In this Maple file, we compute the evolution equations for the Painlevé 6 equations using the compatibility equation of the Lax system. We also obtain the expression of the Lax matrices in the geometric gauge without apparent singularities.
The auxiliary operator is \hbar*\partial_t where t is the position of one pole.
The symplectic reduction has already been done: the simple poles are located at \lambda= 0,1, infinity, t

Lax matrices in the oper gauge from previous Maple files

Summary of previous files: We have the expression for some coefficients of the Lax matrix L and of A.

```
> restart:
  with(LinearAlgebra):
  P011 := t010+t020;
  P111 := t110+t120;
  Pt11 := tt10+tt20;
  P022 := t010*t020;
  P122 := t120*t110;
  Pt22 := tt20*tt10;
  CoherenceEquation1 := tinfty10+tt10+tt20+t010+t020+tinfty20+
  t110+t120;
  CoherenceEquation2 := Pt12+P012+P112;
  CoherenceEquation3 := -tinfty10*tinfty20+tt20*tt10+Pt12*t+t010*
  t020+P112+t120*t110;
  CoherenceEquation4 := a0+a1+at+ h*p;
  CoherenceEquation5 := a1+t*at+t010*t020+t110*t120+tt10*tt20-
  tinfty10*tinfty20+h*p*q -h*tinfty10;

  P1:=lambda->P011/lambda+P111/(lambda-1)+Pt11/(lambda-t);
  dP1dlambda:=unapply(diff(P1(lambda),lambda),lambda):
  P2:=lambda->P022/lambda^2+P012/lambda+P122/(lambda-1)^2+P112/
  (lambda-1)+Pt22/(lambda-t)^2+Pt12/(lambda-t);
  dP2dlambda:=unapply(diff(P2(lambda),lambda),lambda):

  tdP2:=unapply(P2(lambda)-P012/lambda-P112/(lambda-1)-Pt12/
  (lambda-t),lambda);
  dtdP2dlambda:=unapply(diff(tdP2(lambda),lambda),lambda):

  mubis:=q*(q-1)*(q-t)/t/(t-1);
  nuMinus1bis:=mubis/q/(q-1);
  nu0bis:=mubis/q;


  L:=Matrix(2,2,0):
```

```
L[1,1]:=0:
L[1,2]:=1:
L[2,1]:=-t110*t120/(lambda-1)^2 -a1/(lambda-1)   -t010*
t020/lambda^2 -a0/lambda -tt10*tt20/(lambda-t)^2 -at/(lambda-t)
- p*h/(lambda-q):
L[2,2]:=(t110+t120-h)/(lambda-1)+ (t010+t020-h)/lambda+ (tt10+
tt20-h)/(lambda-t)   +h/(lambda-q):
P1(lambda)-h/lambda-h/(lambda-1)-h/(lambda-t)+ h/(lambda-q);
-P2(lambda) -p*h/(lambda-q);
C01:=-a0:
C11:=-a1:
Ct1:=-at:

rho:=q*p*(q-1)*(q-t)/t/(t-1);
A:=Matrix(2,2,0):
A[1,1]:=C-rho/(lambda-q):
A[1,2]:=(a*(lambda-t)^2+b*(lambda-t)+q-t)/(lambda-q):
A[2,1]:=AA21(lambda):
A[2,2]:=AA22(lambda):
dAdlambda:=Matrix(2,2,0):
for i from 1 to 2 do for j from 1 to 2 do dAdlambda[i,j]:=diff
(A[i,j],lambda):  od: od:

L;
A;

nuMinus1:=factor(-residue(A[1,2]/lambda^2,lambda=infinity));
nu0:=factor(-residue(A[1,2]/lambda,lambda=infinity));
mu:=factor(residue(A[1,2],lambda=q));


Q2:=unapply(-p*(q-0)*(q-1)*(q-t),lambda):
J:=Matrix(2,2,0):
J[1,1]:=1:
J[1,2]:=0:
J[2,1]:=Q2(lambda)/(lambda-q):
J[2,2]:=(lambda-0)^1*(lambda-1)^1*(lambda-t)^1/(lambda-q):
dJdlambda:=Matrix(2,2,0):
for i from 1 to 2 do for j from 1 to 2 do dJdlambda[i,j]:=diff
(J[i,j],lambda): od: od:
J:
```

```
LJ:=Matrix(2,2,0):
LJ[1,1]:=0:
LJ[1,2]:=0:
LJ[2,2]:=diff(J[2,2],q)*Lq+diff(J[2,2],p)*Lp+h*diff(J[2,2],t):
LJ[2,1]:=diff(J[2,1],q)*Lq+diff(J[2,1],p)*Lp+h*diff(J[2,1],t):
LJ:



checkL:=simplify(Multiply(Multiply(J,L),J^(-1))+h*Multiply
(dJdlambda,J^(-1))):
checkA:=simplify(Multiply(Multiply(J,A),J^(-1))+Multiply(LJ,J^
(-1))):


V:=Matrix(3,3,0):
V[1,1]:=1:
V[1,2]:=1:
V[1,3]:=1:
V[2,1]:=0:
V[2,2]:=1:
V[2,3]:=t:
V[3,1]:=1/q:
V[3,2]:=1/(q-1):
V[3,3]:=1/(q-t):
V;

B:=Matrix(3,1,0):
B[1,1]:=h*p:
B[2,1]:=h*p*q+t1+t2+t3-tinfty10*(h+tinfty20):
B[3,1]:=p^2 -p*((T1-h)/q+(T2-h)/(q-1)+(T3-h)/(q-t))+t1/q^2+t2/
(q-1)^2+t3/(q-t)^2:
B;
t1:=t010*t020;
t2:=t110*t120;
t3:=tt10*tt20;
T1:=t010+t020;
T2:=t110+t120;
T3:=tt10+tt20;

CMatrix:=Multiply(V^(-1),B):
C01bis:=CMatrix[1,1]:
```

```
C11bis:=CMatrix[2,1]:
Ct1bis:=CMatrix[3,1]:
```

$$P011 := t010 + t020 \tag{1.1}$$
$$P111 := t110 + t120$$
$$Pt11 := tt10 + tt20$$
$$P022 := t010\, t020$$
$$P122 := t120\, t110$$
$$Pt22 := tt20\, tt10$$
$$CoherenceEquation1 := tinfty10 + tt10 + tt20 + t010 + t020 + tinfty20 + t110 + t120$$
$$CoherenceEquation2 := Pt12 + P012 + P112$$
$$CoherenceEquation3 := Pt12\, t + t010\, t020 + t110\, t120 + tt10\, tt20 - tinfty10\, tinfty20 + P112$$
$$CoherenceEquation4 := h\, p + a0 + a1 + at$$
$$CoherenceEquation5 := h\, p\, q + at\, t - h\, tinfty10 + t010\, t020 + t110\, t120 + tt10\, tt20 - tinfty10\, tinfty20 + a1$$
$$P1 := \lambda \rightarrow \frac{P011}{\lambda} + \frac{P111}{\lambda - 1} + \frac{Pt11}{\lambda - t}$$
$$P2 := \lambda \rightarrow \frac{P022}{\lambda^2} + \frac{P012}{\lambda} + \frac{P122}{(\lambda - 1)^2} + \frac{P112}{\lambda - 1} + \frac{Pt22}{(\lambda - t)^2} + \frac{Pt12}{\lambda - t}$$
$$tdP2 := \lambda \rightarrow \frac{t010\, t020}{\lambda^2} + \frac{t120\, t110}{(\lambda - 1)^2} + \frac{tt20\, tt10}{(\lambda - t)^2}$$
$$mubis := \frac{q\,(q-1)\,(q-t)}{t\,(t-1)}$$
$$nuMinus1bis := \frac{q-t}{t\,(t-1)}$$
$$nu0bis := \frac{(q-1)\,(q-t)}{t\,(t-1)}$$
$$\frac{t010 + t020}{\lambda} + \frac{t110 + t120}{\lambda - 1} + \frac{tt10 + tt20}{\lambda - t} - \frac{h}{\lambda} - \frac{h}{\lambda - 1} - \frac{h}{\lambda - t} + \frac{h}{\lambda - q}$$
$$-\frac{t010\, t020}{\lambda^2} - \frac{P012}{\lambda} - \frac{t120\, t110}{(\lambda - 1)^2} - \frac{P112}{\lambda - 1} - \frac{tt20\, tt10}{(\lambda - t)^2} - \frac{Pt12}{\lambda - t} - \frac{p\, h}{\lambda - q}$$
$$\rho := \frac{q\, p\,(q-1)\,(q-t)}{t\,(t-1)}$$
$$\Big[\big[0,\ 1\big],$$
$$\Big[-\frac{t120\, t110}{(\lambda-1)^2} - \frac{a1}{\lambda - 1} - \frac{t010\, t020}{\lambda^2} - \frac{a0}{\lambda} - \frac{tt20\, tt10}{(\lambda - t)^2} - \frac{at}{\lambda - t} - \frac{p\, h}{\lambda - q},$$
$$\frac{t110 + t120 - h}{\lambda - 1} + \frac{t010 + t020 - h}{\lambda} + \frac{tt10 + tt20 - h}{\lambda - t} + \frac{h}{\lambda - q}\Big]\Big]$$

$$\begin{bmatrix} C - \frac{q\,p\,(q-1)\,(q-t)}{t\,(t-1)\,(\lambda-q)} & \frac{a\,(\lambda-t)^2 + b\,(\lambda-t) + q - t}{\lambda - q} \\ AA21(\lambda) & AA22(\lambda) \end{bmatrix}$$

$$nuMinus1 := a$$

$$\nu 0 := a\,q - 2\,a\,t + b$$

$$\mu := (q-t)\,(a\,q - a\,t + b + 1)$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & t \\ \frac{1}{q} & \frac{1}{q-1} & \frac{1}{q-t} \end{bmatrix}$$

$$\begin{bmatrix} h\,p \\ h\,p\,q + t1 + t2 + t3 - tinfty10\,(h + tinfty20) \\ p^2 - p\left(\frac{T1-h}{q} + \frac{T2-h}{q-1} + \frac{T3-h}{q-t}\right) + \frac{t1}{q^2} + \frac{t2}{(q-1)^2} + \frac{t3}{(q-t)^2} \end{bmatrix}$$

$$t1 := t010\,t020$$

$$t2 := t120\,t110$$

$$t3 := tt20\,tt10$$

$$T1 := t010 + t020$$

$$T2 := t110 + t120$$

$$T3 := tt10 + tt20$$

Solving the compatibility equations to obtain the Hamiltonian evolutions.

