Quantization of classical spectral curves and isomonodromic deformations

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General position of the talk

General problem

How to quantize a "**classical spectral curve**" ($[y, \lambda] = 0$)

$$P(\lambda, y) = 0 \ , \ P \text{ rational in } \lambda, \text{ monic polynomial in } y$$

into a **linear differential equation** ($[\hbar\partial_\lambda, \lambda] = \hbar$):

$$\left(\hat{P}\left(\lambda, \hbar \frac{d}{d\lambda}\right)\right) \psi(\lambda, \hbar) = 0\,?$$

$\hat{P}$ rational in λ with **same pole structure as** P.

Key ingredients

Key ingredient 1: Topological recursion [24].
Key ingredient 2: Integrable systems, Lax pairs:

$$\hbar \frac{\partial}{\partial\lambda}\Psi(\lambda, \hbar, \mathbf{t}) = L(\lambda, \hbar, \mathbf{t})\Psi(\lambda, \hbar, \mathbf{t}) \ , \ \hbar \frac{\partial}{\partial t}\Psi(\lambda, \hbar, \mathbf{t}) = A_t(\lambda, \hbar, \mathbf{t})\Psi(\lambda, \hbar, \mathbf{t})$$

Strategy of the construction

1. Define proper initial data to apply topological recursion (TR) $\Rightarrow$ Minor technical restrictions on the classical spectral curve
2. Apply TR to initial data: $\Rightarrow$ Output: $(\boldsymbol{\omega_{h,n}})_{\mathbf{h,n\geq 0}}$: "*TR differentials*".
3. Stack the $\omega_{h,n}$ into some "*perturbative wave functions*" $(\psi_i(z))_{i=1}^{d}$. $\Rightarrow$ **formal WKB series in** $\hbar$.
4. Take kind of "formal Fourier transform" to get "*non-perturbative wave functions*" and regroup them into a wave matrix $\Psi^{\mathrm{NP}}(\lambda;\hbar)$ $\Rightarrow$ **Formal trans-series in** $\hbar$.
5. Prove that $\hbar\partial_\lambda\Psi^{\mathrm{NP}}(\lambda,\hbar)=L(\lambda,\hbar)\Psi^{\mathrm{NP}}(\lambda,\hbar)$ with L rational with controlled pole structure. $\Leftrightarrow$ "*Quantum curve*".
6. Obtain *auxiliary systems* $\hbar\partial_t\Psi^{\mathrm{NP}}(\lambda,\hbar,\mathbf{t})=A_t(\lambda,\hbar,\mathbf{t})\Psi^{\mathrm{NP}}(\lambda,\hbar,\mathbf{t})$ with A_t rational with dominated pole structure $\Rightarrow$ Connection with isomonodromic deformations.

Known results and applications

- Review on TR and quantum curves by P. Norbury [36].
- Elements of the strategy already existing in the literature [9, 18, 20, 23, 24, 35].
- Non-perturbative part is not necessary for **genus 0 classical spectral curves**.
- Several examples worked out in details [14, 15, 16, 17, 28, 30, 39].
- **Reverse approach also exists** [3, 7, 29, 33]:
  [Lax pair: $(L(\lambda,\hbar), A(\lambda,\hbar))$ + Topological type property] $\Rightarrow$
  Ψ reconstructed by TR applied on the associated classical spectral curve $\lim_{\hbar\to 0} \det(yI_d - L(\lambda,\hbar)) = 0$.
- Applications in **enumerative geometry** [2, 5, 6, 10, 11, 12, 19, 37, 38, 40, 25, 26].

Summary of the general results

- Results presented following [32] for $\mathfrak{sl}_2$ case (hyper-elliptic case) and [22] **for the general $\mathfrak{gl}_d$ case**. Similar works for $\mathfrak{sl}_2$ case in [21].
- Connection with isomonodromic deformations only in $\mathfrak{gl}_2$ case (so far) in [34, 31].
- Technical assumptions on the classical spectral curve include
  - **Pole of any degree including infinity.**
  - **Poles may be ramification points.**
  - Ramification points are simple and smooth.
- Main results: Construction of the **matrix wave functions**, **quantum curve and some compatible auxiliary systems with same pole structure as the initial spectral curve.**
- Application of the theory to all genus 1 cases in $\mathfrak{gl}_2(\mathbb{C})$ recovers the six Painlevé Lax pairs.

Classical spectral curve, TR

Classical spectral curve

Classical spectral curve

Let $(\Lambda_1, \dots, \Lambda_N)$ be $N \geq 0$ distinct points on $\mathbb{P}^1 \setminus \{\infty\}$. Let $\mathcal{H}_d(\Lambda_1, \dots, \Lambda_N, \infty)$ be the Hurwitz space of covers $x\colon \Sigma \to \mathbb{P}^1$ of degree d defined as the Riemann surface

$$\Sigma := \overline{\{(\lambda, y) \mid P(\lambda, y) = 0\}},$$

where

$$P(\lambda, y) = \sum_{l=0}^{d} (-1)^l y^{d-l} P_l(\lambda) = 0, \;\; P_0(\lambda) = 1$$

with each coefficient $(P_l)_{l \in [\![1,d]\!]}$ **being a rational function with possible poles at** $\lambda \in \mathcal{P} := \{\Lambda_i\}_{i=1}^N \bigcup \{\infty\}$.

A *classical spectral curve* (Σ, x) is the data of the Riemann surface Σ and its realization as a Hurwitz cover of $\mathbb{P}^1$.

Classical spectral curve with fixed pole structure

Classical spectral curve with fixed pole structure

For $l \in [\![1, d]\!]$, let $r_\infty^{(l)}$ and $\left(r_{\Lambda_i}^{(l)}\right)_{i=1}^N$ be some non-negative integers. We consider the subspace

$$\mathcal{H}_d\left((\Lambda_1, (r_{\Lambda_1}^{(l)})_{l=1}^d), \dots, (\Lambda_N, (r_{\Lambda_N}^{(l)})_{l=1}^d), (\infty, (r_\infty^{(l)})_{l=1}^d)\right) \subset \mathcal{H}_d(\Lambda_1, \dots, \Lambda_N, \infty)$$

of covers x such that the rational functions $(P_l)_{l=1}^d$ are of the form

$$P_l(\lambda) := \sum_{P \in \mathcal{P}} \sum_{k \in S_P^{(l)}} P_{P,k}^{(l)} \, \xi_P(\lambda)^{-k}, \text{ for } l \in [\![1, d]\!],$$

where we have defined

$$\forall\, i \in [\![1, N]\!] \,:\, \mathbf{S}_{\Lambda_i}^{(l)} := [\![1, \mathbf{r}_{\Lambda_i}^{(l)}]\!] \quad \text{and} \quad \mathbf{S}_\infty^{(l)} := [\![0, \mathbf{r}_\infty^{(l)}]\!],$$

and the local coordinates $\{\xi_P(\lambda)\}_{P \in \mathcal{P}}$ around $P \in \mathcal{P}$ are defined by

$$\forall\, i \in [\![1, N]\!] \,:\, \xi_{\Lambda_i}(\lambda) := (\lambda - \Lambda_i) \quad \text{and} \quad \xi_\infty(\lambda) := \lambda^{-1}$$

Canonical local coordinates and spectral times

Canonical local coordinates

Let $P \in \mathbb{P}^1$ and $p \in x^{-1}(P)$. Canonical coordinates on $\mathbb{P}^1$ near P are

$$\xi_P(\lambda) := \lambda - P \quad \text{if } P \neq \infty \ , \ \xi_P(\lambda) := \frac{1}{\lambda} \quad \text{if } P = \infty$$

Canonical local coordinates near any $p \in x^{-1}(P)$ are

$$\zeta_p(z) = \xi_P(x(z))^{\frac{1}{d_p}} \ , \ d_p = \text{order}_p(\xi_P)$$

Spectral times (KP times)

The 1-form ydx has the following expansion:

$$ydx = \sum_{k=0}^{r_p-1} t_{p,k} \zeta_p^{-k-1} d\zeta_p + \text{analytic at } p.$$

$\mathbf{t} = \left(t_{p,k}\right)_{p \in x^{-1}(\mathcal{P}), k \in [\![1, r_p - 1]\!]}$ are called "**irregular or spectral times**".
$\mathbf{t_0} = \left(t_{p,0}\right)_{p \in x^{-1}(\mathcal{P})}$ are called "**monodromies**".

