LB20+LB22(\lambda-t)+\frac{1}{2}LB23(\lambda-t)^2+\frac{1}{3}LB24(\lambda-t)^3+\frac{1}{4}LB25(\lambda-t)^4+\frac{1}{5}LB26(\lambda-t)^5+\frac{1}{6}LB27(\lambda-t)^6-\frac{s20\,Lt}{h(\lambda-t)}-B22\,Lt-B23(\lambda-t)\,Lt-B24(\lambda-t)^2\,Lt\right)$$

$$psi1T := e^{\frac{s10\ln(\lambda-t)}{h}+B10+B12(\lambda-t)+\frac{1}{2}B13(\lambda-t)^2+\frac{1}{3}B14(\lambda-t)^3+\frac{1}{4}B15(\lambda-t)^4+\frac{1}{5}B16(\lambda-t)^5+\frac{1}{6}B17(\lambda-t)^6}$$

$$psi2T := e^{\frac{s20\ln(\lambda-t)}{h}+B20+B22(\lambda-t)+\frac{1}{2}B23(\lambda-t)^2+\frac{1}{3}B24(\lambda-t)^3+\frac{1}{4}B25(\lambda-t)^4+\frac{1}{5}B26(\lambda-t)^5+\frac{1}{6}B27(\lambda-t)^6}$$

$$VT1 := s10\ln(\lambda-t)$$
$$VT2 := s20\ln(\lambda-t)$$

```
> L22T:=factor(h*simplify(diff(WronskianLambdabisT,lambda)
  /WronskianLambdabisT)):
  L22TOrderlambdaMinus3:=factor(residue(L22T*(lambda-t)^2,lambda=
  t));
  L22TOrderlambdaMinus2:=factor(residue(L22T*(lambda-t)^1,lambda=
  t));
  L22TOrderlambdaMinus1:=factor(residue(L22T*(lambda-t)^0,lambda=
  t));
  L22TOrderlambda0:=factor(residue(L22T*(lambda-t)^(-1),lambda=t)
  );
  L22TOrderlambda1:=factor(residue(L22T*(lambda-t)^(-2),lambda=t)
```

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):
L22TOrderlambda2:=factor(residue(L22T*(lambda-t)^(-3),lambda=t)
):
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$$L22TOrderlambdaMinus3 := 0 \tag{1.5}$$
$$L22TOrderlambdaMinus2 := 0$$
$$L22TOrderlambdaMinus1 := s10 - h + s20$$
$$L22TOrderlambda0 := \frac{h\,(B12\,h + B12\,s10 - B12\,s20 - B22\,h + B22\,s10 - B22\,s20)}{s10 - s20}$$

We get that L_{2,2} behaves at lambda=t like (s20+s10-h)/(lambda-t) +O(1)
Combining with the asympotics at infinity and the apparent singularity we get that
L_{2,2}=(s20+s10-h)/(lambda-t) -(t12+t22)*lambda-(t11+t21) +(h-s20-s10-t10-t20)/(lambda-q)
i.e.

L_{2,2}=P_1(lambda)+h/(lambda-q)-h/(lambda-t)

```
> L22:=P1(lambda)+h/(lambda-q)-h/(lambda-t);
```

$$L22 := \frac{s10 + s20}{\lambda - t} - t11 - t21 + (-t12 - t22)\,\lambda + \frac{h}{\lambda - q} - \frac{h}{\lambda - t} \tag{1.6}$$

```
> L21T:=factor(simplify
  (WronskianTildeLambdaT/WronskianLambdabisT)):
  L21TOrderlambdaMinus5:=factor(residue(L21T*(lambda-t)^4,lambda=
  t));
  L21TOrderlambdaMinus4:=factor(residue(L21T*(lambda-t)^3,lambda=
  t));
  L21TOrderlambdaMinus3:=factor(residue(L21T*(lambda-t)^2,lambda=
  t));
  L21TOrderlambdaMinus2:=factor(residue(L21T*(lambda-t)^1,lambda=
  t));
  L21TOrderlambdaMinus1:=factor(residue(L21T*(lambda-t)^0,lambda=
  t));
  L21TOrderlambda0:=factor(residue(L21T*(lambda-t)^(-1),lambda=t)
  );
  L21TOrderlambda1:=factor(residue(L21T*(lambda-t)^(-2),lambda=t)
  ):
  L21TOrderlambda2:=factor(residue(L21T*(lambda-t)^(-3),lambda=t)
  ):
```