The compatibility equation is \mathcal{L}L=h\partial_\lambda A+[A,L]
Since the first line of L is trivial, we may easily obtain A[2,1] et A[2,2] to obtain the full expression for A

```
> LL:=h*dAdlambda+(Multiply(A,L)-Multiply(L,A)):
  Entry11:=LL[1,1]:
  Entry12:=LL[1,2]:
  AA21:=unapply(solve(Entry11=0,AA21(lambda)),lambda):
  AA21bis:=h*dAdlambda[1,1]+A[1,2]*L[2,1]:
  simplify(AA21(lambda)-AA21bis);
  AA22:=unapply(solve(Entry12=0,AA22(lambda)),lambda):
  AA22bis:=h*dAdlambda[1,2]+A[1,1]+A[1,2]*L[2,2]:
  simplify(AA22(lambda)-AA22bis);
  simplify(Entry11);
  simplify(Entry12);
  LL:=h*dAdlambda+(Multiply(A,L)-Multiply(L,A)):
```

$$0 \qquad \textbf{(2.1)}$$

0
0
0

We now compute the action of \mathcal{L} on L[2,2] et L[2,1] to obtain the evolution equations

Evolution of entry L_{2,2}

```
> Entry22:=simplify(LL[2,2]):
  Entry22TermLambdaMinusqSquare:=factor(residue(Entry22*(lambda-
  q),lambda=q)):
  Entry22TermLambdaMinusq:=factor(residue(Entry22,lambda=q));
  Entry22TermLambdaZeroMinus4:=factor(residue(Entry22*lambda^3,
  lambda=0));
  Entry22TermLambdaZeroMinus3:=factor(residue(Entry22*lambda^2,
  lambda=0));
  Entry22TermLambdaZeroMinus2:=factor(residue(Entry22*lambda,
  lambda=0));
  Entry22TermLambdaZeroMinus1:=factor(residue(Entry22,lambda=0));
  Entry22TermLambdaInfty2:=factor(-residue(Entry22/lambda^3,
  lambda=infinity));
  Entry22TermLambdaInfty1:=factor(-residue(Entry22/lambda^2,
  lambda=infinity));
  Entry22TermLambdaInfty0:=factor(-residue(Entry22/lambda,lambda=
  infinity));
  Entry22TermLambdaOneMinus4:=factor(residue(Entry22*(lambda-1)
  ^3,lambda=1));
  Entry22TermLambdaOneMinus3:=factor(residue(Entry22*(lambda-1)
  ^2,lambda=1));
  Entry22TermLambdaOneMinus2:=factor(residue(Entry22*(lambda-1),
  lambda=1));
  Entry22TermLambdaOneMinus1:=factor(residue(Entry22,lambda=1));
  Entry22TermLambdaTMinus4:=factor(residue(Entry22*(lambda-t)^3,
  lambda=t));
  Entry22TermLambdaTMinus3:=factor(residue(Entry22*(lambda-t)^2,
  lambda=t));
  Entry22TermLambdaTMinus2:=factor(residue(Entry22*(lambda-t),
  lambda=t));
  Entry22TermLambdaTMinus1:=factor(residue(Entry22,lambda=t));

  simplify( Entry22-(Entry22TermLambdaMinusqSquare/(lambda-q)^2+
  Entry22TermLambdaMinusq/(lambda-q)
  +Entry22TermLambdaZeroMinus4/lambda^4+
  Entry22TermLambdaZeroMinus3/lambda^3+
  Entry22TermLambdaZeroMinus2/lambda^2+
  Entry22TermLambdaZeroMinus1/lambda
```

```
+Entry22TermLambdaOneMinus4/(lambda-1)^4+
Entry22TermLambdaOneMinus3/(lambda-1)^3+
Entry22TermLambdaOneMinus2/(lambda-1)^2+
Entry22TermLambdaOneMinus1/(lambda-1)
+Entry22TermLambdaTMinus4/(lambda-t)^4+
Entry22TermLambdaTMinus3/(lambda-t)^3+Entry22TermLambdaTMinus2/
(lambda-t)^2+Entry22TermLambdaTMinus1/(lambda-t)
+Entry22TermLambdaInfty0+Entry22TermLambdaInfty1*lambda+
Entry22TermLambdaInfty2*lambda^2) );
L[2,2];
```

$$Entry22TermLambdaMinusq := 0 \tag{2.2}$$

$$Entry22TermLambdaZeroMinus4 := 0$$

$$Entry22TermLambdaZeroMinus3 := 0$$

$$Entry22TermLambdaZeroMinus2 := -\frac{(h - t010 - t020)\left(a\,t^2 - b\,t + q - t\right) h}{q}$$

$$Entry22TermLambdaZeroMinus1 := 0$$

$$Entry22TermLambdaInfty2 := 0$$

$$Entry22TermLambdaInfty1 := 0$$

$$Entry22TermLambdaInfty0 := 0$$

$$Entry22TermLambdaOneMinus4 := 0$$

$$Entry22TermLambdaOneMinus3 := 0$$

$$Entry22TermLambdaOneMinus2 := -\frac{(-t120 + h - t110)\left(a\,t^2 - 2\,a\,t - b\,t + a + b + q - t\right) h}{q-1}$$

$$Entry22TermLambdaOneMinus1 := 0$$

$$Entry22TermLambdaTMinus4 := 0$$

$$Entry22TermLambdaTMinus3 := 0$$

$$Entry22TermLambdaTMinus2 := -(-tt10 - tt20 + h)\,h$$

$$Entry22TermLambdaTMinus1 := 0$$

$$0$$

$$\frac{t110 + t120 - h}{\lambda - 1} + \frac{t010 + t020 - h}{\lambda} + \frac{tt10 + tt20 - h}{\lambda - t} + \frac{h}{\lambda - q}$$

Since the deformation operator is \hbar*partial_t we cannot have double poles at lambda=1 or lambda=0 but only a double pole at lambda=t with a numerator equal to tt10+tt20-h which is indeed the case.

```
> a:=(q-t)/t/(t-1);
b:=(q-t)*(2*t-1)/t/(t-1);
simplify(Entry22TermLambdaZeroMinus2);
simplify(Entry22TermLambdaOneMinus2);
factor(A[1,2]);
factor(nuMinus1);
factor(nu0);
factor(mu);
```

```
simplify(mu-mubis);
simplify(nuMinus1-nuMinus1bis);
simplify(nu0-nu0bis);
```

$$a := \frac{q-t}{t\,(t-1)} \tag{2.3}$$

$$b := \frac{(q-t)\,(2\,t-1)}{t\,(t-1)}$$

$$0$$

$$0$$

$$\frac{\lambda\,(q-t)\,(\lambda-1)}{(t-1)\,t\,(\lambda-q)}$$

$$\frac{q-t}{t\,(t-1)}$$

$$\frac{(q-1)\,(q-t)}{t\,(t-1)}$$

$$\frac{q\,(q-1)\,(q-t)}{t\,(t-1)}$$

$$0$$

$$0$$

$$0$$

```
> Lq:=factor(Entry22TermLambdaMinusqSquare/h):
Lqbis:=2*q*p*(q-1)*(q-t)/t/(t-1)-q*(q-1)*(q-t)/t/(t-1)*P1(q)+q*
(q-1)/t/(t-1)*h;
simplify(Lq-Lqbis);
```

$$Lqbis := \frac{2\,q\,p\,(q-1)\,(q-t)}{t\,(t-1)} - \frac{q\,(q-1)\,(q-t)\left(\frac{t010+t020}{q}+\frac{t110+t120}{q-1}+\frac{tt10+tt20}{q-t}\right)}{t\,(t-1)} + \frac{q\,(q-1)\,h}{t\,(t-1)} \tag{2.4}$$

$$0$$

Let us look at \mathcal{L}[L[2,1]]

```
> Entry21:=simplify(LL[2,1]):
Entry21TermLambdaMinusqCube:=factor(residue(Entry21*(lambda-q)
^2,lambda=q));
Entry21TermLambdaMinusqSquare:=factor(residue(Entry21*(lambda-
q),lambda=q));
Entry21TermLambdaMinusq:=factor(residue(Entry21,lambda=q));
Entry21TermLambdaZeroMinus5:=factor(residue(Entry21*lambda^4,
lambda=0));
Entry21TermLambdaZeroMinus4:=factor(residue(Entry21*lambda^3,
lambda=0));
Entry21TermLambdaZeroMinus3:=factor(residue(Entry21*lambda^2,
lambda=0));
Entry21TermLambdaZeroMinus2:=factor(residue(Entry21*lambda,
lambda=0));
```

```
Entry21TermLambdaZeroMinus1:=factor(residue(Entry21,lambda=0));
Entry21TermLambdaInfty3:=factor(-residue(Entry21/lambda^4,
lambda=infinity));
Entry21TermLambdaInfty2:=factor(-residue(Entry21/lambda^3,
lambda=infinity));
Entry21TermLambdaInfty1:=factor(-residue(Entry21/lambda^2,
lambda=infinity));
Entry21TermLambdaInfty0:=factor(-residue(Entry21/lambda,lambda=
infinity));
Entry21TermLambdaOneMinus5:=factor(residue(Entry21*(lambda-1)
^4,lambda=1));
Entry21TermLambdaOneMinus4:=factor(residue(Entry21*(lambda-1)
^3,lambda=1));
Entry21TermLambdaOneMinus3:=factor(residue(Entry21*(lambda-1)
^2,lambda=1));
Entry21TermLambdaOneMinus2:=factor(residue(Entry21*(lambda-1),
lambda=1));
Entry21TermLambdaOneMinus1:=factor(residue(Entry21,lambda=1));
Entry21TermLambdaTMinus5:=factor(residue(Entry21*(lambda-t)^4,
lambda=t));
Entry21TermLambdaTMinus4:=factor(residue(Entry21*(lambda-t)^3,
lambda=t));
Entry21TermLambdaTMinus3:=factor(residue(Entry21*(lambda-t)^2,
lambda=t));
Entry21TermLambdaTMinus2:=factor(residue(Entry21*(lambda-t),
lambda=t));
Entry21TermLambdaTMinus1:=factor(residue(Entry21,lambda=t));

simplify( Entry21-(Entry21TermLambdaMinusqCube/(lambda-q)^3+
Entry21TermLambdaMinusqSquare/(lambda-q)^2+
Entry21TermLambdaMinusq/(lambda-q)
+Entry21TermLambdaZeroMinus5/lambda^5+
Entry21TermLambdaZeroMinus4/lambda^4+
Entry21TermLambdaZeroMinus3/lambda^3+
Entry21TermLambdaZeroMinus2/lambda^2+
Entry21TermLambdaZeroMinus1/lambda
+Entry21TermLambdaOneMinus5/(lambda-1)^5+
Entry21TermLambdaOneMinus4/(lambda-1)^4+
Entry21TermLambdaOneMinus3/(lambda-1)^3+
Entry21TermLambdaOneMinus2/(lambda-1)^2+
Entry21TermLambdaOneMinus1/(lambda-1)
+Entry21TermLambdaTMinus5/(lambda-t)^5+
```

```
Entry21TermLambdaTMinus4/(lambda-t)^4+Entry21TermLambdaTMinus3/
(lambda-t)^3+ Entry21TermLambdaTMinus2/(lambda-t)^2+
Entry21TermLambdaTMinus1/(lambda-t)
+Entry21TermLambdaInfty0+Entry21TermLambdaInfty1*lambda+
Entry21TermLambdaInfty2*lambda^2+Entry21TermLambdaInfty3*
lambda^3) );
L[2,1];
```