Ramification points and critical values

Ramification points and critical values

We denote by $\mathcal{R}_0$ the set of all ramification points of the cover x, and by $\mathcal{R}$ the set of all ramification points that are not poles (i.e. not in $x^{-1}(\mathcal{P})$),

$$\mathcal{R}_0 := \left\{ p \in \Sigma \,/\, 1 + \mathrm{order}_p\, dx \neq \pm 1 \right\},$$

$$\mathcal{R} := \left\{ p \in \Sigma \,/\, dx(p) = 0 \,, \quad x(p) \notin \mathcal{P} \right\} = \mathcal{R}_0 \setminus x^{-1}(\mathcal{P}).$$

We shall refer to their images $x(\mathcal{R})$ as the *critical values* of x.

Admissible spectral curve

Admissible classical spectral curves

We say that a classical spectral curve (Σ, x) is *admissible* if it satisfies:

- The Riemann surface Σ defined by $P(\lambda, y) = 0$ is an irreducible algebraic curve, i.e. $P(\lambda, y)$ does not factorize.
- All ramification points are simple, i.e. dx has only a simple zero at $a \in \mathcal{R}$.
- Critical values are distinct: for any $(a_i, a_j) \in \mathcal{R} \times \mathcal{R}$ such that $a_i \neq a_j$ then $x(a_i) \neq x(a_j)$.
- Ramification points are smooth: for any $a \in \mathcal{R}$, $dy(a) \neq 0$ (i.e. the tangent vector $(dx(a), dy(a))$ to the immersed curve $\{(\lambda, y) \mid P(\lambda, y) = 0\}$ is not vanishing at a).
- Generic ramified poles: for any pole $p \in x^{-1}(\mathcal{P})$ ramified, the 1-form ydx has a pole of degree $r_p \geq 3$ at p, and the corresponding spectral times satisfy $t_{p,r_p-2} \neq 0$.

Remarks on the technical assumptions

- Topology of admissible spectral curves relatively to spectral times is complicated. $\Rightarrow$ Spectral times are not independent. Tangent space and deformations hard to define for $d \geq 3$.
- Tangent space defined for $d = 2 \leftrightarrow$ Existence of deformations $\partial_{t_{p,k}}$. Split into trivial deformations (Mobius transformations and $\mathfrak{gl}_2 \to \mathfrak{sl}_2$) and g isomonodromic deformations.
- Ingredients to remove some technical assumptions already exist in the literature: simple ramification points, smooth ramification points, reducible algebraic curves.
- Defining properly the tangent space (in the spirit of [34]) would allow to make the connection with isomonodromic deformations for $d \geq 3$.
- Condition that ramified poles are generic allows **to exclude ramified poles in the residues of TR**.

Admissible initial data

Admissible initial data

Given an **admissible spectral curve** (Σ, x) of genus g, we add

- **Choice of Torelli marking** $(\mathcal{A}_i, \mathcal{B}_i)_{i=1}^g$.
  $\Leftrightarrow$ Associated "Bergman" kernel (normalized fundamental second kind differential) $B^{(\mathcal{A}_i,\mathcal{B}_i)_{i=1}^g}$.
- A generic smooth point $o \in \Sigma \setminus x^{-1}(\mathcal{P})$ and some choice of non-intersecting homology chains $\mathcal{C}_{o\to p}$ for each $p \in x^{-1}(\mathcal{P})$ compatible with the Torelli marking:

$$\forall\, p \in x^{-1}(\mathcal{P}),\ \forall\, i \in [\![1, g]\!]\ , \qquad \mathcal{A}_i \cap \mathcal{C}_{o\to p} = 0 = \mathcal{B}_i \cap \mathcal{C}_{o\to p},$$

These three ingredients define some "**admissible initial data**" on which TR can be applied. Denoted $((\Sigma, x), (\mathcal{A}_i, \mathcal{B}_i)_{i=1}^g)$.

General considerations

- Initial version [24] of TR dating back to 2007 is sufficient since ramification points are assumed simple.
- Some generalizations of TR exist to deal with non-simple ramification points, non-irreducible curves [8, 13].
- **TR takes admissible initial data as input and provides some TR differentials $(\omega_{h,n})_{h\geq 0, n\geq 0}$ as output.**
- These differentials are computed by recursion on $s = n + 2h$ starting from

$$\omega_{0,1} := ydx, \qquad \omega_{0,2} := B^{(\mathcal{A}_i,\mathcal{B}_i)_{i=1}^{g}},$$

Definition of TR

Definition of Topological Recursion

We have for $h \geq 0$, $n \geq 0$ with $(h, n) \notin \{(0,0),(0,1)\}$:

$$\omega_{h,n+1}(z_0, \mathbf{z}) := \sum_{a \in \mathcal{R}} \operatorname*{Res}_{z \to a} \frac{1}{2} \frac{\int_{\sigma_a(z)}^{z} \omega_{0,2}(z_0, \cdot)}{\omega_{0,1}(z) - \sigma_a^* \omega_{0,1}(z)} \widetilde{\mathcal{W}}^{(2)}_{h,n+1}(z, \sigma_a(z); \mathbf{z}),$$

with

$$\widetilde{\mathcal{W}}^{(2)}_{h,n+1}(z, z'; \mathbf{z}) := \omega_{h-1,n+2}(z, z', \mathbf{z}) + \sum_{\substack{A \sqcup B = \mathbf{z},\, s \in [\![0, h]\!] \\ (s, |A|) \notin \{(0,0),(h,n)\}}} \omega_{s,|A|+1}(z, A)\, \omega_{h-s,|B|+1}(z', B)$$

and

$$\omega_{h,0} := \frac{1}{2-2h} \sum_{a \in \mathcal{R}} \operatorname*{Res}_{z \to a} \omega_{h,1}(z) \Phi(z), \;\; \forall\, h \geq 2$$

and $(\omega_{0,0}, \omega_{1,0})$ defined by specific formulas (See [24])

Loop equations

- Some combinations of the TR differentials have interesting properties ⇒ "*Loop equations*"
- Following [9], for $(h,n,l) \in \mathbb{N}^3$:

$$Q^{(0)}_{h,n+1}(\lambda;\mathbf{z}) = \hat{Q}^{(0)}_{h,n+1}(\lambda;\mathbf{z}) = \widetilde{Q}^{(0)}_{h,n+1}(\lambda;\mathbf{z}) := \delta_{h,0}\delta_{n,0},$$

$$Q^{(l)}_{h,n+1}(\lambda;\mathbf{z}) := \sum_{\beta \underset{l}{\subseteq} x^{-1}(\lambda)} \sum_{\mu\in\mathcal{S}(\beta)} \sum_{\bigsqcup_{i=1}^{l(\mu)} J_i=\mathbf{z}} \sum_{\sum_{i=1}^{l(\mu)} g_i=h+l(\mu)-l} \left[\prod_{i=1}^{l(\mu)} \omega_{g_i,|\mu_i|+|J_i|}(\mu_i,J_i)\right]$$

$$\hat{Q}^{(l)}_{h,n+1}(z;\mathbf{z}) := \sum_{\beta \underset{l}{\subseteq} \left(x^{-1}(x(z))\setminus\{z\}\right)} \sum_{\mu\in\mathcal{S}(\beta)} \sum_{\bigsqcup_{i=1}^{l(\mu)} J_i=\mathbf{z}} \sum_{\sum_{i=1}^{l(\mu)} g_i=h+l(\mu)-l} \left[\prod_{i=1}^{l(\mu)} \omega_{g_i,|\mu_i|+|J_i|}(\mu_i,J_i)\right]$$

$$\widetilde{Q}^{(l)}_{h,n+1}(\lambda;\mathbf{z}) := \frac{Q^{(l)}_{h,n+1}(\lambda;\mathbf{z})}{(d\lambda)^l} - \sum_{j=1}^{n} d_{z_j}\left(\frac{1}{\lambda-x(z_j)}\,\frac{\hat{Q}^{(l-1)}_{h,n}(z_j;\mathbf{z}\setminus\{z_j\})}{(dx(z_j))^{l-1}}\right)$$

Loop equations

For any $(h,n,l)\in\mathbb{N}^3$ and any $\mathbf{z}\in(\Sigma\setminus\mathcal{R})^n$, the function $\lambda \mapsto \frac{Q^{(l)}_{h,n+1}(\lambda;\mathbf{z})}{(d\lambda)^l}$ has no poles at critical values.