$$L21TOrderlambdaMinus5 := 0 \tag{1.7}$$
$$L21TOrderlambdaMinus4 := 0$$
$$L21TOrderlambdaMinus3 := 0$$
$$L21TOrderlambdaMinus2 := -s10\,s20$$
$$L21TOrderlambdaMinus1 := -\frac{h\,(B12\,h\,s20 + B12\,s10\,s20 - B12\,s20^2 - B22\,h\,s10 + B22\,s10^2 - B22\,s10\,s20)}{s10 - s20}$$
$$L21TOrderlambda0 := \frac{1}{(s10 - s20)^2}\,(h\,(B12^2\,h^2\,s20 - B12\,B22\,h^2\,s10 - B12\,B22\,h^2\,s20$$
$$- B12\,B22\,h\,s10^2 + 2\,B12\,B22\,h\,s10\,s20 - B12\,B22\,h\,s20^2 + B22^2\,h^2\,s10$$

$$-2\,B13\,h\,s10\,s20+2\,B13\,h\,s20^2-B13\,s10^2\,s20+2\,B13\,s10\,s20^2-B13\,s20^3$$
$$+2\,B23\,h\,s10^2-2\,B23\,h\,s10\,s20-B23\,s10^3+2\,B23\,s10^2\,s20-B23\,s10\,s20^2))$$

We get that L_{2,1} behaves at lambda=0 like -s10s20/(lambda-t)^2+ O(1/(lambda-t))
Combining with the asympotics at infinity and the apparent singularity we get that
L_{2,1}=-t22*t11*lambda^2-(t21*t12+t22*t11)*lambda-(t12*t20+t22*t10+t11*t21+h*t12)-s10s20/(lambda-t)^2 +C/(lambda-t) - p*h/(lambda-q)
i.e.

L_{2,1}=-P_2(\lambda)+P012/(lambda-t) +C/(lambda-t) -h*t12 + -h*p/(lambda-q)

```
> L21:=-P2(lambda)+P012/(lambda-t) +H/(lambda-t) -h*t12 -h*p/
  (lambda-q);
  L21:=-tdP2(lambda)+H/(lambda-t) -h*t12 -h*p/(lambda-q);
```

$$L21:=-\frac{s10\,s20}{(\lambda-t)^2}+t12\,s10+t12\,s20+t12\,t10-t10\,t22-t11\,t21-(t11\,t22+t12\,t21)\,\lambda-t12\,t22\,\lambda^2+\frac{H}{\lambda-t}-h\,t12-\frac{h\,p}{\lambda-q} \quad (1.8)$$

$$L21:=-\frac{s10\,s20}{(\lambda-t)^2}+t12\,s10+t12\,s20+t12\,t10-t10\,t22-t11\,t21-(t11\,t22+t12\,t21)\,\lambda-t12\,t22\,\lambda^2+\frac{H}{\lambda-t}-h\,t12-\frac{h\,p}{\lambda-q}$$

Auxiliary matrix A in the oper gauge

```
> WronskianLInfty:=factor(psi1Infty*Lpsi2Infty-psi2Infty*
  Lpsi1Infty):
  WronskianLT:=factor(psi1T*Lpsi2T-psi2T*Lpsi1T):
  A12Infty:=factor(simplify(WronskianLInfty/WronskianLambdaInfty)
  ):
  A12T:=factor(simplify(WronskianLT/WronskianLambdaT)):
  Y1Infty:=h*factor(dpsi1dlambdaInfty/psi1Infty):
  Y2Infty:=h*factor(dpsi2dlambdaInfty/psi2Infty):
  Y1T:=h*factor(dpsi1dlambdaT/psi1T):
  Y2T:=h*factor(dpsi2dlambdaT/psi2T):
  Z1Infty:=factor(Lpsi1Infty/psi1Infty):
  Z2Infty:=factor(Lpsi2Infty/psi2Infty):
  Z1T:=factor(Lpsi1T/psi1T):
  Z2T:=factor(Lpsi2T/psi2T):
  A12bisInfty:=factor(simplify((Z2Infty-Z1Infty)/(Y2Infty-
  Y1Infty))):
  A12bisT:=factor(simplify((Z2T-Z1T)/(Y2T-Y1T))):
  A11Infty:=factor(simplify( (Y2Infty*Z1Infty-Y1Infty*Z2Infty)/
  (Y2Infty-Y1Infty) )):
  A11T:=factor(simplify( (Y2T*Z1T-Y1T*Z2T)/(Y2T-Y1T) )):
```

```
factor(simplify(A12bisInfty-A12Infty));
factor(simplify(A12bisT-A12T));
factor(simplify(A12bisT-A12T));
```