$$\mathit{Entry21TermLambdaMinusqCube} := 0 \qquad \textbf{(2.5)}$$

$$\begin{aligned}
&\mathit{Entry21TermLambdaMinusqSquare} := \frac{1}{t\,(t-1)\,(q-t)\,q\,(q-1)}\Big(\big(5\,h\,p\,q^5-9\,h\,p\,q^4\,t\\
&+4\,h\,p\,q^3\,t^2-p\,q^5\,t010-p\,q^5\,t020-p\,q^5\,t110-p\,q^5\,t120-p\,q^5\,tt10-p\,q^5\,tt20\\
&+2\,p\,q^4\,t\,t010+2\,p\,q^4\,t\,t020+2\,p\,q^4\,t\,t110+2\,p\,q^4\,t\,t120+p\,q^4\,t\,tt10+p\,q^4\,t\,tt20\\
&-p\,q^3\,t^2\,t010-p\,q^3\,t^2\,t020-p\,q^3\,t^2\,t110-p\,q^3\,t^2\,t120+2\,a0\,q^5-4\,a0\,q^4\,t\\
&+2\,a0\,q^3\,t^2+2\,a1\,q^5-4\,a1\,q^4\,t+2\,a1\,q^3\,t^2+2\,at\,q^5-2\,at\,q^4\,t-8\,h\,p\,q^4\\
&+14\,h\,p\,q^3\,t-6\,h\,p\,q^2\,t^2+2\,p\,q^4\,t010+2\,p\,q^4\,t020+p\,q^4\,t110+p\,q^4\,t120\\
&+2\,p\,q^4\,tt10+2\,p\,q^4\,tt20-4\,p\,q^3\,t\,t010-4\,p\,q^3\,t\,t020-2\,p\,q^3\,t\,t110-2\,p\,q^3\,t\,t120\\
&-2\,p\,q^3\,t\,tt10-2\,p\,q^3\,t\,tt20+2\,p\,q^2\,t^2\,t010+2\,p\,q^2\,t^2\,t020+p\,q^2\,t^2\,t110+p\,q^2\,t^2\,t120\\
&+2\,q^4\,t010\,t020+2\,q^4\,t110\,t120+2\,q^4\,tt10\,tt20-4\,q^3\,t\,t010\,t020-4\,q^3\,t\,t110\,t120\\
&+2\,q^2\,t^2\,t010\,t020+2\,q^2\,t^2\,t110\,t120-4\,a0\,q^4+8\,a0\,q^3\,t-4\,a0\,q^2\,t^2-2\,a1\,q^4\\
&+4\,a1\,q^3\,t-2\,a1\,q^2\,t^2-4\,at\,q^4+4\,at\,q^3\,t+3\,h\,p\,q^3-5\,h\,p\,q^2\,t+2\,h\,p\,q\,t^2\\
&-p\,q^3\,t010-p\,q^3\,t020-p\,q^3\,tt10-p\,q^3\,tt20+2\,p\,q^2\,t\,t010+2\,p\,q^2\,t\,t020\\
&+p\,q^2\,t\,tt10+p\,q^2\,t\,tt20-p\,q\,t^2\,t010-p\,q\,t^2\,t020-4\,q^3\,t010\,t020-4\,q^3\,tt10\,tt20\\
&+8\,q^2\,t\,t010\,t020-4\,q\,t^2\,t010\,t020+2\,a0\,q^3-4\,a0\,q^2\,t+2\,a0\,q\,t^2+2\,at\,q^3\\
&-2\,at\,q^2\,t+2\,q^2\,t010\,t020+2\,q^2\,tt10\,tt20-4\,q\,t\,t010\,t020+2\,t^2\,t010\,t020\big)\,h\Big)
\end{aligned}$$

$$\begin{aligned}
&\mathit{Entry21TermLambdaMinusq} := -\frac{1}{t\,(t-1)\,(q-1)^2\,(q-t)^2\,q^2}\Big(h\,\big(4\,h\,p\,q^7-10\,h\,p\,q^6\,t\\
&+9\,h\,p\,q^5\,t^2-3\,h\,p\,q^4\,t^3-p\,q^7\,t010-p\,q^7\,t020-p\,q^7\,t110-p\,q^7\,t120-p\,q^7\,tt10\\
&-p\,q^7\,tt20+3\,p\,q^6\,t\,t010+3\,p\,q^6\,t\,t020+3\,p\,q^6\,t\,t110+3\,p\,q^6\,t\,t120+p\,q^6\,t\,tt10\\
&+p\,q^6\,t\,tt20-3\,p\,q^5\,t^2\,t010-3\,p\,q^5\,t^2\,t020-3\,p\,q^5\,t^2\,t110-3\,p\,q^5\,t^2\,t120\\
&+p\,q^4\,t^3\,t010+p\,q^4\,t^3\,t020+p\,q^4\,t^3\,t110+p\,q^4\,t^3\,t120+a0\,q^7-3\,a0\,q^6\,t+3\,a0\,q^5\,t^2\\
&-a0\,q^4\,t^3+a1\,q^7-3\,a1\,q^6\,t+3\,a1\,q^5\,t^2-a1\,q^4\,t^3+at\,q^7-at\,q^6\,t-9\,h\,p\,q^6\\
&+21\,h\,p\,q^5\,t-18\,h\,p\,q^4\,t^2+6\,h\,p\,q^3\,t^3+3\,p\,q^6\,t010+3\,p\,q^6\,t020+p\,q^6\,t110\\
&+p\,q^6\,t120+3\,p\,q^6\,tt10+3\,p\,q^6\,tt20-9\,p\,q^5\,t\,t010-9\,p\,q^5\,t\,t020-3\,p\,q^5\,t\,t110\\
&-3\,p\,q^5\,t\,t120-3\,p\,q^5\,t\,tt10-3\,p\,q^5\,t\,tt20+9\,p\,q^4\,t^2\,t010+9\,p\,q^4\,t^2\,t020\\
&+3\,p\,q^4\,t^2\,t110+3\,p\,q^4\,t^2\,t120-3\,p\,q^3\,t^3\,t010-3\,p\,q^3\,t^3\,t020-p\,q^3\,t^3\,t110\\
&-p\,q^3\,t^3\,t120+2\,q^6\,t010\,t020+2\,q^6\,t110\,t120+2\,q^6\,tt10\,tt20-6\,q^5\,t\,t010\,t020\\
&-6\,q^5\,t\,t110\,t120+6\,q^4\,t^2\,t010\,t020+6\,q^4\,t^2\,t110\,t120-2\,q^3\,t^3\,t010\,t020\\
&-2\,q^3\,t^3\,t110\,t120-3\,a0\,q^6+9\,a0\,q^5\,t-9\,a0\,q^4\,t^2+3\,a0\,q^3\,t^3-a1\,q^6+3\,a1\,q^5\,t\\
&-3\,a1\,q^4\,t^2+a1\,q^3\,t^3-3\,at\,q^6+3\,at\,q^5\,t+7\,h\,p\,q^5-15\,h\,p\,q^4\,t+12\,h\,p\,q^3\,t^2\\
&-4\,h\,p\,q^2\,t^3-3\,p\,q^5\,t010-3\,p\,q^5\,t020-3\,p\,q^5\,tt10-3\,p\,q^5\,tt20+9\,p\,q^4\,t\,t010\\
&+9\,p\,q^4\,t\,t020+3\,p\,q^4\,t\,tt10+3\,p\,q^4\,t\,tt20-9\,p\,q^3\,t^2\,t010-9\,p\,q^3\,t^2\,t020
\end{aligned}$$

$$+3\,p\,q^2\,t^3\,t010+3\,p\,q^2\,t^3\,t020-6\,q^5\,t010\,t020-6\,q^5\,tt10\,tt20+18\,q^4\,t\,t010\,t020$$
$$-18\,q^3\,t^2\,t010\,t020+6\,q^2\,t^3\,t010\,t020+3\,a0\,q^5-9\,a0\,q^4\,t+9\,a0\,q^3\,t^2-3\,a0\,q^2\,t^3$$
$$+3\,at\,q^5-3\,at\,q^4\,t-2\,h\,p\,q^4+4\,h\,p\,q^3\,t-3\,h\,p\,q^2\,t^2+h\,p\,q\,t^3+p\,q^4\,t010$$
$$+p\,q^4\,t020+p\,q^4\,tt10+p\,q^4\,tt20-3\,p\,q^3\,t\,t010-3\,p\,q^3\,t\,t020-p\,q^3\,t\,tt10$$
$$-p\,q^3\,t\,tt20+3\,p\,q^2\,t^2\,t010+3\,p\,q^2\,t^2\,t020-p\,q\,t^3\,t010-p\,q\,t^3\,t020+6\,q^4\,t010\,t020$$
$$+6\,q^4\,tt10\,tt20-18\,q^3\,t\,t010\,t020+18\,q^2\,t^2\,t010\,t020-6\,q\,t^3\,t010\,t020-a0\,q^4$$
$$+3\,a0\,q^3\,t-3\,a0\,q^2\,t^2+a0\,q\,t^3-at\,q^4+at\,q^3\,t-2\,q^3\,t010\,t020-2\,q^3\,tt10\,tt20$$
$$+6\,q^2\,t\,t010\,t020-6\,q\,t^2\,t010\,t020+2\,t^3\,t010\,t020)\,)$$

$$Entry21TermLambdaZeroMinus5 := 0$$
$$Entry21TermLambdaZeroMinus4 := 0$$
$$Entry21TermLambdaZeroMinus3 := 0$$
$$Entry21TermLambdaZeroMinus2 := 0$$
$$Entry21TermLambdaZeroMinus1 := -\frac{1}{t\,q^2\,(t-1)}\Big(h\,(q-t)\,\big(-h\,p\,q^2+p\,q^2\,t010+p\,q^2\,t020+h\,p\,q-p\,q\,t010-p\,q\,t020-2\,q\,t010\,t020+a0\,q+2\,t010\,t020\big)\Big)$$
$$Entry21TermLambdaInfty3 := 0$$
$$Entry21TermLambdaInfty2 := 0$$
$$Entry21TermLambdaInfty1 := 0$$
$$Entry21TermLambdaInfty0 := 0$$
$$Entry21TermLambdaOneMinus5 := 0$$
$$Entry21TermLambdaOneMinus4 := 0$$
$$Entry21TermLambdaOneMinus3 := 0$$
$$Entry21TermLambdaOneMinus2 := 0$$
$$Entry21TermLambdaOneMinus1 := \frac{1}{(q-1)^2\,(t-1)\,t}\Big((q-t)\,\big(h\,p\,q^2-p\,q^2\,t110-p\,q^2\,t120-h\,p\,q+p\,q\,t110+p\,q\,t120+2\,q\,t110\,t120+a1\,q-a1\big)\,h\Big)$$
$$Entry21TermLambdaTMinus5 := 0$$
$$Entry21TermLambdaTMinus4 := 0$$
$$Entry21TermLambdaTMinus3 := -2\,h\,tt10\,tt20$$
$$Entry21TermLambdaTMinus2 := -at\,h$$
$$Entry21TermLambdaTMinus1 := \frac{1}{t\,(t-1)\,(q-t)^2}\Big(\big(h\,p\,q^3-h\,p\,q^2\,t-p\,q^3\,tt10-p\,q^3\,tt20+p\,q^2\,t\,tt10+p\,q^2\,t\,tt20+2\,at\,q^2\,t-3\,at\,q\,t^2+at\,t^3-h\,p\,q^2+h\,p\,q\,t+p\,q^2\,tt10+p\,q^2\,tt20-p\,q\,t\,tt10-p\,q\,t\,tt20+2\,q^2\,tt10\,tt20-at\,q^2+at\,q\,t-2\,q\,tt10\,tt20\big)\,h\Big)$$
$$0$$
$$-\frac{t120\,t110}{(\lambda-1)^2}-\frac{a1}{\lambda-1}-\frac{t010\,t020}{\lambda^2}-\frac{a0}{\lambda}-\frac{tt20\,tt10}{(\lambda-t)^2}-\frac{at}{\lambda-t}-\frac{p\,h}{\lambda-q}$$