Perturbative wave functions

Generic perturbative wave functions

Perturbative wave functions

$((\Sigma,x),(\mathcal{A}_i,\mathcal{B}_i)_{i=1}^g)$ admissible initial data, $D=\sum_{i=1}^s \alpha_i[p_i]$ generic divisor on Σ. *Perturbative wave functions* associated to D are

$$\psi(D,\hbar) := \exp\left(\sum_{h,n\geq 0}\frac{\hbar^{2h-2+n}}{n!}\int_D\cdots\int_D \omega_{h,n}(\mathbf{z}) - \delta_{h,0}\delta_{n,2}\frac{dx(z_1)dx(z_2)}{(x(z_1)-x(z_2))^2}\right)$$

$$\forall\, i\in[\![1,s]\!]\ :\ \psi_{0,i}(D,\hbar) \ := \ \psi(D,\hbar),$$

$$\forall\, i\in[\![1,s]\!]\,,\ l\geq 1\ :\ \psi_{l,i}(D,\hbar) \ := \ \left[\sum_{h\geq 0}\sum_{n\geq 0}\frac{\hbar^{2h+n}}{n!}\overbrace{\int_D\cdots\int_D}^{n}\frac{\hat{Q}^{(l)}_{h,n+1}(p_i;\cdot)}{(dx(p_i))^l}\right]\psi(D,\hbar).$$

Remark

Definition as a **formal power series in $\hbar$ times exponential terms in finite negative powers of $\hbar$** (formal WKB series):

$$e^{-\hbar^{-2}\omega_{0,0}}e^{-\hbar^{-1}\int_D\omega_{0,1}}\psi(D,\hbar)\in\mathbb{C}[[\hbar]].$$

KZ equations

- Loop equations translate into Knizhnik–Zamolodchikov (KZ) equations [9]

Generic KZ equations

For $i \in [\![1, s]\!]$ and $l \in [\![0, d-1]\!]$, we have

$$\frac{\hbar}{\alpha_i}\frac{d\psi_{l,i}(D,\hbar)}{dx(p_i)} = -\psi_{l+1,i}(D,\hbar) - \hbar \sum_{j\in[\![1,s]\!]\setminus\{i\}} \alpha_j \frac{\psi_{l,i}(D,\hbar)-\psi_{l,j}(D,\hbar)}{x(p_i)-x(p_j)}$$
$$+\sum_{h\geq 0}\sum_{n\geq 0}\frac{\hbar^{2h+n}}{n!}\int_{z_1\in D}\cdots\int_{z_n\in D}\widetilde{Q}^{(l+1)}_{h,n+1}(x(p_i);\mathbf{z})\,\psi(D,\hbar)$$
$$+\left(\frac{1}{\alpha_i}-\alpha_i\right)\left[\sum_{(h,n)\in\mathbb{N}^2}\frac{\hbar^{2h+n+1}}{n!}\overbrace{\int_D\cdots\int_D}^{n}\frac{d}{dx(p_i)}\left(\frac{\hat{Q}^{(l)}_{h,n+1}(p_i;\cdot)}{(dx(p_i))^l}\right)\right]\psi(D,\hbar).$$

- Valid for generic divisors (p_i not a pole or a ramification point).
- Simplification for two points divisors with $(\alpha_1,\alpha_2)\in\{-1,+1\}^2$.

Remarks

- KZ equations allow to obtain PDEs for $\psi(D, \hbar)$.
- Generic divisors provide PDEs with derivatives $\frac{\partial}{\partial x(z)}$ up to order d^2 generically.
- Quantum curve is expected to be of order d and not d^2.
- At least two specific choices of divisors allow for order d: $D = [z] - [\infty^{(\alpha)}]$ or $D = [z] - [\sigma(z)]$.
- Open question: are there other choices that provide PDEs of order d?

Regularization of perturbative wave functions for $D = [z] - [\infty^{(\alpha)}]$

Infinity is a pole of the classical spectral curve $\Rightarrow D = [z] - [\infty^{(\alpha)}]$ is **not** a generic divisor $\Rightarrow$ Some quantities ($\omega_{0,1}$ and $\omega_{0,2}$) require regularization obtained from $\lim_{p \to \infty^{(\alpha)}} ([z] - [p])$

Definition of regularized wave function

$$\psi^{\text{reg}}(D = [z] - [\infty^{(\alpha)}], \hbar) := \exp\left(\hbar^{-1}\left(V_{\infty^{(\alpha)}}(z) + \int_{\infty^{(\alpha)}}^{z} (ydx - dV_{\infty^{(\alpha)}})\right)\right)$$

$$\frac{1}{E(z, \infty^{(\alpha)})\sqrt{dx(z) d\zeta_{\infty^{(\alpha)}}(\infty^{(\alpha)})}} \exp\left(\sum_{n \geq 3\delta_{h,0}} \frac{\hbar^{2h-2+n}}{n!} \int_{\infty^{(\alpha)}}^{z} \cdots \int_{\infty^{(\alpha)}}^{z} \omega_{h,n}\right)$$

$$\psi_l^{\text{reg}}(D = [z] - [\infty^{(\alpha)}], \hbar) :=$$

$$\left(\sum_{n \geq 3\delta_{h,0}} \frac{\hbar^{2h+n}}{n!} \int_{\infty^{(\alpha)}}^{z} \cdots \int_{\infty^{(\alpha)}}^{z} \frac{\hat{Q}^{(l)}_{h,n+1}(z; z_1, \ldots, z_n)}{dx(z)^l}\right) \psi^{\text{reg}}(D = [z] - [\infty^{(\alpha)}], \hbar)$$

KZ equations for regularized wave functions

KZ equations for regularized wave functions

$$\hbar \frac{d}{dx(z)} \psi_l^{\mathrm{reg}}(D = [z] - [\infty^{(\alpha)}], \hbar) + \psi_{l+1}^{\mathrm{reg}}(D = [z] - [\infty^{(\alpha)}], \hbar)$$
$$= \Bigg[\sum_{h\geq 0} \sum_{n\geq 0} \frac{\hbar^{2h+n}}{n!} \sum_{P\in\mathcal{P}} \sum_{k\in S_P^{(l+1)}} \xi_P(x(z))^{-k} \operatorname*{Res}_{\lambda\to P} d\xi_P(\lambda)\, \xi_P(\lambda)^{k-1}$$
$$\int_{z_1=\infty^{(\alpha)}}^{z_1=z} \cdots \int_{z_n=\infty^{(\alpha)}}^{z_n=z} \frac{Q_{h,n+1}^{(l+1)}(\lambda; \mathbf{z})}{(d\lambda)^{l+1}} \Bigg] \psi^{\mathrm{reg}}(D = [z] - [\infty^{(\alpha)}], \hbar)$$

Comments and technical issue

- RHS of KZ equations uses residues, i.e. integrals.
- RHS may be rewritten using generalized integrals, i.e. **linear operators** $\mathcal{I}_{\mathcal{C}_{p,k}}$.
- $\mathcal{I}_{\mathcal{C}_{p,k}}$ is expected to correspond to $\partial_{t_{p,k}}$. Valid for $d = 2$ and examples.
- Action of these operators is defined only on a sub-algebra generated by $\int_{\mathcal{C}_1} \cdots \int_{\mathcal{C}_n} \omega_{h,n}$. $\Leftrightarrow$ **Algebra of symbols**
- One need to check that these operators never act on something else.