$$0 \quad (2.1)$$
$$0$$
$$0$$

The deformation operator is defined as h*(alpha12 \partial_{t_{\infty^{(1)},2}}+ alpha22 \partial_{t_{\infty^{(2)},2}}+alpha11\partial_{t_{\infty^{(1)},1}}+ alpha21\partial_{t_{\infty^{(2)},1}} + Lt *partial_t)

```
> Lt20:=0:
  Lt10:=0:
  Lt12:=h*alpha12:
  Lt22:=h*alpha22:
  Lt11:=h*alpha11:
  Lt21:=h*alpha21:
  Ls10:=0:
  Ls20:=0:
```

```
> A12InftyLambda3:=factor(-residue(A12Infty/lambda^4,lambda=
  infinity));
  A12InftyLambda2:=factor(-residue(A12Infty/lambda^3,lambda=
  infinity));
  A12InftyLambda1:=factor(-residue(A12Infty/lambda^2,lambda=
  infinity));
  A12InftyLambda0:=factor(-residue(A12Infty/lambda^1,lambda=
  infinity));
  A12InftyLambdaMinus1:=factor(-residue(A12Infty/lambda^0,lambda=
  infinity)):
```

$$A12InftyLambda3 := 0 \quad (2.2)$$
$$A12InftyLambda2 := 0$$
$$A12InftyLambda1 := \frac{1}{2}\,\frac{\alpha 12-\alpha 22}{-t22+t12}$$
$$A12InftyLambda0 := \frac{1}{2}\,\frac{1}{(-t22+t12)^2}\left(2\,\alpha 11\,t12-2\,\alpha 11\,t22-\alpha 12\,t11+\alpha 12\,t21-2\,\alpha 21\,t12+2\,\alpha 21\,t22+\alpha 22\,t11-\alpha 22\,t21\right)$$

```
> A12InftyLambda0bis:=(1/2)*(t21-t11)/(t12-t22)^2*(alpha12-
  alpha22)+(alpha11-alpha21)/(t12-t22):
  simplify(series(A12InftyLambda0-A12InftyLambda0bis,alpha));
```

$$0 \quad (2.3)$$

```
> A12TLambdaMinus3:=factor(residue(A12T*(lambda-t)^2,lambda=t));
  A12TLambdaMinus2:=factor(residue(A12T*(lambda-t)^1,lambda=t));
  A12TLambdaMinus1:=factor(residue(A12T*(lambda-t)^0,lambda=t));
  A12TLambda0:=factor(residue(A12T*(lambda-t)^(-1),lambda=t));
  A12ZeroLambda1:=factor(residue(A12Zero*lambda^(-2),lambda=0));
```

$$A12TLambdaMinus3 := 0 \quad (2.4)$$
$$A12TLambdaMinus2 := 0$$
$$A12TLambdaMinus1 := 0$$
$$A12TLambda0 := -\frac{Lt}{h}$$
$$A12ZeroLambda1 := 0$$

> A_{1,2} behaves like $-\frac{Lt}{h}$ at \lambda=t and A_{1,2}=(alpha-alpha22)/2/(tinfty12-tinfty22)*lambda+(1/2)*(t21-t11)/(t12-t22)^2*(alpha12-alpha22)+(alpha11-alpha21)/(t12-t22) + O(1/lambda) at \lambda->\infty

Thus, we get that

A_{1,2}=(alpha12-alpha22)/2/(tinfty12-tinfty22)*lambda+nu0+q*mu/(lambda-q)

nu0=(1/2)*(t21-t11)/(t12-t22)^2*(alpha12-alpha22)+(alpha11-alpha21)/(t12-t22)

avec mu=(1/2)*(t21-t11)/(t12-t22)^2*(alpha12-alpha22)+(alpha11-alpha21)/(t12-t22)

```
> nuMinus1:=(alpha12-alpha22)/(2*(-t22+t12));
  nu0:=(1/2)*(t21-t11)/(t12-t22)^2*(alpha12-alpha22)+(alpha11-alpha21)/(t12-t22);
  A12Form:=lambda->nuMinus1*lambda+nu0+mu/(lambda-q);
```