Choice of mu removes the cubic pole at lambda=q. Term in (lambda-t)^{-3} is coherent with the deformation operator \hbar*\partial_t. Term in (lambda-t)^{-2} is also coherent. Term (lambda)^{-1} at infinity should be -h*\partial_t a_0 -h*\partial_t a_1-h*\partial_t a_t- h^2\partial_t p

```
> factor(simplify(-residue(Entry21,lambda=infinity)));
  CoherenceEquation4;
  CoherenceEquation5;
```

$$-\frac{(h\,p+a0+a1+at)\,(q-t)\,h}{t\,(t-1)} \tag{2.6}$$

$$\frac{h\,p + a0 + a1 + at}{h\,p\,q + at\,t - h\,tinfty10 + t010\,t020 + t110\,t120 + tt10\,tt20 - tinfty10\,tinfty20 + a1}$$

```
> La0:=-factor(Entry21TermLambdaZeroMinus1):
  La1:=-factor(Entry21TermLambdaOneMinus1):
  Lat:=-factor(Entry21TermLambdaTMinus1):
  LpFunction:=unapply(-Entry21TermLambdaMinusq/h,a0,a1,at):
> Equation1:=simplify(Entry21TermLambdaMinusqSquare-(-p*h*Lq)):
  Equation1bis:=2*(q-1)*(q-t)*h*a0/((t-1)*t)+2*q*(q-t)*h*a1/(
  (t-1)*t)+2*q*(q-1)*h*at/((t-1)*t)+2*q*(q-1)*(q-t)*h*p^2/((t-1)*
  t)
  +2*((h-t010-t020)*t+(-2*h*t+t*t010+t*t020+t*t110+t*t120-2*h+
  t010+t020+tt10+tt20)*q+(3*h-t010-t020-t110-t120-tt10-tt20)*q^2)
  *h*p/((t-1)*t)
  +2*h/t/(t-1)/q/(q-1)/(q-t)*(t^2*t010*t020-2*t*t010*t020*(t+1)*
  q+(t^2*t010*t020+t^2*t110*t120+4*t*t010*t020+t010*t020+tt10*
  tt20)*q^2+(-2*t*t010*t020-2*t*t110*t120-2*t010*t020-2*tt10*
  tt20)*q^3+(t010*t020+t110*t120+tt10*tt20)*q^4);
  factor(series(Equation1-Equation1bis,p));
```

$$Equation1bis := \frac{2\,(q-1)\,(q-t)\,h\,a0}{(t-1)\,t} + \frac{2\,q\,(q-t)\,h\,a1}{(t-1)\,t} + \frac{2\,q\,(q-1)\,h\,at}{(t-1)\,t} + \frac{2\,q\,(q-1)\,(q-t)\,h\,p^2}{(t-1)\,t} + \frac{1}{(t-1)\,t}\Big(2\,\big((h - t010 - t020)\,t + (-2\,h\,t + t\,t010 + t\,t020 + t\,t110 + t\,t120 - 2\,h + t010 + t020 + tt10 + tt20)\,q + (3\,h - t010 - t020 - t110 - t120 - tt10 - tt20)\,q^2\big)\,h\,p\Big) + \frac{1}{t\,(t-1)\,q\,(q-1)\,(q-t)}\Big(2\,h\,\big(t^2\,t010\,t020 - 2\,t\,t010\,t020\,(t+1)\,q + \big(t^2\,t010\,t020 + t^2\,t110\,t120 + 4\,t\,t010\,t020 + t010\,t020 + tt10\,tt20\big)\,q^2 + (-2\,t\,t010\,t020 - 2\,t\,t110\,t120 - 2\,t010\,t020 - 2\,tt10\,tt20)\,q^3 + (t010\,t020 + t110\,t120 + tt10\,tt20)\,q^4\big)\Big) \tag{2.7}$$

$$0$$

```
> Coeffp:=((2*((h-t010-t020)*t+(-2*h*t+t*t010+t*t020+t*t110+t*
  t120-2*h+t010+t020+tt10+tt20)*q+(3*h-t010-t020-t110-t120-tt10-
  tt20)*q^2))*h/((t-1)*t))/(2*q*(q-1)*(q-t)*h/t/(t-1)):
  Coeffpbis:=-(t010+t020-h)/q-(t120-h+t110)/(q-1)-(tt10+tt20-h)/
  (q-t);
  factor(Coeffp-Coeffpbis);
```

$$Coeffpbis := -\frac{t010 + t020 - h}{q} - \frac{t110 + t120 - h}{q-1} - \frac{tt10 + tt20 - h}{q-t} \tag{2.8}$$

$$0$$

```
> CoeffConstant:=(2*h*(t^2*t010*t020-2*t*t010*t020*(t+1)*q+(t^2*
  t010*t020+t^2*t110*t120+4*t*t010*t020+t010*t020+tt10*tt20)*q^2+
  (-2*t*t010*t020-2*t*t110*t120-2*t010*t020-2*tt10*tt20)*q^3+
  (t010*t020+t110*t120+tt10*tt20)*q^4)/(t*(t-1)*q*(q-1)*(q-t)))/
```

```
(2*q*(q-1)*(q-t)*h/((t-1)*t));
CoeffConstantbis:=(t010*t020/q^2+t110*t120/(q-1)^2+tt10*tt20/
(q-t)^2);
simplify(CoeffConstant-CoeffConstantbis);
```

$$CoeffConstant := \frac{1}{(q-t)^2\, q^2\, (q-1)^2} \Big(t^2\, t010\, t020 - 2\, t\, t010\, t020\, (t+1)\, q + \big(t^2\, t010\, t020 + t^2\, t110\, t120 + 4\, t\, t010\, t020 + t010\, t020 + tt10\, tt20\big)\, q^2 + \big(-2\, t\, t010\, t020 - 2\, t\, t110\, t120 - 2\, t010\, t020 - 2\, tt10\, tt20\big)\, q^3 + \big(t010\, t020 + t110\, t120 + tt10\, tt20\big)\, q^4\Big) \quad \textbf{(2.9)}$$

$$CoeffConstantbis := \frac{t010\, t020}{q^2} + \frac{t120\, t110}{(q-1)^2} + \frac{tt20\, tt10}{(q-t)^2}$$

$$0$$

```
> simplify(CoherenceEquation4);
  simplify(CoherenceEquation5);
```

$$h\, p + a0 + a1 + at \quad \textbf{(2.10)}$$

$$(-h - tinfty20)\, tinfty10 + h\, p\, q + t120\, t110 + tt20\, tt10 + at\, t + t010\, t020 + a1$$

```
> a1:=factor((-q^4*tinfty10*tinfty20-q^4*h*tinfty10+2*q^3*
  tinfty10*tinfty20+2*q^3*h*tinfty10-2*q^2*t*tinfty10*tinfty20-2*
  q^2*t*h*tinfty10+t*q^3*tinfty10*tinfty20+t*q^3*h*tinfty10+t*
  t120*t110*q^3-2*t*q^4*p*t110-2*t*q^4*t120*p+t120*t110*q^2+2*
  tt20*tt10*q^2*t-tt20*tt10*q^3*t+2*q^5*p^2*t+2*q^5*p^2-q^6*p^2-
  q^4*p^2-t010*t020*t^2*q^2-4*q^4*p^2*t+2*q^3*p^2*t-q^2*tinfty10*
  tinfty20-q^2*h*tinfty10-h*q^5*p+p*t010*q^3-2*p*t010*q^4+p*t020*
  q^3-2*p*t020*q^4+h*q^4*p-q^4*p^2*t^2+2*t^2*t010*t020*q+2*h*q^4*
  p*t-t010*t020*t^2-2*q^2*p*t020*t^2-2*q^2*p*t010*t^2-q^2*p^2*
  t^2+q*p*t020*t^2+q*p*t010*t^2-q^3*p*h*t^2+q*tinfty10*tinfty20*
  t+q*h*tinfty10*t-tt20*tt10*q*t-2*t120*t110*q^3-2*q^4*p*tt10-2*
  q^4*p*tt20-q^4*p*t110-q^4*t120*p+q^5*p*t110+q^5*t120*p+2*p*
  t110*t*q^3+p*t110*t^2*q^3+2*t120*p*t*q^3+t120*p*t^2*q^3+2*p*
  tt10*t*q^3+4*p*t020*t*q^3+p*t020*t^2*q^3+p*t010*q^5+p*t020*q^5+
  q^5*p*tt10+q^5*p*tt20+2*t120*t110*t*q^2-t120*t110*t^2*q^2-2*
  t010*t020*t*q^2+t010*t020*t*q^3-2*q^4*p*t010*t-q^4*p*tt10*t-
  q^4*p*tt20*t-2*q^4*p*t020*t+p*tt10*q^3+t010*t020*t*q+q^2*p*h*
  t^2-2*q^3*p*h*t-2*q^2*p*t010*t-q^2*p*tt10*t-q^2*p*tt20*t-2*q^2*
  p*t020*t+p*tt20*q^3+2*p*tt20*t*q^3-q*t120*t110*t-q^2*p*t110*t^2
  -q^2*t120*p*t^2+4*p*t010*t*q^3+p*t010*t^2*q^3+2*q^3*p^2*t^2)/
  (t^2*q-t^2-q^2*t+t-q+q^2)/q);
  a0:=factor(-(-q^4*tinfty10*tinfty20-q^4*h*tinfty10+q^3*
  tinfty10*tinfty20+q^3*h*tinfty10-q^2*t*tinfty10*tinfty20-q^2*t*
  h*tinfty10+t*q^3*tinfty10*tinfty20+t*q^3*h*tinfty10+t*t120*
  t110*q^3-2*t*q^4*p*t110-2*t*q^4*t120*p+tt20*tt10*q^2*t-tt20*
  tt10*q^3*t-tt20*tt10*q^2+2*q^5*p^2*t+2*q^5*p^2-q^6*p^2-q^4*p^2-
  t010*t020*t^2*q^2-4*q^4*p^2*t+2*q^3*p^2*t-h*q^5*p+p*t010*q^3-2*
```

```
p*t010*q^4+p*t020*q^3-2*p*t020*q^4+2*h*q^4*p-h*q^3*p-q^4*p^2*
t^2+2*t^2*t010*t020*q+2*h*q^4*p*t-t010*t020*t^2-2*q^2*p*t020*
t^2-2*q^2*p*t010*t^2-q^2*p^2*t^2+q*p*t020*t^2+q*p*t010*t^2-q^3*
p*h*t^2-t120*t110*q^3+tt20*tt10*q^3+t010*t020*q^3-2*q^4*p*tt10
-2*q^4*p*tt20-q^4*p*t110-q^4*t120*p+q^5*p*t110+q^5*t120*p+2*p*
t110*t*q^3+p*t110*t^2*q^3+2*t120*p*t*q^3+t120*p*t^2*q^3+2*p*
tt10*t*q^3+4*p*t020*t*q^3+p*t020*t^2*q^3+p*t010*q^5+p*t020*q^5+
q^5*p*tt10+q^5*p*tt20+t120*t110*t*q^2-t120*t110*t^2*q^2-3*t010*
t020*t*q^2+t010*t020*t*q^3-2*q^4*p*t010*t-q^4*p*tt10*t-q^4*p*
tt20*t-2*q^4*p*t020*t+p*tt10*q^3+2*t010*t020*t*q-q*p*h*t^2+2*
q^2*p*h*t+2*q^2*p*h*t^2-4*q^3*p*h*t-2*q^2*p*t010*t-q^2*p*tt10*
t-q^2*p*tt20*t-2*q^2*p*t020*t+p*tt20*q^3+2*p*tt20*t*q^3-q^2*p*
t110*t^2-q^2*t120*p*t^2+4*p*t010*t*q^3+p*t010*t^2*q^3+2*q^3*
p^2*t^2-t010*t020*q^2)/q/t/(t*q-t+q-q^2));
at:=factor((q^4*tinfty10*tinfty20+q^4*h*tinfty10-q^3*tinfty10*
tinfty20-q^3*h*tinfty10+2*q^2*t*tinfty10*tinfty20+2*q^2*t*h*
tinfty10-2*t*q^3*tinfty10*tinfty20-2*t*q^3*h*tinfty10-t^2*q*
tinfty10*tinfty20-t^2*q*h*tinfty10+q^2*t^2*tinfty10*tinfty20+
q^2*t^2*h*tinfty10+2*t*q^4*p*t110+2*t*q^4*t120*p-2*tt20*tt10*
q^2*t+q*t120*t110*t^2-t^2*tt20*tt10*q^2+2*tt20*tt10*q^3*t+tt20*
tt10*q^2-2*q^5*p^2*t-2*q^5*p^2+q^6*p^2+q^4*p^2+4*q^4*p^2*t-2*
q^3*p^2*t+h*q^5*p-p*t010*q^3+2*p*t010*q^4-p*t020*q^3+2*p*t020*
q^4-2*h*q^4*p+h*q^3*p+q^4*p^2*t^2-t^2*t010*t020*q-h*q^4*p*t+
t010*t020*t^2+2*q^2*p*t020*t^2+2*q^2*p*t010*t^2+q^2*p^2*t^2-q*
p*t020*t^2-q*p*t010*t^2+t120*t110*q^3-tt20*tt10*q^3-t010*t020*
q^3+2*q^4*p*tt10+2*q^4*p*tt20+q^4*p*t110+q^4*t120*p-q^5*p*t110-
q^5*t120*p-2*p*t110*t*q^3-p*t110*t^2*q^3-2*t120*p*t*q^3-t120*p*
t^2*q^3-2*p*tt10*t*q^3-4*p*t020*t*q^3-p*t020*t^2*q^3-p*t010*q^5
-p*t020*q^5-q^5*p*tt10-q^5*p*tt20-2*t120*t110*t*q^2+2*t010*
t020*t*q^2+2*q^4*p*t010*t+q^4*p*tt10*t+q^4*p*tt20*t+2*q^4*p*
t020*t-p*tt10*q^3-2*t010*t020*t*q-q^2*p*h*t+2*q^3*p*h*t+2*q^2*
p*t010*t+q^2*p*tt10*t+q^2*p*tt20*t+2*q^2*p*t020*t+t^2*tt20*
tt10*q-p*tt20*q^3-2*p*tt20*t*q^3+q^2*p*t110*t^2+q^2*t120*p*t^2
-4*p*t010*t*q^3-p*t010*t^2*q^3-2*q^3*p^2*t^2+t010*t020*q^2)/
(t^2*q-t^2-q^2*t+t-q+q^2)/t/q);
```