PDE form of KZ equations

PDE form of KZ equations

$$\hbar \frac{d}{dx(z)} \psi_l^{\mathrm{reg}}([z]-[\infty^{(\alpha)}]) + \psi_{l+1}^{\mathrm{reg}}([z]-[\infty^{(\alpha)}]) = \mathrm{ev.}\, \widetilde{\mathcal{L}}_l(x(z)) \left[\psi^{\mathrm{reg\ symb}}([z]-[\infty^{(\alpha)}])\right]$$

with

$$\widetilde{\mathcal{L}}_l(x(z)) = \sum_{P\in\mathcal{P}} \sum_{k\in S_P^{(l+1)}} \xi_P(x(z))^{-k} \widetilde{\mathcal{L}}_{P,k,l}$$

Definition of the operators

Definition of the operators $\widetilde{\mathcal{L}}_{P,k,l}$

$$
\begin{aligned}
\widetilde{\mathcal{L}}_{P,k,l} \quad := \quad & \epsilon_P^{l+1}\Bigg[\xi_P(x(z))^{-(l+1)\epsilon_P} \sum_{\ell'=0}^{l+1} \sum_{\nu' \subset_{\ell'} [\![1,d]\!]} \prod_{j\in\nu'} \left(\sum_{m=0}^{r_{P(j)}-1} \frac{t_{P(j),m}}{d_{P(j)}} \xi_P^{-\frac{m}{d_{P(j)}}} \right) \\
& \sum_{0\leq \ell'' \leq \frac{l+1-\ell'}{2}} \sum_{\substack{\nu'' \in \mathcal{S}^{(2)}([\![1,d]\!]\setminus \nu') \\ l(\nu'')=\ell''}} \prod_{i=1}^{\ell''} \frac{\hbar^2 R(P)_{\nu''_i}}{d_{P(\nu''_{i,+})} d_{P(\nu''_{i,-})}} \\
& \sum_{\nu \subseteq_{l+1-\ell'-2\ell''} [\![1,d]\!]\setminus(\nu'\cup\nu'')} \prod_{j\in\nu} \left(\hbar^2 \sum_{m=1}^{\infty} \frac{\xi_P^{\frac{m}{d_{P(j)}}}}{d_{P(j)}} \mathcal{I}_{\mathcal{C}_{P(j),k}} \right)\Bigg]_{-k} \\
+ \quad & \hbar \delta_{P,\infty} \frac{\epsilon_\infty^{l+1}}{d_{\infty(\alpha)}} \Bigg[\xi_\infty(x(z))^{-(l+1)\epsilon_\infty} \sum_{\ell'=0}^{l+1} \sum_{\nu' \subset_{\ell'} [\![1,d]\!]\setminus\{\alpha\}} \prod_{j\in\nu'} \left(\sum_{m=0}^{r_{\infty(j)}-1} \frac{t_{\infty(j),k}}{d_{\infty(j)}} \xi_\infty^{-\frac{m}{d_{\infty(j)}}} \right) \\
& \sum_{0\leq \ell'' \leq \frac{l+1-\ell'}{2}} \sum_{\substack{\nu'' \in \mathcal{S}^{(2)}([\![1,d]\!]\setminus(\nu'\cup\{\alpha\})) \\ l(\nu'')=\ell''}} \prod_{i=1}^{\ell''} \frac{\hbar^2 R(\infty)_{\nu''_i}}{d_{\infty(\nu''_{i,+})} d_{\infty(\nu''_{i,-})}} \\
& \sum_{\nu \subseteq_{l-\ell'-2\ell''} [\![1,d]\!]\setminus(\nu'\cup\nu''\cup\{\alpha\})} \prod_{j\in\nu} \left(\hbar^2 \sum_{m=1}^{\infty} \frac{\xi_\infty^{\frac{m}{d_{\infty(j)}}}}{d_{\infty(j)}} \mathcal{I}_{\mathcal{C}_{\infty(j),m}} \right)\Bigg]_{-k}
\end{aligned}
$$

Monodromies

- Perturbative wave functions have bad monodromies on $\mathcal{B}$-cycles.
- Monodromies are directly connected to a shift of the filling fractions $\epsilon_i = \oint_{\mathcal{A}_i} \omega_{0,1}$ by $\hbar$.
- Monodromies issues only arise for genus $g > 0$ classical spectral curves.
- Solution is to "sum over filling fractions" $\Rightarrow$ Formal Fourier transform $\Rightarrow$ non-perturbative corrections.

Non-perturbative wave functions

Non perturbative wave functions

Integrals of TR differentials

For any divisor D, let us define $G^{(0)}_{(i_1,\dots,i_k)}(D) = \delta_{k,0}$ and for $r \geq 1$:

$$G^{(r)}_{(i_1,i_2,\dots,i_k)}(D) = \sum_{\ell=1}^{r} \frac{1}{\ell!} \sum_{(h_1,n_1),\dots,(h_\ell,n_\ell)} \delta\left(r = k + \sum_{j=1}^{\ell} 2h_j - 2 + n_j\right)$$

$$\left(\mathcal{I}_{\mathcal{B}_{i_1}} \dots \mathcal{I}_{\mathcal{B}_{i_k}} \prod_{j=1}^{\ell} \left(\frac{1}{n_j!} \overbrace{\int_D \cdots \int_D}^{n_j} \omega_{h_j,n_j}\right)\right)_{\text{stable}}$$

$$G^{(1)}_{\emptyset}(D) = \int_D \omega_{1,1} + \frac{1}{6}\int_D\int_D\int_D \omega_{0,3}\,,\ G^{(1)}_{(i_1)}(D) = \int_{\mathcal{B}_{i_1}} \omega_{1,1} + \frac{1}{2}\int_D\int_D\int_{\mathcal{B}_{i_1}} \omega_{0,3}$$

$$\begin{aligned} G^{(2)}_{(i_1,i_2)}(D) &= \frac{1}{2}\int_{\mathcal{B}_{i_1}} \omega_{1,1} \int_{\mathcal{B}_{i_2}} \omega_{1,1} + \frac{1}{2}\int_D\int_D\int_{\mathcal{B}_{i_2}}\int_{\mathcal{B}_{i_1}} \omega_{0,4} \\ &+ \frac{1}{2}\int_{\mathcal{B}_{i_1}} \omega_{1,1} \int_D\int_D\int_{\mathcal{B}_{i_2}} \omega_{0,3} + \frac{1}{2}\int_{\mathcal{B}_{i_2}} \omega_{1,1} \int_D\int_D\int_{\mathcal{B}_{i_1}} \omega_{0,3} \\ &+ \frac{1}{8}\int_D\int_D\int_{\mathcal{B}_{i_1}} \omega_{0,3} \int_D\int_D\int_{\mathcal{B}_{i_2}} \omega_{0,3} \end{aligned}$$

Non perturbative wave functions

Non perturbative wave functions

$$\psi_{\mathrm{NP}}(D;\hbar,\boldsymbol{\rho}) := e^{\hbar^{-2}\omega_{0,0}+\omega_{1,0}}\, e^{\hbar^{-1}\int_D \omega_{0,1}} \frac{1}{E(D)} \sum_{r=0}^{\infty} \hbar^r G^{(r)}(D;\boldsymbol{\rho})$$

where E prime form on Σ and

$$G^{(r)}(D;\boldsymbol{\rho}) := \sum_{k=0}^{3r} \sum_{(i_1,\dots,i_k)\in[\![1,g]\!]^k} \Theta^{(i_1,\dots,i_k)}(\mathbf{v},\tau)\, G^{(r)}_{(i_1,\dots,i_k)}(D)$$

with

$$v_j := \frac{\rho_j+\phi_j}{\hbar} + \mu_j^{(\alpha)}(z),\;\; \phi_j := \frac{1}{2\pi \mathrm{i}} \oint_{\mathcal{B}_j} \omega_{0,1},\;\; \mu_j^{(\alpha)}(z) := \frac{1}{2\pi \mathrm{i}} \int_D \oint_{\mathcal{B}_j} \omega_{0,2}.$$