$$nuMinus1 := \frac{\alpha 12 - \alpha 22}{-2\, t22 + 2\, t12} \quad (2.5)$$
$$\nu 0 := \frac{1}{2} \frac{(t21 - t11)\,(\alpha 12 - \alpha 22)}{(-t22 + t12)^2} + \frac{\alpha 11 - \alpha 21}{-t22 + t12}$$
$$A12Form := \lambda \rightarrow nuMinus1\, \lambda + \nu 0 + \frac{\mu}{\lambda - q}$$

```
> mu:=factor(solve(A12Form(t)=-Lt/h,mu)):
  mubis:=(1/2)*(t*(t12-t22)-t11+t21)*(q-t)/(-t22+t12)^2*(alpha12-alpha22)+(alpha11-alpha21)*(q-t)/(t12-t22)+(q-t)*Lt/h;
  factor(series(mu-mubis,alpha=0));
```

$$mubis := \frac{1}{2} \frac{(t\,(-t22 + t12) - t11 + t21)\,(-t + q)\,(\alpha 12 - \alpha 22)}{(-t22 + t12)^2} + \frac{(\alpha 11 - \alpha 21)\,(-t + q)}{-t22 + t12} + \frac{(-t + q)\,Lt}{h} \quad (2.6)$$
$$0$$

```
> A12Form:=nuMinus1*lambda+nu0+mubis/(lambda-q):
  series(simplify(series(A12Form-A12InftyLambda1*
```

```
lambda-A12InftyLambda0,lambda=infinity)),lambda=infinity,2);
series(simplify(series(A12Form,lambda=t,2)),lambda=t,1);
```

$$O\left(\frac{1}{\lambda}\right) \tag{2.7}$$

$$-\frac{Lt}{h}+O(\lambda-t)$$

Asymptotics of A_{1,1} at infinity and \lambda=t.

```
> A11InftyLambda3:=factor(-residue(A11Infty/lambda^4,lambda=
  infinity));
  A11InftyLambda2:=factor(-residue(A11Infty/lambda^3,lambda=
  infinity));
  A11InftyLambda1:=factor(-residue(A11Infty/lambda^2,lambda=
  infinity));
  A11InftyLambda0:=factor(-residue(A11Infty/lambda^1,lambda=
  infinity));
  A11InftyLambdaMinus1:=factor(-residue(A11Infty/lambda^0,lambda=
  infinity)):

  A11TLambdaMinus3:=factor(residue(A11T*(lambda-t)^2,lambda=t));
  A11TLambdaMinus2:=factor(residue(A11T*(lambda-t)^1,lambda=t));
  A11TLambdaMinus1:=factor(residue(A11T*(lambda-t)^0,lambda=t));
  A11TLambda0:=factor(residue(A11T*(lambda-t)^(-1),lambda=t));
  A11TLambda1:=factor(residue(A11T*(lambda-t)^(-2),lambda=t)):
```

$$A11InftyLambda3 := 0 \tag{2.8}$$

$$A11InftyLambda2 := \frac{1}{2}\,\frac{\alpha 12\,t22-\alpha 22\,t12}{-t22+t12}$$

$$A11InftyLambda1 := \frac{1}{2}\,\frac{1}{(-t22+t12)^2}\left(2\,\alpha 11\,t12\,t22-2\,\alpha 11\,t22^2-\alpha 12\,t11\,t22 + \alpha 12\,t12\,t21-2\,\alpha 21\,t12^2+2\,\alpha 21\,t12\,t22+\alpha 22\,t11\,t22-\alpha 22\,t12\,t21\right)$$

$$A11InftyLambda0 := -\frac{1}{2}\,\frac{1}{(-t22+t12)^3}\left(2\,LA10\,t12^2\,t22-4\,LA10\,t12\,t22^2 + 2\,LA10\,t22^3-2\,LA20\,t12^3+4\,LA20\,t12^2\,t22-2\,LA20\,t12\,t22^2+2\,\alpha 11\,t11\,t12\,t22 - 2\,\alpha 11\,t11\,t22^2-2\,\alpha 11\,t12^2\,t21+2\,\alpha 11\,t12\,t21\,t22-\alpha 12\,h\,t12^2+\alpha 12\,h\,t12\,t22 + \alpha 12\,t10\,t12\,t22-\alpha 12\,t10\,t22^2-\alpha 12\,t11^2\,t22+\alpha 12\,t11\,t12\,t21+\alpha 12\,t11\,t21\,t22 - \alpha 12\,t12^2\,t20+\alpha 12\,t12\,t20\,t22-\alpha 12\,t12\,t21^2-2\,\alpha 21\,t11\,t12\,t22+2\,\alpha 21\,t11\,t22^2 + 2\,\alpha 21\,t12^2\,t21-2\,\alpha 21\,t12\,t21\,t22+\alpha 22\,h\,t12^2-\alpha 22\,h\,t12\,t22-\alpha 22\,t10\,t12\,t22 + \alpha 22\,t10\,t22^2+\alpha 22\,t11^2\,t22-\alpha 22\,t11\,t12\,t21-\alpha 22\,t11\,t21\,t22+\alpha 22\,t12^2\,t20 - \alpha 22\,t12\,t20\,t22+\alpha 22\,t12\,t21^2\right)$$

$$A11TLambdaMinus3 := 0$$

$$A11TLambdaMinus2 := 0$$

$$A11TLambdaMinus1 := 0$$

$$A11TLambda0 := -\frac{LB10\,s20-LB20\,s10}{s10-s20}$$

```
> A11InftyLambda1bis:=(1/2)*(t12*t21-t11*t22)/(t12-t22)^2*
  (alpha12-alpha22)+ (alpha11*t22-alpha21*t12)/(t12-t22):
  factor(simplify(series(A11InftyLambda1-A11InftyLambda1bis,
  alpha22)));
```

$$0 \tag{2.9}$$

A_{1,1} behaves like O(1) en \lambda=t and A_{1,1}=(alpha12*t22-alpha22*t12)/(2*(t12-t22))*lambda^2+(1/2)*(t12*t21-t11*t22)/(t12-t22)^2*(alpha12-alpha22)+ (alpha11*t22-alpha21*t12)/(t12-t22)*lambda+ O(1) quand \lambda->\infty
Thus,

A_{1,1}=c2*lambda^2 +c1*lambda+ c0+ rho/(lambda-q)
with c_2=(alpha12*t22-alpha22*t12)/(2*(t12-t22)) and c1=(1/2)*(t12*t21-t11*t22)/(t12-t22)^2*(alpha12-alpha22)+ (alpha11*t22-alpha21*t12)/(t12-t22)

```
> c2:=(alpha12*t22-alpha22*t12)/(2*(t12-t22));
  c1:=(1/2)*(t12*t21-t11*t22)/(t12-t22)^2*(alpha12-alpha22)+
  (alpha11*t22-alpha21*t12)/(t12-t22);
  A11Form:=c2*lambda^2+c1*lambda+rho/(lambda-q);
```

$$c2 := \frac{\alpha 12\, t22 - \alpha 22\, t12}{-2\, t22 + 2\, t12} \tag{2.10}$$

$$c1 := \frac{1}{2}\, \frac{(-t11\, t22 + t12\, t21)\,(\alpha 12 - \alpha 22)}{(-t22 + t12)^2} + \frac{\alpha 11\, t22 - \alpha 21\, t12}{-t22 + t12}$$

$$A11Form := \frac{(\alpha 12\, t22 - \alpha 22\, t12)\, \lambda^2}{-2\, t22 + 2\, t12} + \left(\frac{1}{2}\, \frac{(-t11\, t22 + t12\, t21)\,(\alpha 12 - \alpha 22)}{(-t22 + t12)^2} + \frac{\alpha 11\, t22 - \alpha 21\, t12}{-t22 + t12} \right) \lambda + \frac{\rho}{\lambda - q}$$

```
> series(simplify(series(A11Form-A11InftyLambda2*lambda^2-
  A11InftyLambda1*lambda,lambda=infinity)),lambda=infinity,2);
  series(simplify(series(A11Form,lambda=t,1)),lambda=t,1);
```

$$\mathrm{O}\left(\frac{1}{\lambda}\right) \tag{2.11}$$

$$\frac{1}{2}\, \frac{1}{(-t22 + t12)^2\,(-t + q)} \Bigg((-t22 + t12)\,(-\alpha 12\, t22 + \alpha 22\, t12)\, t^3 + \Big((-\alpha 22\, q + 2\, \alpha 21)\, t12^2 + \big(((\alpha 22 + \alpha 12)\, q - 2\, \alpha 21 - 2\, \alpha 11)\, t22 + t21\,(-\alpha 12 + \alpha 22) \big)\, t12 - t22\, \big((\alpha 12\, q - 2\, \alpha 11)\, t22 + t11\,(-\alpha 12 + \alpha 22) \big) \Big)\, t^2 - 2 \left(\alpha 21\, t12^2 + \left((-\alpha 21 - \alpha 11)\, t22 + \frac{1}{2}\, t21\,(-\alpha 12 + \alpha 22) \right) t12 - \frac{1}{2}\, t22\, (-2\, \alpha 11\, t22 + t11\,(-\alpha 12 + \alpha 22)) \right) q\, t - 2\, \rho\,(-t22 + t12)^2 \Bigg) + \mathrm{O}(\lambda - t)$$