$$
\begin{aligned}
a1 := {} & \frac{1}{(t-1)\,(q-1)\,(q-t)\,q}\big(p^2 q^6 - 2 p^2 q^5 t + p^2 q^4 t^2 + h p q^5 - 2 h p q^4 t \\
& + h p q^3 t^2 - 2 p^2 q^5 + 4 p^2 q^4 t - 2 p^2 q^3 t^2 - p q^5\, t010 - p q^5\, t020 - p q^5\, t110 \\
& - p q^5\, t120 - p q^5\, tt10 - p q^5\, tt20 + 2 p q^4 t\, t010 + 2 p q^4 t\, t020 + 2 p q^4 t\, t110 \\
& + 2 p q^4 t\, t120 + p q^4 t\, tt10 + p q^4 t\, tt20 - p q^3 t^2\, t010 - p q^3 t^2\, t020 - p q^3 t^2\, t110 \\
& - p q^3 t^2\, t120 - h p q^4 + 2 h p q^3 t - h p q^2 t^2 + h q^4\, tinfty10 - h q^3 t\, tinfty10 + p^2 q^4 \\
& - 2 p^2 q^3 t + p^2 q^2 t^2 + 2 p q^4\, t010 + 2 p q^4\, t020 + p q^4\, t110 + p q^4\, t120 + 2 p q^4\, tt10 \\
& + 2 p q^4\, tt20 - 4 p q^3 t\, t010 - 4 p q^3 t\, t020 - 2 p q^3 t\, t110 - 2 p q^3 t\, t120 - 2 p q^3 t\, tt10
\end{aligned}
\tag{2.11}
$$

$$
\begin{aligned}
&-2\,p\,q^{3}\,t\,\mathit{tt20}+2\,p\,q^{2}\,t^{2}\,\mathit{t010}+2\,p\,q^{2}\,t^{2}\,\mathit{t020}+p\,q^{2}\,t^{2}\,\mathit{t110}+p\,q^{2}\,t^{2}\,\mathit{t120}\\
&+q^{4}\,\mathit{tinfty10}\,\mathit{tinfty20}-q^{3}\,t\,\mathit{t010}\,\mathit{t020}-q^{3}\,t\,\mathit{t110}\,\mathit{t120}+q^{3}\,t\,\mathit{tt10}\,\mathit{tt20}\\
&-q^{3}\,t\,\mathit{tinfty10}\,\mathit{tinfty20}+q^{2}\,t^{2}\,\mathit{t010}\,\mathit{t020}+q^{2}\,t^{2}\,\mathit{t110}\,\mathit{t120}-2\,h\,q^{3}\,\mathit{tinfty10}\\
&+2\,h\,q^{2}\,t\,\mathit{tinfty10}-p\,q^{3}\,\mathit{t010}-p\,q^{3}\,\mathit{t020}-p\,q^{3}\,\mathit{tt10}-p\,q^{3}\,\mathit{tt20}+2\,p\,q^{2}\,t\,\mathit{t010}\\
&+2\,p\,q^{2}\,t\,\mathit{t020}+p\,q^{2}\,t\,\mathit{tt10}+p\,q^{2}\,t\,\mathit{tt20}-p\,q\,t^{2}\,\mathit{t010}-p\,q\,t^{2}\,\mathit{t020}+2\,q^{3}\,\mathit{t110}\,\mathit{t120}\\
&-2\,q^{3}\,\mathit{tinfty10}\,\mathit{tinfty20}+2\,q^{2}\,t\,\mathit{t010}\,\mathit{t020}-2\,q^{2}\,t\,\mathit{t110}\,\mathit{t120}-2\,q^{2}\,t\,\mathit{tt10}\,\mathit{tt20}\\
&+2\,q^{2}\,t\,\mathit{tinfty10}\,\mathit{tinfty20}-2\,q\,t^{2}\,\mathit{t010}\,\mathit{t020}+h\,q^{2}\,\mathit{tinfty10}-h\,q\,t\,\mathit{tinfty10}-q^{2}\,\mathit{t110}\,\mathit{t120}\\
&+q^{2}\,\mathit{tinfty10}\,\mathit{tinfty20}-q\,t\,\mathit{t010}\,\mathit{t020}+q\,t\,\mathit{t110}\,\mathit{t120}+q\,t\,\mathit{tt10}\,\mathit{tt20}-q\,t\,\mathit{tinfty10}\,\mathit{tinfty20}\\
&+t^{2}\,\mathit{t010}\,\mathit{t020}\big)
\end{aligned}
$$

$$
\begin{aligned}
\mathit{a0}:=&-\frac{1}{q\,t\,(q-t)\,(q-1)}\big(p^{2}\,q^{6}-2\,p^{2}\,q^{5}\,t+p^{2}\,q^{4}\,t^{2}+h\,p\,q^{5}-2\,h\,p\,q^{4}\,t+h\,p\,q^{3}\,t^{2}\\
&-2\,p^{2}\,q^{5}+4\,p^{2}\,q^{4}\,t-2\,p^{2}\,q^{3}\,t^{2}-p\,q^{5}\,\mathit{t010}-p\,q^{5}\,\mathit{t020}-p\,q^{5}\,\mathit{t110}-p\,q^{5}\,\mathit{t120}\\
&-p\,q^{5}\,\mathit{tt10}-p\,q^{5}\,\mathit{tt20}+2\,p\,q^{4}\,t\,\mathit{t010}+2\,p\,q^{4}\,t\,\mathit{t020}+2\,p\,q^{4}\,t\,\mathit{t110}+2\,p\,q^{4}\,t\,\mathit{t120}\\
&+p\,q^{4}\,t\,\mathit{tt10}+p\,q^{4}\,t\,\mathit{tt20}-p\,q^{3}\,t^{2}\,\mathit{t010}-p\,q^{3}\,t^{2}\,\mathit{t020}-p\,q^{3}\,t^{2}\,\mathit{t110}-p\,q^{3}\,t^{2}\,\mathit{t120}\\
&-2\,h\,p\,q^{4}+4\,h\,p\,q^{3}\,t-2\,h\,p\,q^{2}\,t^{2}+h\,q^{4}\,\mathit{tinfty10}-h\,q^{3}\,t\,\mathit{tinfty10}+p^{2}\,q^{4}-2\,p^{2}\,q^{3}\,t\\
&+p^{2}\,q^{2}\,t^{2}+2\,p\,q^{4}\,\mathit{t010}+2\,p\,q^{4}\,\mathit{t020}+p\,q^{4}\,\mathit{t110}+p\,q^{4}\,\mathit{t120}+2\,p\,q^{4}\,\mathit{tt10}+2\,p\,q^{4}\,\mathit{tt20}\\
&-4\,p\,q^{3}\,t\,\mathit{t010}-4\,p\,q^{3}\,t\,\mathit{t020}-2\,p\,q^{3}\,t\,\mathit{t110}-2\,p\,q^{3}\,t\,\mathit{t120}-2\,p\,q^{3}\,t\,\mathit{tt10}\\
&-2\,p\,q^{3}\,t\,\mathit{tt20}+2\,p\,q^{2}\,t^{2}\,\mathit{t010}+2\,p\,q^{2}\,t^{2}\,\mathit{t020}+p\,q^{2}\,t^{2}\,\mathit{t110}+p\,q^{2}\,t^{2}\,\mathit{t120}\\
&+q^{4}\,\mathit{tinfty10}\,\mathit{tinfty20}-q^{3}\,t\,\mathit{t010}\,\mathit{t020}-q^{3}\,t\,\mathit{t110}\,\mathit{t120}+q^{3}\,t\,\mathit{tt10}\,\mathit{tt20}\\
&-q^{3}\,t\,\mathit{tinfty10}\,\mathit{tinfty20}+q^{2}\,t^{2}\,\mathit{t010}\,\mathit{t020}+q^{2}\,t^{2}\,\mathit{t110}\,\mathit{t120}+h\,p\,q^{3}-2\,h\,p\,q^{2}\,t+h\,p\,q\,t^{2}\\
&-h\,q^{3}\,\mathit{tinfty10}+h\,q^{2}\,t\,\mathit{tinfty10}-p\,q^{3}\,\mathit{t010}-p\,q^{3}\,\mathit{t020}-p\,q^{3}\,\mathit{tt10}-p\,q^{3}\,\mathit{tt20}\\
&+2\,p\,q^{2}\,t\,\mathit{t010}+2\,p\,q^{2}\,t\,\mathit{t020}+p\,q^{2}\,t\,\mathit{tt10}+p\,q^{2}\,t\,\mathit{tt20}-p\,q\,t^{2}\,\mathit{t010}-p\,q\,t^{2}\,\mathit{t020}\\
&-q^{3}\,\mathit{t010}\,\mathit{t020}+q^{3}\,\mathit{t110}\,\mathit{t120}-q^{3}\,\mathit{tt10}\,\mathit{tt20}-q^{3}\,\mathit{tinfty10}\,\mathit{tinfty20}+3\,q^{2}\,t\,\mathit{t010}\,\mathit{t020}\\
&-q^{2}\,t\,\mathit{t110}\,\mathit{t120}-q^{2}\,t\,\mathit{tt10}\,\mathit{tt20}+q^{2}\,t\,\mathit{tinfty10}\,\mathit{tinfty20}-2\,q\,t^{2}\,\mathit{t010}\,\mathit{t020}+q^{2}\,\mathit{t010}\,\mathit{t020}\\
&+q^{2}\,\mathit{tt10}\,\mathit{tt20}-2\,q\,t\,\mathit{t010}\,\mathit{t020}+t^{2}\,\mathit{t010}\,\mathit{t020}\big)
\end{aligned}
$$