Moreover

$$\psi_{l,\mathrm{NP}}^{\infty^{(\alpha)}}(z,\hbar,\boldsymbol{\rho}) := \sum_{\beta \subseteq_l (x^{-1}(x(z))\setminus\{z\})} \frac{1}{l!}\,\mathrm{ev.}\left(\prod_{j=1}^{l} \mathcal{I}_{\mathcal{C}_{\beta_j,1}}\right) \psi_{\mathrm{NP}}^{\mathrm{symbol}}([z]-[\infty^{(\alpha)}];\hbar,\boldsymbol{\rho})$$

and $d\times d$ wave functions matrix

$$\widehat{\Psi}_{\mathrm{NP}}(\lambda,\hbar,\boldsymbol{\rho}) := \left[\psi_{l-1,\mathrm{NP}}^{\infty^{(\alpha)}}(z^{(\alpha)}(\lambda),\hbar,\boldsymbol{\rho})\right]_{1\le l,\alpha\le d},$$

Trans-series in $\hbar$

- Non-perturbative quantities are **formal trans-series in** $\hbar$ of the form

$$\sum_{r=0}^{\infty} \sum_{\mathbf{n}\in\mathbb{Z}^g} \hbar^r e^{\frac{1}{\hbar}\sum_{j=1}^{g} n_j \phi_j} F_{r,\mathbf{n}},$$

- Equalities should only be considered coefficients by coefficients in the trans-monomials.
- Non-perturbative wave functions satisfy same KZ equations as the perturbative wave functions.
- Non-perturbative wave functions have good monodromies. $\Rightarrow$ **rational functions** of λ.

Lax pairs

Lax systems

Lax systems

We have the Lax systems

$$\begin{aligned} \hbar \frac{d\widehat{\Psi}_{\mathrm{NP}}(\lambda,\hbar)}{d\lambda} &= \widehat{L}(\lambda,\hbar)\widehat{\Psi}_{\mathrm{NP}}(\lambda,\hbar) \\ \hbar^{-1}\mathrm{ev}.\mathcal{L}_{P,k,l}\widehat{\Psi}^{\mathrm{symbol}}_{\mathrm{NP}}(\lambda,\hbar) &= \widehat{A}_{P,k,l}(\lambda,\hbar)\widehat{\Psi}_{\mathrm{NP}}(\lambda,\hbar) \end{aligned}$$

with

$$\begin{aligned} \widehat{L}(\lambda,\hbar) &= \left[-\widehat{P}(\lambda) + \hbar \sum_{P\in\mathcal{P}}\sum_{k\in\mathbb{N}} \xi_P^{-k}(\lambda)\widehat{\Delta}_{P,k}(\lambda,\hbar)\right] \\ \left[\widehat{\Delta}_{P,k}(\lambda,\hbar)\right]_{2,j} &= \left[\widehat{A}_{P,k,l}(\lambda,\hbar)\right]_{1,j},\ \forall j\in[\![1,d]\!], \end{aligned}$$

and

$$\widehat{P}(\lambda) := \begin{bmatrix} -P_1(\lambda) & 1 & 0 & \dots & 0 \\ -P_2(\lambda) & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -P_{d-1}(\lambda) & 0 & 0 & \dots & 1 \\ -P_d(\lambda) & 0 & 0 & \dots & 0 \end{bmatrix}$$

Gauge transformation to recover companion-like matrix when $\hbar \to 0$

Define

$$G(\lambda) := \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ P_1(\lambda) & -1 & 0 & \dots & 0 & 0 \\ P_2(\lambda) & -P_1(\lambda) & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ P_{d-2}(\lambda) & -P_{d-3}(\lambda) & P_{d-4}(\lambda) & \dots & (-1)^{d-2} & 0 \\ P_{d-1}(\lambda) & -P_{d-2}(\lambda) & P_{d-3}(\lambda) & \dots & (-1)^{d-2}P_1(\lambda) & (-1)^{d-1} \end{bmatrix}$$

and

$$\begin{aligned} \widetilde{\Psi}(\lambda,\hbar) &:= (G(\lambda))^{-1}\,\widehat{\Psi}_{\mathrm{NP}}(\lambda,\hbar) \\ \hbar\frac{d\widetilde{\Psi}(\lambda,\hbar)}{d\lambda} &= \widetilde{L}(\lambda,\hbar)\widetilde{\Psi}(\lambda,\hbar) \\ \hbar^{-1}\mathrm{ev}.\mathcal{L}_{P,k,l}\widetilde{\Psi}(\lambda,\hbar) &= \widetilde{A}_{P,k,l}(\lambda,\hbar)\widetilde{\Psi}(\lambda) \end{aligned}$$

with

$$\widetilde{L}(\lambda,\hbar) = \left[\widetilde{P}(\lambda) + \hbar\sum_{P\in\mathcal{P}}\sum_{k\in\mathbb{N}}\xi_P^{-k}(\lambda)\widetilde{\Delta}_{P,k}(\lambda,\hbar)\right]$$

$\widetilde{\mathrm{P}}(\lambda)$ **companion-like matrix associated to the classical spectral curve**.

Main result: pole structure of the Lax system

Pole structure of the Lax system

Matrices $\widetilde{A}_{P,k,l}(\lambda,\hbar)$ are **rational functions** of λ with **no pole at critical values** $u \in x(\mathcal{R})$.
Matrices $\widetilde{L}(\lambda,\hbar)$ and $\hat{L}(\lambda,\hbar)$ are **rational functions** of λ with possible **poles only at** $\lambda \in \mathcal{P}$ and at **zeros of the Wronskian** $\det \widehat{\Psi}_{\mathrm{NP}}(\lambda,\hbar)$ (i.e. *apparent singularities*).

- Long and technical proof by induction relatively to the order in the trans-series.
- Proof uses some of admissibility conditions (distinct critical values, smooth and simple ramification points).
- Proof should adapt without the admissibility conditions but involving more technical computations.

Quantum curve

Quantum curve

$\forall j \in [\![1,d]\!]$, $\psi^{\infty,(\alpha)}_{0,\mathrm{NP}}(z^{(j)}(\lambda),\hbar)$ is solution to a degree d ODE of the form

$$\forall j \in [\![1,d]\!] \,:\, \sum_{k=0}^{d} b_{d-k}(\lambda,\hbar)\left(\hbar\frac{\partial}{\partial\lambda}\right)^{k}\psi^{\infty,(\alpha)}_{0,\mathrm{NP}}(z^{(j)}(\lambda),\hbar)=0,$$

Coefficients $(b_l(\lambda,\hbar))_{l\in[\![0,d]\!]}$ with $b_0(\lambda,\hbar)=1$ are **rational functions** of λ with **poles only at** $\lambda\in\mathcal{P}$ **and zeros of the Wronskian.**

$\Leftrightarrow$ Matrix form: $\Psi(\lambda,\hbar):=\left[\left(\hbar\frac{\partial}{\partial\lambda}\right)^{i-1}\psi^{\infty,(\alpha)}_{0,\mathrm{NP}}(z^{(j)}(\lambda),\hbar)\right]_{1\leq i,j\leq d}$ satisfies:

$$\begin{aligned}\hbar\frac{\partial}{\partial\lambda}\Psi(\lambda,\hbar) &= \begin{bmatrix} 0 & 1 & \dots & 0\\ \vdots & \ddots & \ddots & \ddots\\ 0 & 0 & & 1\\ -b_d(\lambda,\hbar) & -b_{d-1}(\lambda,\hbar) & \dots & -b_1(\lambda,\hbar)\end{bmatrix}\Psi(\lambda,\hbar)\\ &:= L(\lambda,\hbar)\Psi(\lambda,\hbar)\end{aligned}$$

Gauge transformation to remove apparent singularities

- Apparent singularities $\Leftrightarrow$ zeros of Wronskian:

$$W(\lambda,\hbar) := \det \Psi(\lambda,\hbar) = \kappa \frac{\prod_{i=1}^{G} (\lambda - q_i(\hbar))}{\prod_{i=1}^{N} (\lambda - \Lambda_i)^{G_{\Lambda_i}}} \exp\left(\hbar^{-1} \int_0^{\lambda} P_1(\lambda) d\lambda\right),$$