$$
\begin{aligned}
\mathit{at}:=&-\frac{1}{t\,(t-1)\,(q-t)\,q\,(q-1)}\big(p^{2}\,q^{6}-2\,p^{2}\,q^{5}\,t+p^{2}\,q^{4}\,t^{2}+h\,p\,q^{5}-h\,p\,q^{4}\,t-2\,p^{2}\,q^{5}\\
&+4\,p^{2}\,q^{4}\,t-2\,p^{2}\,q^{3}\,t^{2}-p\,q^{5}\,\mathit{t010}-p\,q^{5}\,\mathit{t020}-p\,q^{5}\,\mathit{t110}-p\,q^{5}\,\mathit{t120}-p\,q^{5}\,\mathit{tt10}\\
&-p\,q^{5}\,\mathit{tt20}+2\,p\,q^{4}\,t\,\mathit{t010}+2\,p\,q^{4}\,t\,\mathit{t020}+2\,p\,q^{4}\,t\,\mathit{t110}+2\,p\,q^{4}\,t\,\mathit{t120}+p\,q^{4}\,t\,\mathit{tt10}\\
&+p\,q^{4}\,t\,\mathit{tt20}-p\,q^{3}\,t^{2}\,\mathit{t010}-p\,q^{3}\,t^{2}\,\mathit{t020}-p\,q^{3}\,t^{2}\,\mathit{t110}-p\,q^{3}\,t^{2}\,\mathit{t120}-2\,h\,p\,q^{4}\\
&+2\,h\,p\,q^{3}\,t+h\,q^{4}\,\mathit{tinfty10}-2\,h\,q^{3}\,t\,\mathit{tinfty10}+h\,q^{2}\,t^{2}\,\mathit{tinfty10}+p^{2}\,q^{4}-2\,p^{2}\,q^{3}\,t\\
&+p^{2}\,q^{2}\,t^{2}+2\,p\,q^{4}\,\mathit{t010}+2\,p\,q^{4}\,\mathit{t020}+p\,q^{4}\,\mathit{t110}+p\,q^{4}\,\mathit{t120}+2\,p\,q^{4}\,\mathit{tt10}+2\,p\,q^{4}\,\mathit{tt20}\\
&-4\,p\,q^{3}\,t\,\mathit{t010}-4\,p\,q^{3}\,t\,\mathit{t020}-2\,p\,q^{3}\,t\,\mathit{t110}-2\,p\,q^{3}\,t\,\mathit{t120}-2\,p\,q^{3}\,t\,\mathit{tt10}\\
&-2\,p\,q^{3}\,t\,\mathit{tt20}+2\,p\,q^{2}\,t^{2}\,\mathit{t010}+2\,p\,q^{2}\,t^{2}\,\mathit{t020}+p\,q^{2}\,t^{2}\,\mathit{t110}+p\,q^{2}\,t^{2}\,\mathit{t120}\\
&+q^{4}\,\mathit{tinfty10}\,\mathit{tinfty20}+2\,q^{3}\,t\,\mathit{tt10}\,\mathit{tt20}-2\,q^{3}\,t\,\mathit{tinfty10}\,\mathit{tinfty20}-q^{2}\,t^{2}\,\mathit{tt10}\,\mathit{tt20}\\
&+q^{2}\,t^{2}\,\mathit{tinfty10}\,\mathit{tinfty20}+h\,p\,q^{3}-h\,p\,q^{2}\,t-h\,q^{3}\,\mathit{tinfty10}+2\,h\,q^{2}\,t\,\mathit{tinfty10}\\
&-h\,q\,t^{2}\,\mathit{tinfty10}-p\,q^{3}\,\mathit{t010}-p\,q^{3}\,\mathit{t020}-p\,q^{3}\,\mathit{tt10}-p\,q^{3}\,\mathit{tt20}+2\,p\,q^{2}\,t\,\mathit{t010}\\
&+2\,p\,q^{2}\,t\,\mathit{t020}+p\,q^{2}\,t\,\mathit{tt10}+p\,q^{2}\,t\,\mathit{tt20}-p\,q\,t^{2}\,\mathit{t010}-p\,q\,t^{2}\,\mathit{t020}-q^{3}\,\mathit{t010}\,\mathit{t020}\\
&+q^{3}\,\mathit{t110}\,\mathit{t120}-q^{3}\,\mathit{tt10}\,\mathit{tt20}-q^{3}\,\mathit{tinfty10}\,\mathit{tinfty20}+2\,q^{2}\,t\,\mathit{t010}\,\mathit{t020}-2\,q^{2}\,t\,\mathit{t110}\,\mathit{t120}\\
&-2\,q^{2}\,t\,\mathit{tt10}\,\mathit{tt20}+2\,q^{2}\,t\,\mathit{tinfty10}\,\mathit{tinfty20}-q\,t^{2}\,\mathit{t010}\,\mathit{t020}+q\,t^{2}\,\mathit{t110}\,\mathit{t120}\\
&+q\,t^{2}\,\mathit{tt10}\,\mathit{tt20}-q\,t^{2}\,\mathit{tinfty10}\,\mathit{tinfty20}+q^{2}\,\mathit{t010}\,\mathit{t020}+q^{2}\,\mathit{tt10}\,\mathit{tt20}-2\,q\,t\,\mathit{t010}\,\mathit{t020}\\
&+t^{2}\,\mathit{t010}\,\mathit{t020}\big)
\end{aligned}
$$

```
> simplify(C01-C01bis);
```

```
simplify(C11-C11bis);
simplify(Ct1-Ct1bis);
```

0 **(2.12)**
0
0

```
> a0bis:=-q*(q-1)*(q-t)/t*p^2-p*(q-1)*(q-t)/t*h-q*tinfty10/t*h
-p/t*( (tinfty10+tinfty20)*q*(q-1)+t*(t010+t020)*(q-1)+(t110+
t120)*(t-1)*q )
-t010*t020/q+t110*t120*(t-1)/t/(q-1)-tt10*tt20*(t-1)/(q-t)-
tinfty10*tinfty20/t*q+(t010*t020-t120*t110+tt20*tt10)/t+(t010*
t020+t120*t110-tt20*tt10)
+p*q*(q-1)/t*CoherenceEquation1
:
factor(a0-a0bis);
```

0 **(2.13)**

```
> a1bis:=q*(q-1)*(q-t)/(t-1)*p^2+p*q*(q-t)/(t-1)*h+(q-1)*
tinfty10/(t-1)*h
+p/(t-1)*( (tinfty10+tinfty20)*q*(q-1)+(t-1)*(t110+t120)*q+
(t010+t020)*t*(q-1))
+t010*t020*t/(t-1)/q-t110*t120/(q-1)+tt10*tt20*t/(q-t)+
tinfty10*tinfty20/(t-1)*(q-1)-(t010*t020-t110*t120-tt10*tt20)/
(t-1)+(-t010*t020-t120*t110+tt20*tt10)
-p*q*(q-1)/(t-1)*CoherenceEquation1
:
factor(a1-a1bis);
```

0 **(2.14)**

```
> atbis:=-q*(q-1)*(q-t)/t/(t-1)*p^2-p*q*(q-1)/t/(t-1)*h-tinfty10*
(q-t)/t/(t-1)*h
-p/t/(t-1)*( (tinfty10+tinfty20)*q*(q-1)+(t-1)*(t110+t120)*q+
(t010+t020)*t*(q-1) )
-t010*t020/(t-1)/q+t110*t120/t/(q-1)-tt10*tt20/(q-t)-tinfty10*
tinfty20/t/(t-1)*(q-t)-(t010*t020-t120*t110+tt20*tt10)/t+(t010*
t020-t120*t110-tt20*tt10)/(t-1)
+p*q*(q-1)/t/(t-1)*CoherenceEquation1
:
factor(at-atbis);
```

0 **(2.15)**

```
> simplify(CoherenceEquation4);
simplify(CoherenceEquation5);
simplify(Equation1);
```

0 **(2.16)**

0
0

```
> Lp:=factor(simplify(LpFunction(a0,a1,at))):
Lpbis:=-diff(q*(q-1)*(q-t),q)/t/(t-1)*p^2
+((t110+tt10+tt20+t120)*q+(t010+t020+tt10+tt20)*(q-1)+(t020+
t110+t120+t010)*(q-t)-(2*q-1)*h    )* p/t/(t-1)
+t010*t020/(t-1)/q^2-t110*t120/t/(q-1)^2+tt10*tt20/(q-t)^2-
tinfty10*tinfty20/t/(t-1)
-tinfty10/t/(t-1)*h:

Lpter:=-diff(q*(q-1)*(q-t),q)/t/(t-1)*p^2
+((P111+Pt11)*q+(P011+Pt11)*(q-1)+(P011+P111)*(q-t)-(2*q-1)*h
)* p/t/(t-1)
+P022/(t-1)/q^2-P122/t/(q-1)^2+Pt22/(q-t)^2-(Pt22+Pt12*t+P022+
P112+P122)/t/(t-1)
-tinfty10/t/(t-1)*h
+CoherenceEquation3/t/(t-1)
:

factor(Lp-Lpbis);
factor(Lp-Lpter);
```

0 **(2.17)**
0

```
> Lqquater:=2*mu*(p-P1(q)/2+h/2/q+h/2/(q-1)+h/2/(q-t))-h*nu0 -h*
nuMinus1*q:
Lpquater:=mu*(p*diff(P1(q),q)+h*p*(1/q^2+1/(q-1)^2+1/(q-t)^2)-
diff(tdP2(q),q)-C01/q^2-C11/(q-1)^2-Ct1/(q-t)^2 )
+h*nuMinus1*p:
simplify(Lq-Lqquater);
factor(simplify(Lp-Lpquater));
```

0 **(2.18)**
0

We thus obtain the evolution of the Darboux coordinates:

\hbar \partial_t q= 2*q*(q-1)*(q-t)/t/(t-1)*p-q*(q-1)*(q-t)/t/(t-1)*P1(q)+q*(q-1)/t/(t-1)*h

\hbar \partial_t p = -diff(q*(q-1)*(q-t),q)/t/(t-1)*p^2
+((t110+tt10+tt20+t120)*q+(t010+t020+tt10+tt20)*(q-1)+(t020+t110+t120+t010)*(q-t)-(2*q-1)*h
)* p/t/(t-1)
+t010*t020/(t-1)/q^2-t110*t120/t/(q-1)^2+tt10*tt20/(q-t)^2-tinfty10*tinfty20/t/(t-1)
-tinfty10/t/(t-1)*h:

```
> Hamiltonian:= q*(q-1)*(q-t)/t/(t-1)*p^2-q*(q-1)*(q-t)/t/(t-1)*
P1(q)*p+q*(q-1)/t/(t-1)*h*p
+q*(q-1)*(q-t)*P2(q)/t/(t-1)+h*tinfty10/t/(t-1)*q;
```

```
factor(simplify(diff(Hamiltonian,p)-Lq));
factor(simplify(diff(Hamiltonian,q)+Lp -CoherenceEquation2*(2*
q-t-1)/t/(t-1)-CoherenceEquation3/t/(t-1)
));

Hamiltonianbis:= mu*(p^2-P1(q)*p+h*p*(1/q+1/(q-1)+1/(q-t)) +
tdP2(q) )-h*nu0*p-h*nuMinus1*q*p
-nuMinus1*(t010*t020+t110*t120+tt10*tt20-tinfty10*(tinfty20+h))
;
factor(series(simplify(Lp-(-diff(Hamiltonianbis,q))),h=0));
simplify(Lq-(diff(Hamiltonianbis,p)));

simplify(Hamiltonianbis-Ct1);
```

$$Hamiltonian := \frac{q(q-1)(q-t)p^2}{t(t-1)} - \frac{q(q-1)(q-t)\left(\frac{t010+t020}{q}+\frac{t110+t120}{q-1}+\frac{tt10+tt20}{q-t}\right)p}{t(t-1)} + \frac{p\,q(q-1)h}{t(t-1)} + \frac{1}{t(t-1)}\left(q(q-1)(q-t)\left(\frac{t010\,t020}{q^2}+\frac{P012}{q}+\frac{t120\,t110}{(q-1)^2}+\frac{P112}{q-1} + \frac{tt20\,tt10}{(q-t)^2}+\frac{Pt12}{q-t}\right)\right) + \frac{h\,tinfty10\,q}{t(t-1)} \tag{2.19}$$