- Explicit gauge transformation $J(\lambda,\hbar)$ to **remove apparent singularities**

$$\check{\Psi}(\lambda,\hbar) := \begin{bmatrix} 1 & \dots & 0 & 0 \\ \ddots & \ddots & & \vdots \\ 0 & \dots & 1 & 0 \\ \frac{Q_d(\lambda,\hbar)}{\prod_{i=1}^{G}(\lambda-q_i(\hbar))} & \dots & \frac{Q_2(\lambda,\hbar)}{\prod_{i=1}^{G}(\lambda-q_i(\hbar))} & \frac{Q_1(\lambda,\hbar)}{\prod_{i=1}^{G}(\lambda-q_i(\hbar))} \end{bmatrix} \Psi(\lambda,\hbar)$$

- Q_j: polynomial of degree $G-1$ at most defined by interpolation.
- Gauge transformation does not introduce new poles because

$$\det J(\lambda,\hbar) = \left(\prod_{k=1}^{N} (\lambda - \Lambda_k)^{G_{\Lambda_k}}\right) \left(\prod_{i=1}^{G} (\lambda - q_i(\hbar))\right)^{-1}$$

Remarks

4 equivalent gauges:

- Gauge $\hat{\Psi}(\lambda,\hbar)$: Natural gauge from KZ equations and provides compatible auxiliary systems. But leading order in $\hbar$ of $\hat{L}(\lambda,\hbar)$ is not companion-like $\Rightarrow$ Classical spectral curve is not easily recovered. Contains apparent singularities.
- Gauge $\widetilde{\Psi}(\lambda,\hbar)$: Same properties as the previous gauge ($\hbar^0$ gauge transformation) except leading order in $\hbar$ is companion-like and recovers the classical spectral curve.
- Gauge $\Psi(\lambda,\hbar)$: $L(\lambda,\hbar)$ is companion-like $\Rightarrow$ Quantum curve is directly read in the last line of $L(\lambda,\hbar)$. Classical spectral curve directly obtained as $\hbar \to 0$ limit of $L(\lambda,\hbar)$. But contains apparent singularities. Natural framework for Darboux coordinates and isomonodromic deformations.
- Gauge $\check{\Psi}$: $\check{L}(\lambda,\hbar)$ has no apparent singularity. But no longer companion like so less adapted to read the classical and quantum curves.

Lax systems and isomonodromic deformations

Meromorphic connections in $\mathfrak{gl}_d(\mathbb{C})$

- Start from a differential system $\hbar\partial_\lambda \tilde{\Psi} = \tilde{L}(\lambda)\tilde{\Psi}$ with $\tilde{L}(\lambda)$ rational in λ with poles in $\mathcal{P} = \{\infty, \Lambda_1, \dots, \Lambda_N\}$
- Classical spectral curve is defined by $\lim_{\hbar\to 0} \det(yI_d - \tilde{L}(\lambda)) = 0$
- Choose orders of poles $(r_\infty, r_1, \dots, r_N)$ to get same type of classical spectral curve and define

$$F_{\mathcal{R},\mathbf{r}} := \left\{ \hat{L}(\lambda) = \sum_{k=1}^{r_\infty - 1} \hat{L}^{[\infty,k]}\lambda^{k-1} + \sum_{s=1}^{n}\sum_{k=0}^{r_s-1} \frac{\hat{L}^{[X_s,k]}}{(\lambda - X_s)^{k+1}} \right\} / \mathrm{GL_d}(\mathbb{C})$$

- $F_{\mathcal{R},\mathbf{r}}$ has a Poisson structure (loop algebra [27, 1]). Representative normalized at infinity $\tilde{L}$ ($\tilde{L}^{[\infty,r_\infty-1]}$ diagonal and $\left[\tilde{L}^{[\infty,r_\infty-2]}\right]_{1,j} = 1$ for $j \geq 2$).
- Irregular times $\mathbf{t} = (t_{p,k})_{p\in\mathcal{P},1\leq k\leq r_p-1}$ and monodromies $\mathbf{t}_0 = (t_{p,0})_{p\in\mathcal{P}}$ are given as singular part of the local diagonalization of $\tilde{L}$ at each pole.
- Symplectic manifold of dimension $2g$ (g genus of the spectral curve):

$$\hat{\mathcal{M}}_{\mathcal{R},\mathbf{r},\mathbf{t},\mathbf{t_0}} := \left\{ \hat{L}(\lambda) \in \hat{F}_{\mathcal{R},\mathbf{r}} \ / \ \hat{L}(\lambda) \text{ has irregular times } \mathbf{t} \text{ and monodromies } \mathbf{t_0} \right\}$$

Isomonodromic deformations

- Existence of g isomonodromic deformations

$$\hbar \partial_{\tau_i} \tilde{\Psi} = A_{\tau_i}(\lambda) \tilde{\Psi}$$

A_{τ_i} rational in λ with dominated pole structure [4].

- Existence of a Hamiltonian system and $2g$ Darboux coordinates $(x_i, y_i)_{1 \leq i \leq g}$ parametrizing the Lax pairs.

$$\hbar \partial_{\tau_i} x_j = \frac{\partial H_i}{\partial y_j} \,,\; \hbar \partial_{\tau_i} y_j = -\frac{\partial H_i}{\partial x_j}$$

- Explicit expression of the Lax pairs, Hamiltonians in $\mathfrak{gl}_2$ in terms of the apparent singularities q_i and dual coordinates p_i in [34, 31].
- Lax matrices are the same as the one produced by the quantization procedure in $\mathfrak{gl}_2$.
- Monodromies and Stokes matrices are independent of the isomonodromic times. Help for analytic understanding?

Determinantal formulas

For any differential system $\hbar\partial_\lambda \Psi = L(\lambda,\hbar)\Psi$ with L rational in λ and formal Taylor series in $\hbar$, one may associate determinantal formulas.

Definition

Define $(\Sigma, x(z))$ the classical spectral curve: $\lim_{\hbar\to 0} \det(yI_d - L(\lambda)) = 0$.
Let $E_i = \operatorname{diag}(\mathbf{0}_{i-1}, 1, \mathbf{0}_{d-1-i})$ for $i \in [\![1,d]\!]$ and

$$M(z;E_i) := \Psi(z)E_i\Psi(z)^{-1}$$

Set of correlators W_n ("**determinantal formulas**") for $n \geq 1$:

$$W_1(z_1 \otimes E_{i_1}) = \mathrm{Tr}\left[L(x(z_1))M(z_1;E_{i_1})\right] dx(z_1)$$
$$W_n(z_1 \otimes E_{i_1}, \dots, z_n \otimes E_{i_n})$$
$$= \frac{(-1)^n}{n} \sum_{\sigma\in\mathcal{S}_n} \frac{\mathrm{Tr}\left[M(z_{\sigma(1)};E_{i_{\sigma(1)}})\dots M(z_{\sigma(n)};E_{i_{\sigma(n)}})\right]}{\prod_{i=1}^{n}\left(x(z_{\sigma(i)})-x(z_{\sigma(i+1)})\right)} \prod_{i=1}^{n} dx(z_i)$$

Properties of determinantal formulas

- Correlators W_n satisfy loop equations.
- Correlators $(W_n)_{n\geq 1}$ can be defined for any linear differential systems (no need for Lax pairs)
- If correlators W_n satisfy "Topological Type Property" then they are reconstructed by $(\omega_{h,n})_{h,n\geq 0}$ TR differentials:

$$W_n(\lambda_1,\dots,\lambda_n;\hbar)=\sum_{k=0}^{\infty}\omega_{k,n}(\lambda_1,\dots,\lambda_n)\hbar^{n-2+2k}$$

- Topological Type Property requires genus 0 and some additional pole structure. Can it be generalized to higher genus? Ψ (and W_n) are expected to be transseries in $\hbar$, how is the reconstruction formula adapted?