$$0$$

$$0$$

$$Hamiltonianbis := (q-t)\left(\frac{(q-t)q}{t(t-1)}-\frac{q-t}{t-1}+\frac{(q-t)(2t-1)}{t(t-1)}+1\right)\left(p^2 - \left(\frac{t010+t020}{q}+\frac{t110+t120}{q-1}+\frac{tt10+tt20}{q-t}\right)p + h\,p\left(\frac{1}{q}+\frac{1}{q-1}+\frac{1}{q-t}\right) + \frac{t010\,t020}{q^2}+\frac{t120\,t110}{(q-1)^2}+\frac{tt20\,tt10}{(q-t)^2}\right) - h\left(\frac{(q-t)q}{t(t-1)}-\frac{2(q-t)}{t-1} + \frac{(q-t)(2t-1)}{t(t-1)}\right)p - \frac{h(q-t)q\,p}{t(t-1)} - \frac{(q-t)(t010\,t020+t120\,t110+tt20\,tt10-tinfty10\,(h+tinfty20))}{t(t-1)}$$

$$0$$

$$0$$

$$0$$

```
> simplify(series(a0bis-(-(t-1)*Hamiltonian+P012+p*(q-1)*h)-q*(q-
t)/t*CoherenceEquation3-(q-1)*(q-t)/t*CoherenceEquation2+(q-1)*
(q-t)/t*CoherenceEquation3-CoherenceEquation3+
CoherenceEquation2,P112=0) );
```

```
simplify(series(a1bis-(t*Hamiltonian+P112-p*q*h-h*tinfty10/
(t-1))+CoherenceEquation2*q^2/(t-1)-(t*q+q-t)/(t-1)*
CoherenceEquation2+(q-1)/(t-1)*CoherenceEquation3,P112=0) );

simplify(series(atbis-(-Hamiltonian+Pt12+h*tinfty10/(t-1))-
CoherenceEquation2*q^2/t/(t-1)+(t*q+q-t)/t/(t-1)*
CoherenceEquation2-(q-t)/t/(t-1)*CoherenceEquation3,P112=0) );
```

$$0 \tag{2.20}$$
$$0$$
$$0$$

```
> nuMinus1;
factor(nu0);
simplify((0*C01+1*C11+t*Ct1));
simplify((C01+C11+Ct1));
factor(series(simplify((C01/(q-0)+C11/(q-1)+Ct1/(q-t))),p=0)):
Quantity:=p^2-P1(q)*p+p*(h*1/(q-0)+h*1/(q-1)+h*1/(q-t))+tdP2(q)
:
simplify((C01/(q-0)+C11/(q-1)+Ct1/(q-t))-Quantity);
CoherenceEquation3;
CoherenceEquation2;
CoherenceEquation1;
simplify(CoherenceEquation4);
simplify(CoherenceEquation5);
```

$$\frac{q-t}{t\,(t-1)} \tag{2.21}$$
$$\frac{(q-1)\,(q-t)}{t\,(t-1)}$$
$$(-h-tinfty20)\,tinfty10+h\,p\,q+t010\,t020+t120\,t110+tt20\,tt10$$
$$h\,p$$
$$0$$
$$Pt12\,t+t010\,t020+t110\,t120+tt10\,tt20-tinfty10\,tinfty20+P112$$
$$Pt12+P012+P112$$
$$tinfty10+tt10+tt20+t010+t020+tinfty20+t110+t120$$
$$0$$
$$0$$

Computation of the Lax matrices in the geometric gauge without apparent singularities

```
> G1:=Matrix(2,2,0):
G1[1,1]:=1:
G1[1,2]:=0:
G1[2,1]:=tinfty10*lambda+eta0:
G1[2,2]:=1:
```

```
dG1dlambda:=Matrix(2,2,0):
for i from 1 to 2 do for j from 1 to 2 do dG1dlambda[i,j]:=diff
(G1[i,j],lambda):  od: od:

dG1dt:=Matrix(2,2,0):
for i from 1 to 2 do for j from 1 to 2 do dG1dt[i,j]:=diff(G1
[i,j],t)+diff(G1[i,j],q)*dqdt+diff(G1[i,j],p)*dpdt:  od: od:

Ltilde:=simplify(Multiply(Multiply(G1,checkL),G1^(-1))+h*
Multiply(dG1dlambda,G1^(-1))):

eta0:=factor(solve(factor(-residue(Ltilde[2,1],lambda=infinity)
),eta0)):

eta0theo:=1/(tinfty10-tinfty20)*(
-(2*0*P022+2*1*P122+2*t*Pt22) +(0^2*C01+1^2*C11+t^2*Ct1) -h*p*
q^2 -tinfty10*(0*(t010+t020)+1*(t110+t120)+t*(tt10+tt20))
+tinfty10*(tinfty10-tinfty20-h)*(q-1*0-1*1-t*1)):

factor(simplify(eta0-eta0theo+eta0theo*((tinfty10+tt10+tt20+
t010+t020+tinfty20+t110+t120)/(t010+t020+t110+t120+tt10+tt20+2*
tinfty10))));

factor(-residue(Ltilde[2,1]/lambda,lambda=infinity)+tinfty10*
CoherenceEquation1);
factor(-residue(Ltilde[2,1],lambda=infinity));
```

$$0 \quad \textbf{(3.1)}$$
$$0$$
$$0$$

Expression of the Lax matrix in the geometric gauge

```
> checkL;
simplify(checkL[1,1]);
simplify(checkL[1,2]);
simplify(checkL[2,2]):
checkL22bis:=((t010+t020)- q*(q-1)*(q-t)*p/t)/lambda + (t110+
t120+q*(q-1)*(q-t)/(t-1)*p )/(lambda-1)+(tt10+tt20+(-q*(q-1)*
(q-t)/(t-1)/t)*p)/(lambda-t):
factor(checkL[2,2]-checkL22bis);

checkL21bis:=(t010*t020*t/q-(t010+t020)*(q-1)*(q-t)*p+q*(q-1)
^2*(q-t)^2/t*p^2)/lambda
-(t120*t110*(t-1)/(q-1)+(t110+t120)*q*(q-t)*p+(q-t)^2*(q-1)*
q^2/(t-1)*p^2)/(lambda-1)
```

```
+(tt10*tt20*t*(t-1)/(q-t)-(tt10+tt20)*q*(q-1)*p+(q-1)^2*(q-t)*
q^2/(t-1)/t*p^2)/(lambda-t)
-tinfty10*tinfty20 -h*tinfty10
:
factor((checkL[2,1]-checkL21bis));

factor(-residue(checkL[2,1]/lambda,lambda=infinity));
factor(-residue(checkL[2,1],lambda=infinity)):
factor(residue(1/2*Trace(checkL^2)*(lambda-1)^0,lambda=1)):
simplify(residue(1/2*Trace(checkL^2)*(lambda-1)^0,lambda=1)-
C11);

checkL11bis:=-Q2(lambda)/(lambda-0)/(lambda-1)/(lambda-t);
checkL12bis:=(lambda-q)/(lambda-0)/(lambda-1)/(lambda-t);
checkL22bis:=P1(lambda)+Q2(lambda)/(lambda-0)/(lambda-1)/
(lambda-t);
checkL21bis:=h*diff(Q2(lambda)/(lambda-q),lambda)+L[2,1]*
(lambda-0)*(lambda-1)*(lambda-t)/(lambda-q)-P1(lambda)*Q2
(lambda)/(lambda-q)
-Q2(lambda)^2/(lambda-q)/(lambda-0)/(lambda-1)/(lambda-t):
simplify(checkL[1,1]-checkL11bis);
simplify(checkL[1,2]-checkL12bis);
simplify(checkL[2,2]-checkL22bis);
simplify(checkL[2,1]-checkL21bis);
```

$$\left[\left[\frac{p\,q\,(q-1)\,(q-t)}{\lambda\,(\lambda-1)\,(\lambda-t)}, \frac{-\lambda+q}{\lambda\,(\lambda-1)\,(-\lambda+t)}\right],\right. \tag{3.2}$$

$$\left[\frac{1}{\lambda\,(\lambda-t)\,(\lambda-1)\,(\lambda-q)}\Big((-h\,p-a0-a1-at)\,\lambda^5+(-h\,(q-2\,t-2)\,p\right.$$

$$+(2\,a0+2\,a1+at)\,t-t120\,t110-tt20\,tt10-t010\,t020+2\,a0+a1+2\,at)\,\lambda^4+($$

$$-\big(q^2+(-2\,t-2)\,q+t^2+4\,t+1\big)\,h\,p+(-a0-a1)\,t^2+(2\,t010\,t020+2\,t110\,t120$$

$$-4\,a0-2\,a1-2\,at)\,t+2\,tt20\,tt10+2\,t010\,t020-a0-at\big)\,\lambda^3+\big(\big((t010+t020$$

$$+t110+t120+tt10+tt20)\,q^3+(t+1)\,(h-t110-t120-tt10-tt20-t010$$

$$-t020)\,q^2+\big(-h\,t^2+(-3\,h+t110+t120+tt10+tt20+t010+t020)\,t-h\big)\,q$$

$$+2\,h\,t\,(t+1)\big)\,p+(-t010\,t020-t110\,t120+2\,a0+a1)\,t^2+(-4\,t010\,t020+2\,a0$$

$+ at)\, t - tt20\, tt10 - t010\, t020\big)\, \lambda^2 + (-(((t110 + t120 + t010 + t020)\, t + tt10 + tt20$

$+ t010 + t020)\, q + h\, t)\, (q - t)\, (q - 1)\, p - ((-2\, t010\, t020 + a0)\, t - 2\, t010\, t020)\, t)$

$\lambda - (p\, q\, (q - 1)\, (q - t) - t\, t020)\, (p\, q\, (q - 1)\, (q - t) - t\, t010)),$

$\frac{1}{\lambda\,(\lambda - t)\,(\lambda - 1)}\Big((t010 + t020 + t110 + t120 + tt10 + tt20)\,\lambda^2 + ((-t010 - t020$
$- t110 - t120)\, t - t010 - t020 - tt10 - tt20)\,\lambda + \big(t010 + t020 + (q^2 - q)\, p\big)\, t$
$- p\, q^2\, (q - 1)\Big)\Big]\Big]$

$$\frac{p\, q\, (q - 1)\, (q - t)}{\lambda\, (\lambda - 1)\, (\lambda - t)}$$

$$\frac{-\lambda + q}{\lambda\, (\lambda - 1)\, (-\lambda + t)}$$

$$0$$

$$0$$

$$-tinfty10\, (h + tinfty20)$$

$\frac{1}{t - 1}\,(((t010 + t020)\, t - tt20 - t010 - t020 - tt10)\, t110 + ((t010 + t020)\, t - tt20$
$- t010 - t020 - tt10)\, t120 + p\, q\, (q - t)\, h)$

$$checkL11bis := \frac{p\, q\, (q - 1)\, (q - t)}{\lambda\, (\lambda - 1)\, (\lambda - t)}$$

$$checkL12bis := \frac{\lambda - q}{\lambda\, (\lambda - 1)\, (\lambda - t)}$$

$$checkL22bis := \frac{t010 + t020}{\lambda} + \frac{t110 + t120}{\lambda - 1} + \frac{tt10 + tt20}{\lambda - t} - \frac{p\, q\, (q - 1)\, (q - t)}{\lambda\, (\lambda - 1)\, (\lambda - t)}$$