Several types of solutions

- Hamiltonian system, Lax pairs, spectral curves are independent of the type of solution Ψ (formal WKB, formal trans-series, etc.)
- Formal WKB solutions of the Lax system $\Leftrightarrow$ 0-instanton solutions of the Hamiltonian system (i.e. formal power series in $\hbar$) $\Leftrightarrow$ Classical spectral curve genus drops to 0 $\Leftrightarrow$ Degenerate classical spectral curve: ramification points coincide by pairs
- Auxiliary matrices are crucial to prove "topological type property" (no pole at double zeros) in the formal WKB solutions case.
- General solutions are expected to be $\hbar$-transseries (k-instanton solutions, etc.). How auxiliary matrices could help in this case?

Example

Classical spectral curve

Classical spectral curve

We take $d = 2$, $N = 0$, $r_\infty^{(1)} = 2$ and $r_\infty^{(2)} = 4$. Two points above infinity denoted by $\infty^{(1)}$ and $\infty^{(2)}$ non-ramified.

$$y^2 - P_1(\lambda) y + P_2(\lambda) = 0,$$

with

$$\begin{aligned} P_1(\lambda) &= P_{\infty,2}^{(1)}\lambda^2 + P_{\infty,1}^{(1)}\lambda + P_{\infty,0}^{(1)} \\ P_2(\lambda) &= P_{\infty,4}^{(2)}\lambda^4 + P_{\infty,3}^{(2)}\lambda^3 + P_{\infty,2}^{(2)}\lambda^2 + P_{\infty,1}^{(2)}\lambda + P_{\infty,0}^{(2)} \end{aligned}$$

Six spectral times $(t_{i,j})_{1\leq i\leq 2, 0\leq j\leq 3}$ are defined by $\forall\, i \in \{1,2\}$:

$$y(z) = -t_{i,3}x(z)^2 - t_{i,2}x(z) - t_{i,1} - t_{i,0}x(z)^{-1} + O\left(x(z)^{-2}\right), \text{ as } z \to \infty^{(i)}$$

Associated isomonodromic system is the Painlevé 2 Lax pair.

Connection with spectral times

Relations between spectral times and coefficients of the classical spectral curve:

$$
\begin{aligned}
P^{(1)}_{\infty,2} &= -t_{1,3} - t_{2,3} \\
P^{(1)}_{\infty,1} &= -t_{1,2} - t_{2,2} \\
P^{(1)}_{\infty,0} &= -t_{1,1} - t_{2,1} \\
P^{(2)}_{\infty,4} &= t_{1,3}t_{2,3} \\
P^{(2)}_{\infty,3} &= t_{1,2}t_{2,3} + t_{1,3}t_{2,2} \\
P^{(2)}_{\infty,2} &= t_{1,2}t_{2,2} + t_{1,3}t_{2,1} + t_{1,1}t_{2,3} \\
P^{(2)}_{\infty,1} &= t_{1,3}t_{2,0} + t_{1,0}t_{2,3} + t_{1,2}t_{2,1} + t_{1,1}t_{2,2} \\
0 &= -t_{1,0} - t_{2,0}
\end{aligned}
$$

Only $P^{(2)}_{\infty,0}$ remains undetermined (genus 1).

KZ equations

Using the general theory, we get:

KZ equations

$$\begin{cases} \hbar \frac{\partial \psi^{\infty(1)}_{0,\mathrm{NP}}(z,\hbar)}{\partial x(z)} + \psi^{\infty(1)}_{1,\mathrm{NP}}(z,\hbar) = P_1(x(z))\psi^{\infty(1)}_{0,\mathrm{NP}}(z,\hbar), \\ \hbar \frac{\partial \psi^{\infty(1)}_{1,\mathrm{NP}}(z,\hbar)}{\partial x(z)} = P_2(x(z))\psi^{\infty(1)}_{0,\mathrm{NP}}(z,\hbar) + \hbar\, \mathrm{ev.}\, \mathcal{L}_{\mathrm{KZ}}(x(z)) \left[\psi^{\infty(1),\,\mathrm{symbol}}_{0,\mathrm{NP}}(z,\hbar)\right] \end{cases}$$

where

$$\mathcal{L}_{\mathrm{KZ}}(\lambda) := \hbar \mathbf{t}_{1,3} \mathcal{I}_{\mathcal{C}_{\infty^{(2)},1}} + \hbar \mathbf{t}_{2,3} \mathcal{I}_{\mathcal{C}_{\infty^{(1)},1}} - \mathbf{t}_{2,3}\lambda - \mathbf{t}_{2,2}$$

Lax pair from KZ equations

Define $\Psi(\lambda,\hbar) = \begin{pmatrix} \psi^{\infty(\alpha)}_{0,\mathrm{NP}}(z^{(1)}(\lambda),\hbar) & \psi^{\infty(\alpha)}_{0,\mathrm{NP}}(z^{(2)}(\lambda),\hbar) \\ \hbar\partial_\lambda \psi^{\infty(\alpha)}_{0,\mathrm{NP}}(z^{(1)}(\lambda),\hbar) & \hbar\partial_\lambda \psi^{\infty(\alpha)}_{0,\mathrm{NP}}(z^{(2)}(\lambda),\hbar) \end{pmatrix}$

KZ equations are equivalent to

$$\hbar\partial_\lambda \Psi(\lambda,\hbar) = \begin{pmatrix} 0 & 1 \\ -P_2(\lambda) + \hbar P_1'(\lambda) + H - \frac{p}{\lambda-q} + \hbar\alpha\lambda & P_1(\lambda) + \frac{\hbar}{\lambda-q} \end{pmatrix} \Psi(\lambda,\hbar)$$

$$\mathrm{ev}.\mathcal{L}_{KZ}(\lambda)[\Psi^{\mathrm{symbol}}(\lambda,\hbar)] = \begin{pmatrix} -\alpha\lambda - \frac{H}{\hbar} + \frac{p}{\hbar(\lambda-q)} & -\frac{1}{\lambda-q} \\ [A_{KZ}]_{2,1}(\lambda,\hbar) & [A_{KZ}]_{2,2}(\lambda,\hbar) \end{pmatrix} \Psi(\lambda,\hbar)$$

for $\alpha = t_{1,3} + 2t_{2,3}$ and some unknown H.
Equivalently defining

$$\mathcal{L} := \mathcal{L}_{\mathbf{KZ}}(\lambda) + \mathbf{t}_{2,3}\lambda + \mathbf{t}_{2,2} = \hbar\mathbf{t}_{1,3}\mathcal{I}_{\infty^{(2)},1} + \hbar\mathbf{t}_{2,3}\mathcal{I}_{\infty^{(1)},1}$$

we have

$$\begin{aligned} \mathrm{ev}.\mathcal{L}[\Psi^{\mathrm{symbol}}(\lambda,\hbar)] &= \begin{pmatrix} P^{(1)}_{\infty,2}\lambda + t_{2,2} - \frac{H}{\hbar} + \frac{p}{\hbar(\lambda-q)} & -\frac{1}{\lambda-q} \\ A_{2,1}(\lambda,\hbar) & A_{2,2}(\lambda,\hbar) \end{pmatrix} \Psi(\lambda,\hbar) \\ &:= A(\lambda,\hbar)\Psi(\lambda,\hbar) \end{aligned}$$

Evolution equations

- Compatibility equations $\mathcal{L}[L(\lambda,\hbar)] = \hbar\partial_\lambda A(\lambda,\hbar) + [A(\lambda,\hbar), L(\lambda,\hbar)]$:

$$\begin{aligned}
\mathcal{L}[P^{(2)}_{\infty,4}] &= \mathcal{L}[P^{(2)}_{\infty,3}] = 0\\
\mathcal{L}[P^{(2)}_{\infty,2}] &= -2\hbar P^{(2)}_{\infty,4} + \hbar\left[P^{(1)}_{\infty,2}\right]^2\\
\mathcal{L}[P^{(2)}_{\infty,1}] &= -\hbar P^{(2)}_{\infty,3} + \hbar P^{(1)}_{\infty,1}P^{(1)}_{\infty,2}\\
\mathcal{L}[P^{(2)}_{\infty,0}] - \mathcal{L}[H] &= 2\hbar P^{(2)}_{\infty,4}q^2 + \hbar P^{(2)}_{\infty,3}q - P^{(1)}_{\infty,2}p + \hbar P^{(1)}_{\infty,0}P^{(1)}_{\infty,2}\\
H &= \frac{p^2}{\hbar^2} - P_1(q)\frac{p}{\hbar} + P_2(q) - \hbar P_1'(q) + \hbar(P^{(1)}_{\infty,2} - t_{2,3})q\\
\mathcal{L}[q] &= P_1(q) - 2\frac{p}{\hbar}\\
\mathcal{L}[p] &= -P_1'(q)p + \hbar P_2'(q) + \hbar^2 t_{2,3}
\end{aligned}$$

- Equivalent to

$$\mathcal{L}[t_{1,3}] = \mathcal{L}[t_{2,3}] = \mathcal{L}[t_{1,2}] = \mathcal{L}[t_{1,0}] = \mathcal{L}[t_{2,0}] = 0\,,\ \mathcal{L}[t_{1,1}] = \hbar t_{2,3}\,,\ \mathcal{L}[t_{2,1}] = \hbar t_{1,3}$$

- Equivalent to $\mathcal{L} = \hbar t_{2,3}\partial_{t_{1,1}} + \hbar t_{1,3}\partial_{t_{2,1}}$

Hamiltonian evolution

Hamiltonian evolution

"Time" ($\mathcal{L}$)-evolution is Hamiltonian $\Leftrightarrow$ (p,q) are Darboux coordinates

$$\mathcal{L}[q] = -\hbar \frac{\partial H_0}{\partial p}\,, \quad \mathcal{L}[p] = \hbar \frac{\partial H_0}{\partial q}$$

for Hamiltonian $H_0(p,q,\hbar)$:

$$H_0(p,q,\hbar) = \frac{p^2}{\hbar^2} - P_1(q)\frac{p}{\hbar} + P_2(q) - \hbar P_1'(q) + \hbar q(2P^{(1)}_{\infty,2} - t_{2,3})$$

giving $H = H_0(p,q,\hbar) + \hbar(t_{1,3} + t_{2,3})q$.

Connection with the Painlevé 2 equation

- q satisfies the evolution equation:

$$\begin{aligned}\mathcal{L}^2[q] &= 2(t_{1,3}-t_{2,3})^2 q^3 + 3(t_{1,3}-t_{2,3})(t_{1,2}-t_{2,2})q^2 \\ &+ \left((t_{1,2}-t_{2,2})^2 + 2(t_{1,3}-t_{2,3})(t_{1,1}-t_{2,1})\right) q \\ &+(t_{1,2}-t_{2,2})(t_{1,1}-t_{2,1}) + (2t_{1,0}-\hbar)(t_{1,3}-t_{2,3})\end{aligned}$$

- Change of variables $(t_{1,1}, t_{2,1}) \leftrightarrow (\tau, \tilde{\tau})$ and affine rescaling:

$$\begin{aligned}\tau &= \frac{1}{t_{1,3}-t_{2,3}}(t_{2,1}-t_{1,1}), \quad \tilde{\tau} = \frac{1}{t_{1,3}-t_{2,3}}(t_{1,3}t_{1,1}-t_{2,3}t_{2,1}) \\ t &= \left(-2(t_{1,3}-t_{2,3})^2\right)^{\frac{1}{3}}\left(\tau + \frac{(t_{1,2}-t_{2,2})^2}{4(t_{1,3}-t_{2,3})^2}\right) \\ \tilde{q} &= \left(\frac{-(t_{1,3}-t_{2,3})}{2}\right)^{\frac{1}{3}}\left(q + \frac{t_{1,2}-t_{2,2}}{2(t_{1,3}-t_{2,3})}\right)\end{aligned}$$

Then $\tilde{q}(t,\hbar)$ satisfies the Painlevé 2 equation

$$\hbar^2 \partial_{t^2}^2 \tilde{q} = 2\tilde{q}^3 + t\tilde{q} - \left(t_{1,0} - \frac{\hbar}{2}\right)$$

Gauge without apparent singularities

- Gauge transformation to remove apparent singularity:

$$\check{\Psi}(\lambda,\hbar) = \begin{pmatrix} 1 & 0 \\ -\frac{p}{\hbar(\lambda-q)} & \frac{1}{\lambda-q} \end{pmatrix} \Psi(\lambda,\hbar) := J(\lambda,\hbar)\Psi(\lambda,\hbar)$$

- Provides another Lax pair (**Jimbo-Miwa type**) without apparent singularity:

$$\begin{aligned}
\check{L}(\lambda,\hbar) &= \begin{pmatrix} \frac{p}{\hbar} & \lambda - q \\ -((\lambda+q)(t_{1,3}+t_{2,3}) + t_{2,2} + t_{1,2})\frac{p}{\hbar} + Q_3(\lambda,\hbar) & -\frac{p}{\hbar} + P_1(\lambda) \end{pmatrix} \\
\check{A}(\lambda,\hbar) &= \begin{pmatrix} -(t_{1,3}+t_{2,3})\lambda - \frac{H}{\hbar} + t_{2,2} & -1 \\ (t_{1,3}+t_{2,3})\frac{p}{\hbar} + Q_2(\lambda,\hbar) & (t_{1,3}+t_{2,3})q + t_{1,2} + 2t_{2,2} - \frac{H}{\hbar} \end{pmatrix}
\end{aligned}$$

where

$$\begin{aligned}
Q_3(\lambda,\hbar) &= -P^{(2)}_{\infty,4}\lambda^3 - (P^{(2)}_{\infty,4}q + P^{(2)}_{\infty,3})\lambda^2 - (P^{(2)}_{\infty,4}q^2 + P^{(2)}_{\infty,3}q + P^{(2)}_{\infty,2})\lambda \\
&\quad + P^{(2)}_{\infty,4}q^3 + P^{(2)}_{\infty,3}q^2 + P^{(2)}_{\infty,2}q + P^{(2)}_{\infty,1} + \hbar t_{1,3}) \\
Q_2(\lambda,\hbar) &= P^{(2)}_{\infty,4}\lambda^2 + 2P^{(2)}_{\infty,4}q\lambda + P^{(2)}_{\infty,3}\lambda + (3P^{(2)}_{\infty,4}q^2 + 2P^{(2)}_{\infty,3}q + P^{(2)}_{\infty,2})
\end{aligned}$$

Connection with isomonodromic deformations

- Classical spectral curve corresponds to isomonodromic deformations of
$$\tilde{L}(\lambda) = L^{[\infty,3]}\lambda^2 + L^{[\infty,2]}\lambda^1 + L^{[\infty,1]}$$
- Results of [34] immediately recovers the previous Lax matrices and the Hamiltonian system.
- Hamiltonian system is proved invariant under the choice of trivial times. A canonical choice is equivalent to $t_{i,2} = -t_{i,1}$ for all $0 \leq i \leq 3$ ($\mathfrak{gl}_2 \to \mathfrak{sl}_2$) and $t_{3,1} = 1$, $t_{2,1} = 0$ (Mobius transformations). Equivalent to the invariance of TR under rational reparametrization and $y \to y + f(\lambda)$.
- Results of [34] provide explicit formulas for the quantum curve, Lax pairs and Hamiltonians in terms of the apparent singularities $(q_i)_{1\leq i\leq g}$ and dual coordinates $(p_i)_{1\leq i\leq g}$. Results only available for $\mathfrak{gl}_2$ connections so far.

Open questions

Open questions

- Can we give some analytic meaning to the formal WKB solutions or formal $\hbar$-transseries? Compute Stokes matrices? Write a Riemann-Hilbert-Problem for Ψ?
- What about the underlying Hamiltonian structure coming from isomonodromic deformations? How is it helpful?
- How can the auxiliary matrices be useful to describe analytic solutions of the quantum curve?
- Are determinantal formulas (reverse approach) easier to use than the wave matrix?
- Can we obtain explicit expressions for the isomonodromic deformations for $\mathfrak{gl}_d$?

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