$$0$$

$$0$$

$$0$$

$$0$$

```
> C:=0:
  factor(simplify(checkA[1,1])):
  checkA11bis:=C-q*(q-1)*(q-t)/t/(t-1)*p/(lambda-t);
  simplify(checkA[1,1]-checkA11bis);
  checkA[1,2]:=simplify(checkA[1,2]);
  factor(simplify(checkA[2,2]+checkA[1,1])):
  checkA22bis:=C +(-(tt10+tt20)+q*(q-1)*(q-t)/t/(t-1)*p)/(lambda-
  t)-(q-t)/t/(t-1)*h-(tinfty10+tinfty20)*(q-t)/t/(t-1);
  factor(simplify(checkA[2,2]-checkA22bis-(q-t)/t/(t-1)*
  CoherenceEquation1));

  checkA21:=simplify(checkA[2,1]):
  checkA21bis:=(-tt10*tt20*t*(t-1)/(q-t)+(q-1)*q*(tt10+tt20)*p-
```

```
(q-1)^2*(q-t)*q^2/(t-1)/t*p^2)/(lambda-t)
-tinfty10*tinfty20*(q-t)/t/(t-1)*lambda
+(q-1)*q*(q-t)^2/t/(t-1)*p^2
+(q-t)/t/(t-1)*((tinfty10+tinfty20)*q*(q-1)+((t110+t120)*(t-1)+
t*(t010+t020))*q-(t010+t020)*t)*p
-t/(t-1)*t010*t020/q+(t-1)/t*t110*t120/(q-1)-tinfty10*
tinfty20/t/(t-1)*(q-t)^2-tinfty10*tinfty20/t*(q-t)+tt10*tt20-
t110*t120/t+t010*t020/(t-1)
-(q-t)*tinfty10*lambda/t/(t-1)*h+(q-t)*(q-1)/t/(t-1)*(p*q-
tinfty10)*h
;
factor(checkA21-checkA21bis+(q-1)*(q-t)/t/(t-1)*q*p*
CoherenceEquation1);
```

$$checkA11bis := -\frac{p\, q\, (q-1)\, (q-t)}{(t-1)\, t\, (\lambda-t)} \tag{3.3}$$

$$0$$

$$checkA_{1,2} := \frac{q-t}{(t-1)\, t\, (\lambda-t)}$$

$$checkA22bis := \frac{-tt10 - tt20 + \frac{q\, p\, (q-1)\, (q-t)}{t\, (t-1)}}{\lambda-t} - \frac{(q-t)\, h}{t\, (t-1)} - \frac{(tinfty10 + tinfty20)\, (q-t)}{t\, (t-1)}$$

$$0$$

$$checkA21bis := \frac{1}{\lambda-t}\left(-\frac{tt10\, tt20\, t\, (t-1)}{q-t} + (tt10+tt20)\, q\, (q-1)\, p - \frac{(q-1)^2\, (q-t)\, q^2\, p^2}{(t-1)\, t}\right) - \frac{tinfty10\, tinfty20\, (q-t)\, \lambda}{t\, (t-1)} + \frac{(q-1)\, q\, (q-t)^2\, p^2}{t\, (t-1)} + \frac{1}{t\, (t-1)}\big((q-t)\, ((tinfty10+tinfty20)\, q\, (q-1) + ((t110+t120)\, (t-1) + (t010+t020)\, t)\, q - (t010+t020)\, t)\, p\big) - \frac{t010\, t020\, t}{(t-1)\, q} + \frac{t110\, t120\, (t-1)}{t\, (q-1)} - \frac{tinfty10\, tinfty20\, (q-t)^2}{t\, (t-1)} - \frac{tinfty10\, tinfty20\, (q-t)}{t} + tt20\, tt10 - \frac{t110\, t120}{t} + \frac{t010\, t020}{t-1} - \frac{(q-t)\, tinfty10\, \lambda\, h}{t\, (t-1)} + \frac{(q-t)\, (q-1)\, (p\, q - tinfty10)\, h}{t\, (t-1)}$$

$$0$$

```
> factor(-residue(checkA[2,2]/lambda^2,lambda=infinity));
factor(-residue(checkA21/lambda^2,lambda=infinity));

factor(-residue((checkA[1,1]-C)/lambda,lambda=infinity));
```

```
factor(-residue(checkA[1,2]/lambda,lambda=infinity));
factor(-residue((checkA[2,2]-C)/lambda,lambda=infinity));
ConstantCoeffcheckA21:=factor(-residue(checkA21/lambda,lambda=
infinity)):
ConstantCoeffcheckA21bis:=q*(q-1)*(q-t)^2/t/(t-1)*p^2
+p*(q-t)/t/(t-1)*((tinfty10+tinfty20)*q^2+(t010+t020)*(t*(q-1)+
q)+(t110+t120)*t*q+(tt10+tt20)*q)
-t010*t020*t/(t-1)/q+t110*t120*(t-1)/t/(q-1)
-tinfty10*tinfty20*(q-t)*(q-1)/t/(t-1)
+t010*t020/(t-1)-t110*t120/t+tt10*tt20
-(q-1)*(q-t)*(-p*q+tinfty10)/(t-1)/t*h
-CoherenceEquation1*p*q^2*(q-t)/t/(t-1);
factor(ConstantCoeffcheckA21-ConstantCoeffcheckA21bis);
```

$$0 \tag{3.4}$$

$$-\frac{tinfty10\,(q-t)\,(h+tinfty20)}{(t-1)\,t}$$

$$0$$

$$0$$

$$-\frac{(h-t110-t120-tt10-tt20-t010-t020)\,(q-t)}{(t-1)\,t}$$

$$ConstantCoeffcheckA21bis := \frac{(q-1)\,q\,(q-t)^2\,p^2}{t\,(t-1)} + \frac{1}{t\,(t-1)}\Big(p\,(q-t)\,\big((tinfty10+tinfty20)\,q^2+(t010+t020)\,(t\,(q-1)+q)+(t110+t120)\,t\,q+(tt10+tt20)\,q\big)\Big) - \frac{t010\,t020\,t}{(t-1)\,q} + \frac{t110\,t120\,(t-1)}{t\,(q-1)} - \frac{tinfty10\,tinfty20\,(q-t)\,(q-1)}{t\,(t-1)} + \frac{t010\,t020}{t-1} - \frac{t110\,t120}{t} + tt20\,tt10 - \frac{(q-1)\,(q-t)\,(-p\,q+tinfty10)\,h}{(t-1)\,t} - \frac{(tinfty10+tt10+tt20+t010+t020+tinfty20+t110+t120)\,p\,q^2\,(q-t)}{t\,(t-1)}$$

$$0$$

Getting the Painlevé 6 equation

```
> L2q:=diff(Lq,q)*Lq+diff(Lq,p)*Lp+ diff(Lq,t)*h:
PainleveEquation:=simplify( L2q- 1/2*(1/q+1/(q-1)+1/(q-t))*Lq^2
+h*Lq*(1/t+1/(t-1)+1/(q-t))-q*(q-1)*(q-t)/t^2/(t-1)^2*(alpha+
beta*t/q^2+gammaa*(t-1)/(q-1)^2+delta*t*(t-1)/(q-t)^2)):
> beta:=-1/2*(t010-t020)^2;
gammaa:=1/2*(t110-t120)^2;
delta:=-(tt10-tt20)^2/2+h^2/2;
alpha:=1/2*(tinfty10-tinfty20-h)^2:
factor(PainleveEquation+q*(q-1)*(q-t)/2/t^2/(t-1)^2*(tinfty20-
t010-tt20-t120-tt10-t020-t110+tinfty10)*CoherenceEquation1
+q*(q-1)*(q-t)*h/t^2/(t-1)^2*CoherenceEquation1
```

```
);
```

$$\beta := -\frac{1}{2}\,(t010 - t020)^2 \tag{4.1}$$

$$gammaa := \frac{1}{2}\,(t110 - t120)^2$$

$$\delta := -\frac{1}{2}\,(tt10 - tt20)^2 + \frac{1}{2}\,h^2$$

$$0$$

```
> RHSP6:=-q*(q-1)*(q-t)/t^2/(t-1)^2*(alpha+beta*t/q^2+gammaa*
(t-1)/(q-1)^2+delta*t*(t-1)/(q-t)^2);
RHSP6bis:=-2*q*(q-1)*(q-t)/t^2/(t-1)^2*diff(q*(q-1)*(q-t)*(P1
(q)^2/4-P2(q)),q)
+q*(q-1)*(q-t)*(tinfty10-tinfty20)/t^2/(t-1)^2*h-q*(q-1)*(q-t)
/2/t^2/(t-1)^2*h^2-q*(q-1)/(q-t)/2/t/(t-1)*h^2:
factor(RHSP6-RHSP6bis+4*(Pt12+P012+P112)/t^2/(t-1)^2*q^4
+(-2*(1+t)/t/(t-1)^2*CoherenceEquation2+2/t/(t-1)^2*
CoherenceEquation3+1/2*(tinfty20-t010-t020-t110-tt10-tt20-t120+
tinfty10)/t/(t-1)^2*CoherenceEquation1)*q
-(-2*(1+4*t+t^2)/t^2/(t-1)^2*CoherenceEquation2+2*(1+t)/t^2/
(t-1)^2*CoherenceEquation3+(t+1)/2*(tinfty20-t010-t020-t110-
tt10-tt20-t120+tinfty10)/t^2/(t-1)^2*CoherenceEquation1)*q^2
-(6*(1+t)/t^2/(t-1)^2*CoherenceEquation2-2/t^2/(t-1)^2*
CoherenceEquation3-1/2*(tinfty20-t010-t020-t110-tt10-tt20-t120+
tinfty10)/t^2/(t-1)^2*CoherenceEquation1)*q^3
);
Termq:=factor(residue((RHSP6-RHSP6bis+4*(Pt12+P012+P112)/t^2/
(t-1)^2*q^4)/q^2,q=0)):
Termq2:=factor(residue((RHSP6-RHSP6bis+4*(Pt12+P012+P112)/t^2/
(t-1)^2*q^4)/q^3,q=0)):
Termq3:=factor(residue((RHSP6-RHSP6bis+4*(Pt12+P012+P112)/t^2/
(t-1)^2*q^4)/q^4,q=0)):
factor(Termq-2*(1+t)/t/(t-1)^2*CoherenceEquation2+2/t/(t-1)^2*
CoherenceEquation3+1/2*(tinfty20-t010-t020-t110-tt10-tt20-t120+
tinfty10)/t/(t-1)^2*CoherenceEquation1);
factor(Termq3-6*(1+t)/t^2/(t-1)^2*CoherenceEquation2+2/t^2/
(t-1)^2*CoherenceEquation3+1/2*(tinfty20-t010-t020-t110-tt10-
tt20-t120+tinfty10)/t^2/(t-1)^2*CoherenceEquation1);
factor(Termq2+2*(1+4*t+t^2)/t^2/(t-1)^2*CoherenceEquation2-2*
(1+t)/t^2/(t-1)^2*CoherenceEquation3-(t+1)/2*(tinfty20-t010-
t020-t110-tt10-tt20-t120+tinfty10)/t^2/(t-1)^2*
CoherenceEquation1);
```

(4.2)

$$RHSP6 := -\frac{1}{t^2(t-1)^2}\left(q(q-1)(q-t)\left(\frac{1}{2}(tinfty10 - tinfty20 - h)^2 - \frac{1}{2}\frac{(t010 - t020)^2 t}{q^2} + \frac{1}{2}\frac{(t110 - t120)^2 (t-1)}{(q-1)^2} + \frac{\left(-\frac{1}{2}(tt10 - tt20)^2 + \frac{1}{2}h^2\right)t(t-1)}{(q-t)^2}\right)\right) \tag{4.2}$$

$$\begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix}$$

Conclusion: q satisfies the Painlevé 6 equation whose r.h.s. may be rewritten as
-q*(q-1)*(q-t)/t^2/(t-1)^2*(alpha+beta*t/q^2+gammaa*(t-1)/(q-1)^2+delta*t*(t-1)/(q-t)^2)
=-2*q*(q-1)*(q-t)/t^2/(t-1)^2*diff(q*(q-1)*(q-t)*(P1(q)^2/4-P2(q)),q) +q*(q-1)*(q-t)*(tinfty10-tinfty20)/t^2/(t-1)^2*h-q*(q-1)*(q-t)/2/t^2/(t-1)^2*h^2-q*(q-1)/(q-t)/2/t/(t-1)*h^2: