Quantization of spectral curves and integrable systems
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1 Introduction: the Airy curve

Let us consider the Airy curve $y^2 = x$ which defines a genus zero spectral curve with a rational parameterization

$$\left\{ \begin{array}{l} x(z) = z^2 \\ y(z) = z \end{array} \right. . \tag{1-1}$$

This curve has a local involution $z \leftrightarrow -z$ such that $x(z) = x(-z)$ and $y(-z) = -y(z)$.

Let us consider as initial condition for the topological recursion

$$\left\{ \begin{array}{l} \omega_{0,1}(z) := y(z)dx(z) = 2z^2 dz \\ \omega_{0,2}(z_1, z_2) := \frac{dz_1\, dz_2}{(z_1-z_2)^2} \end{array} \right. . \tag{1-2}$$

Remark that the spectral curve having vanishing genus, one does not have any freedom for choosing $\omega_{0,2}$.

The topological recursion then defines differential forms $\omega_{h,n}$ by induction

$$\omega_{h,n+1}(z_0, z_1, \dots, z_n) := \operatorname*{Res}_{z\to 0} \frac{\int_{-z}^{z} \omega_{0,2}(z_0, \cdot)}{2(y(z) - y(-z))dx(z)} \left[\omega_{h-1,n+2}(z, -z, z_1, \dots, z_n) + \sum \omega_{h_1,|A|+1}(z, A)\omega_{h_1,|B|+1}(- \tag{1-3}$$

Let us now define the primitives (there is a choice of sheets for the function $z(x)$ to be made)

$$S^{\pm}_{-1}(x) := \pm\frac{1}{2} \int_{-z(x)}^{z(x)} \omega_{0,1}, \tag{1-4}$$

$$S^{\pm}_{0}(x) := \frac{1}{4} \left[\frac{1}{2} \int_{z_1}^{z_2} \int_{z_1}^{z_2} \omega_{0,2} - \ln(x(z_1) - x(z_2))\right]_{\substack{z_1 = -z(x) \\ z_2 = z(x)}} \tag{1-5}$$

and

$$\forall m \geq 1\,,\ S^{\pm}_{m}(x) := \sum_{\substack{h \geq 0,\, n \geq 1 \\ 2h-2+n=m}} \frac{(\pm 1)^n}{n!\, 2^n} \int_{-z(x)}^{z(x)} \dots \int_{-z(x)}^{z(x)} \omega_{h,n}. \tag{1-6}$$

With these functions, one can define the perturbative wave functions

$$\psi_{\pm}(x, \hbar) := \exp\left[\sum_{m\geq -1} \hbar^m S^{\pm}_{m}(x)\right]. \tag{1-7}$$

Theorem 1.1. *One has the quantum curve equation*

$$\left[\hat{y}^2 - \hat{x}\right] \psi_{\pm}(x, \hbar) = 0 \tag{1-8}$$

where

$$\hat{x} := x \qquad , \qquad \hat{y} := \hbar\frac{d}{dx} \tag{1-9}$$

satisfy

$$[\hat{y}, \hat{x}] = \hbar. \tag{1-10}$$

Proof. Move the integration contour in the TR formule to get contributions from the poles of ydx and the coinciding points from $\omega_{0,2}$. After evaluation integration and specialization, this gives the loop equations which imply the quantum curve formula.

This is a very simple case where the spectral curve has genus 0. Bouchard and Eynard proved that such a quantization procedure is possible for any genus 0 spectral curve. However, the higher genus case is much more intricate because of other contributions to the loop equations when moving the integration contours and non-trivial monodromies for the wave functions.

We shall consider such higher genera curves in the present context.

Remark that the Airy equation can be linearized into

$$\hbar\frac{d}{dx}\begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix} = L(x)\begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix} \tag{1-11}$$

with

$$L(x) := \begin{pmatrix}0 & 1\\ x & 0\end{pmatrix}. \tag{1-12}$$

$(\psi_1, \psi_2) = (\psi_\pm, \hbar d_x \psi_\pm)$ are solutions to these equations (independent away from the branch points).

The associated spectral curve (locus of the eigenvalues) reads

$$\det(y - L(x)) = y^2 - x = 0 \tag{1-13}$$

and is independent of $\hbar$. This is a feature of the quantization of genus 0 curves.

One can interpret this as defining a connection $\hbar d - L(x)dx$ on the base curve $\mathbb{P}^1$.

2 Space of spectral curves and TR

2.1 Space of spectral curves

We shall consider curves Σ defined by an equation of the form

$$y^2 = \phi(x) \tag{2-1}$$

where $\phi(x)$ is a rational function. Choosing such a curve is equivalent to choosing a quadratic differential $\phi(x)(dx)^2$ on the base curve $\mathbb{P}^1$. In the following, we fix the poles and degree at poles of $\phi(x)$ so that it reads

$$\phi(x) = \sum_{k=0}^{2(r_\infty-2)-n_\infty} H_{\infty,k}x^k + \sum_{\nu=1}^{n}\sum_{k=1}^{2r_\nu}\frac{H_{\nu,k}}{(x-X_\nu)^k} \tag{2-2}$$

where $n_\infty \in \{0,1\}$. In the following, unless stated explicitly, we consider curves which are not ramified above infinity meaning that $n_\infty = 0$. The notations H refer to Hamiltonians when equipping the space fo quadratic differentials with a Poisson structure.

For such a curve, there exist two points $(\alpha_\nu^+, \alpha_\nu^-)$ (resp. $\infty^\pm$) above X_ν (resp. ∞) where one has an expansion of the one form ydx of the form

$$ydx = \pm\sum_{k=1}^{r_\nu} T_{\nu,k}\frac{dx}{(x-X_\nu)^k} + O(dx) \tag{2-3}$$

around $\alpha_\nu^\pm$ and

$$ydx = \mp \sum_{k=1}^{r_\infty} T_{\infty,k} x^{k-2} dx + O(x^{-2}dx) \tag{2-4}$$

where the coefficients of the singular part are the KP times discussed in Bertrand's lecture. Let us denote the set of poles of ydx by $\mathcal{P} := \{\infty^\pm, \alpha_\nu^\pm\}$.

If the spectral curve does not have genus 0, fixing the KP times does not fix ydx completely as a one form. One way to fix it is obtained by choosing a Torelli marking by fixing a set of cycles $(\mathcal{A}_i, \mathcal{B}_i)_{i=1}^g$ so that it forms a basis of $H_1(\Sigma \setminus \mathcal{P}, \mathbb{Z})$ when completed by small circles around the poles in $\mathcal{P}$ and such that

$$\mathcal{A}_i \bigcap \mathcal{B}_j = \delta_{i,j}. \tag{2-5}$$

With such a choice, one can fix

$$\forall i = 1, \dots g\,,\ \epsilon_i = \oint_{\mathcal{A}_i} ydx \tag{2-6}$$

called the filling fractions.

ydx is uniquely defined by the values of the KP times and the filling fractions. However, it does not depend on the choice of Torelli marking.

Exercise Show that $H_{\infty,k}$ does not depend on the filling fractions for $k \geq r_\infty - 3$. Obtain a similar result for the poles X_ν for $k \geq r_\nu + 1$. Prove that the genus of the spectral curve is equal to

$$g = r_\infty + \sum_{\nu=1}^n r_\nu - 3 \tag{2-7}$$

which is the number of coefficients depending on the filling fractions.

From the integrable system perspective, KP-times fix a set of Casimirs and thus a symplectic leaf in a Poisson manifold while the filling fractions provide local coordinates in the corresponding symplectic space.

Example

Consider $n = 0$, i.e. a single pole at infinity and $r_\infty = 4$. This leads to an expansion of the form

$$ydx = T_{\infty,4} x^2 dx + T_{\infty,3} x dx + T_{\infty,2} dx + T_{\infty,1} x^{-1} dx + O(x^{-2}dx) \tag{2-8}$$

and the spectral curve equation

$$y^2 = T_{\infty,4}^2 x^4 + 2T_{\infty,3}T_{\infty,4} x^3 + \left(2T_{\infty,4}T_{\infty,2} + T_{\infty,3}^2\right) x^2 + \left(2T_{\infty,4}T_{\infty,1} + 2T_{\infty,3}T_{\infty,2}\right) x + H_0. \tag{2-9}$$

Using Newton's polytope (or the picture of a cover), one can see that one has a genus 1 spectral curve. Let us denote

$$\epsilon = \oint_{\mathcal{A}} ydx. \tag{2-10}$$

2.2 Topological recursion and variational formulae

In order to define TR, one needs a set of two initial data. On the one hand,

$$\omega_{0,1} = ydx \tag{2-11}$$

is provided by the spectral curve. We shall now choose $\omega_{0,2}$ by imposing that

$$\forall i = 1, \ldots, g\,, \oint_{\mathcal{A}_i} \omega_{0,2}(z_1, \cdot) = 0. \tag{2-12}$$

With such initial data, we define by induction

$$\begin{aligned}\omega_{h,n}(z_1, \ldots, z_n) \quad := \quad & \sum_{p \in \mathcal{R}} \operatorname*{Res}_{z \to p} \frac{\int_{\sigma(z)}^{z} \omega_{0,2}(z_1, \cdot)}{2\,(\omega_{0,1}(z) - \omega_{0,1}(\sigma(z)))} \Bigg[\omega_{h-1,n+1}(z, \sigma(z), z_2, \ldots, z_n) \\ & + \sum_{\substack{h_1 + h_2 = h \\ A \sqcup B = \{z_2, \ldots, z_n\} \\ (h_1, |A|) \notin \{(0,0), (h, n-1)\}}} \omega_{h_1, |A|+1}(z, A)\, \omega_{h_2, |B|+1}(z, B) \Bigg]\end{aligned}$$

where $\sigma : \Sigma \to \Sigma$ is the hyper-elliptic involution, namely, it is defined by

$$\forall\, z \in \Sigma \setminus \mathcal{R}\,,\ x(z) = x(\sigma(z)) \quad \text{and} \quad \sigma(z) \neq z. \tag{2-13}$$

$\mathcal{R}$ denotes the set of branch points of the spectral curve.

For $h \geq 2$, we define the free energies by

$$\omega_{h,0} := \frac{1}{2 - 2h} \sum_{p \in \mathcal{R}} \operatorname*{Res}_{z \to p} \omega_{h,1}(z) \int_{o}^{z} \omega_{0,1}$$

where $o \in \Sigma$ is an arbitrary base point of which $\omega_{h,0}$ is independent.

Lemma 2.1. *From the general theory of TR, one has*

$$\forall\, k \geq 2\,,\ \frac{\partial \omega_{h,n}(\mathbf{z})}{\partial T_{\infty,k}} = \operatorname*{Res}_{p \to \infty^+} \omega_{h,n+1}(p, \mathbf{z}) \frac{x(p)^{k-1}}{k-1} - \operatorname*{Res}_{p \to \infty^-} \omega_{h,n+1}(p, \mathbf{z}) \frac{x(p)^{k-1}}{k-1}, \tag{2-14}$$

$$\forall\, k \geq 2\,,\ \frac{\partial \omega_{h,n}(\mathbf{z})}{\partial T_{\nu,k}} = \operatorname*{Res}_{p \to \alpha_\nu^+} \omega_{h,n+1}(p, \mathbf{z}) \frac{(x(p) - X_\nu)^{-k+1}}{k-1} - \operatorname*{Res}_{p \to \alpha_\nu^-} \omega_{h,n+1}(p, \mathbf{z}) \frac{(x(p) - X_\nu)^{-k+1}}{k-1} \tag{2-15}$$

and

$$\frac{\partial \omega_{h,n}(\mathbf{z})}{\partial T_{\nu,1}} = \int_{\alpha_\nu^+}^{p} \omega_{h,n+1}(\cdot, \mathbf{z}) - \int_{\alpha_\nu^-}^{p} \omega_{h,n+1}(\cdot, \mathbf{z}). \tag{2-16}$$

For the filling fractions, one has

$$\forall\, j \in [\![1, g]\!]\ :\ \frac{\partial \omega_{h,n}(\mathbf{z})}{\partial \epsilon_j} = \frac{1}{2\pi i} \oint_{\mathcal{B}_j} \omega_{h,n+1}(\cdot, \mathbf{z}). \tag{2-17}$$

Exercise

Show that The expansion of $\omega_{0,1}$ in local coordinates around its poles reads

- around $\alpha_\nu^{\pm}$,

$$\omega_{0,1} = \pm \sum_{k=1}^{r_\nu} T_{\nu,k} \frac{dx}{(x - X_\nu)^k} \pm \sum_{k=2}^{r_\nu} \frac{k-1}{2} \frac{\partial \omega_{0,0}}{\partial T_{\nu,k}} (x - X_\nu)^{k-2} dx + O((x - X_\nu)^{r_\nu - 1} dx)$$

- around $\infty^\pm$,

$$\omega_{0,1} = \mp \sum_{k=1}^{r_\infty} T_{\infty,k} x^{k-2} dx \mp \sum_{k=2}^{r_\infty} \frac{k-1}{2} \frac{\partial \omega_{0,0}}{\partial T_{\infty,k}} x^{-k} dx + O(x^{-r_\infty - 1} dx).$$

Show that it implies that

$$\begin{aligned} y^2 &= \left[\left(\sum_{k=1}^{r_\infty} T_{\infty,k} x^{k-2}\right)^2\right]_{\infty,+} + \sum_{\nu=1}^{n} \left[\left(\sum_{k=1}^{r_\nu} T_{\nu,k} \frac{dx}{(x-X_\nu)^k}\right)^2\right]_{X_\nu,-} \\ &\quad + \sum_{k\in K_\infty} U_{\infty,k}(x) \frac{\partial \omega_{0,0}}{\partial T_{\infty,k}} + \sum_{\nu=1}^{n} \sum_{k\in K_\nu} U_{\nu,k}(x) \frac{\partial \omega_{0,0}}{\partial T_{\nu,k}} \end{aligned} \tag{2-18}$$

Here, $[f(x)]_{\infty,+}$ (resp. $[f(x)]_{X_\nu,-}$) refers to the positive part of the expansion in x of a function $f(x)$ around ∞, including the constant term, (resp. the strictly negative part of the expansion in $(x - X_\nu)$ around X_ν) and we have defined

- $K_\infty = [\![2, r_\infty - 2]\!]$ and $\forall\, k \in K_\infty$:

$$U_{\infty,k}(x) := (k-1) \sum_{l=k+2}^{r_\infty} T_{\infty,l}\, x^{l-k-2}. \tag{2-19}$$

- $K_\nu = [\![2, r_\nu + 1]\!]$ and $\forall\, k \in K_\nu$:

$$U_{\nu,k}(x) := (k-1) \sum_{l=k-1}^{r_\nu} T_{\nu,l}\, (x - X_\nu)^{-l+k-2}. \tag{2-20}$$

Example
In the case above this gives

$$H_0 = 2T_{\infty,1}T_{\infty,3} + T_{\infty,2}^2 + T_{\infty,4} \frac{\partial \omega_{0,0}}{\partial T_{\infty,2}}. \tag{2-21}$$

We shall now prove that this formula is the leading order in $\hbar$ of a PDE.

2.3 Symmetries (Exercise)

In addition to the variational formulas, the output of the topological recursion is skew-symmetric under the hyper-elliptic involution σ, i.e.

$$\forall\, h \geq 0\,,\, \forall\, n \geq 1\,:\, \omega_{h,n}(z_1, \ldots, z_n) + \omega_{h,n}(\sigma(z_1), z_2, \ldots, z_n) = \delta_{h,0}\delta_{n,2} \frac{dx(z_1)\, dx(z_2)}{(x(z_1) - x(z_2))^2}. \tag{2-22}$$

One also has

$$\forall\, (z_1, z_2) \in (\Sigma)^2 \setminus \Delta\,,\ \omega_{0,2}(z_1, z_2) = \omega_{0,2}(\sigma(z_1), \sigma(z_2)) \tag{2-23}$$

where $\Delta := \{(z, z) \in \Sigma^2, z \in \Sigma\}$.

One can use these symmetry properties to easily obtain a few equalities that we shall use repetitively in the following:

$$\forall\,(z_1,z_2)\in\Sigma^2\setminus\Delta\,,\ \int_{\sigma(z_1)}^{z_1}\omega_{0,2}(z_2,\cdot) = -\int_{\sigma(z_1)}^{z_1}\omega_{0,2}(\sigma(z_2),\cdot) \tag{2-24}$$

which implies, for any ramification point a (thus satisfying $\sigma(a)=a$),

$$\forall\,(z_1,z_2)\in\Sigma^2\setminus\Delta\,,\ \int_a^{z_1}\int_{\sigma(z_2)}^{z_2}\omega_{0,2} = -\int_a^{\sigma(z_1)}\int_{\sigma(z_2)}^{z_2}\omega_{0,2}. \tag{2-25}$$

3 PDE from TR

3.1 Definitions

Definition 3.1 (Perturbative partition function)**.** *Given an admissible initial data, one defines the perturbative partition function as a function of a formal parameter $\hbar$ and the initial data by*

$$Z^{pert}(\hbar,\mathbf{T},\boldsymbol{\epsilon}) := \exp\left(\sum_{h=0}^{\infty}\hbar^{2h-2}\omega_{h,0}(\mathbf{T},\boldsymbol{\epsilon})\right). \tag{3-1}$$

Definition 3.2 (Definition of $(F_{h,n})_{h\geq 0,n\geq 1}$ by integration of the correlators)**.** *For $n\geq 1$ and $h\geq 0$ such that $2h-2+n\geq 1$, let us define*

$$F_{h,n}(z_1,\dots,z_n) = \frac{1}{2^n}\int_{\sigma(z_1)}^{z_1}\cdots\int_{\sigma(z_n)}^{z_n}\omega_{h,n}$$

where one integrates each of the n variables along paths linking two Gallois conjugate points inside a fundamental domain cut out by the chosen symplectic basis $(\mathcal{A}_j,\mathcal{B}_j)_{1\leq j\leq g(\Sigma)}$.
For $(h,n)=(0,1)$, we define similarly

$$F_{0,1}(z) := \frac{1}{2}\int_{\sigma(z)}^{z}\omega_{0,1}.$$

Finally, for $(h,n)=(0,2)$, one cannot define $F_{0,2}$ in the exact same way since $\omega_{0,2}$ has poles on the diagonal Δ. One thus needs to regularize it by removing the polar part. Hence, we define

$$F_{0,2}(z_1,z_2) := \frac{1}{4}\int_{\sigma(z_1)}^{z_1}\int_{\sigma(z_2)}^{z_2}\omega_{0,2} - \frac{1}{2}\ln\left(x(z_1)-x(z_2)\right).$$

Exercice
The definition above seems to depend heavily on an integration path. However, it does only through the first few orders. To see that, prove that, for any $p\in\mathcal{R}$,

$$\operatorname*{Res}_{z\to p}\omega_{h,n+1}(z,z_1,\dots,z_n) = 0. \tag{3-2}$$

Definition 3.3 (Definition of the perturbative wave functions)**.** *We define:*

$$S_{-1}^{\pm\,pert}(x) \;:=\; \pm F_{0,1}(z(x))$$

$$
\begin{aligned}
S_0^{\pm\ pert}(x) &:= \frac{1}{2} F_{0,2}(z(x), z(x)) \\
\forall\, k \geq 1\,,\ S_k^{\pm\ pert}(x) &:= \sum_{\substack{h\geq 0, n\geq 1 \\ 2h-2+n=k}} \frac{(\pm 1)^n}{n!} F_{h,n}(z(x), \dots, z(x))
\end{aligned}
$$

where, for any $x \in \mathbb{P}^1$, we define $z(x) \in \Sigma$ as the unique point such that $x(z(\lambda)) = \lambda$ and $\omega_{0,1}(z(\lambda)) = \sqrt{\phi(\lambda)} d\lambda$. Remark that the $\pm$ sign refers to the choice of sheet for choosing a point in the pre-image of λ.
Eventually, we define the perturbative wave functions $\psi_\pm$ by:

$$
\psi_\pm(x, \hbar, \mathbf{T}, \boldsymbol{\epsilon}) := \exp\left(\sum_{k\geq -1} \hbar^k S_k^{\pm\ pert}(x) \right)
$$

3.2 Equation for the wave function

Let us now obtain a PDE for the wave function. For this purpose, let us move the integration contour for the TR formula. First, let us integrate the recursion formula along the path chosen above for $z_2, \dots, z_n$ and then move the integration contour for z getting contributions from the boundary of a chosen fundamental domain $\mathcal{D}$. One gets

$$
\begin{aligned}
\frac{1}{2\pi i} \oint_{z\in\delta\mathcal{D}} K(z_1, z) R_{h,n}(z, z_2, \dots, z_n) &= \sum_{a\in\mathcal{R}} \operatorname*{Res}_{z\to a} K(z_1, z) R_{h,n}(z, z_2, \dots, z_n) \\
&+ \sum_{i=1}^{n} \operatorname*{Res}_{z\to z_i, \sigma(z_i)} K(z_1, z) R_{h,n}(z, \dots, z_n)
\end{aligned} \tag{3-3}
$$

where, for $2h - 2 + n \geq 1$

$$
\begin{aligned}
R_{h,n}(z_1, \dots, z_n) &:= d_{u_1} d_{u_2} \Bigg[F_{h-1,n+1}(u_1, u_2, z_2, \dots, z_n) \\
&+ \sum^{\text{stable}}_{\substack{h_1+h_2=h \\ A\sqcup B=\{z_2,\dots,z_n\}}} F_{h_1,|A|+1}(u_1, A) F_{h_2,|B|+1}(u_2, B) \Bigg] \Bigg|_{u_1=z_1\,,\ u_2=\sigma(z_1)} \\
&+ \sum_{j=2}^{n} \frac{1}{2} \int_{\sigma(z_j)}^{z_j} \omega_{0,2}(z_1, \cdot)\ d_{\sigma(z_1)} F_{h,n-1}(\sigma(z_1), \mathbf{z}_{\{2,\dots,n\}\setminus\{j\}}) \\
&+ \sum_{j=2}^{n} \frac{1}{2} \int_{\sigma(z_j)}^{z_j} \omega_{0,2}(\sigma(z_1), \cdot)\ d_{z_1} F_{h,n-1}(z_1, \mathbf{z}_{\{2,\dots,n\}\setminus\{j\}})
\end{aligned} \tag{3-4}
$$

where d_u refers to the exterior derivative with respect to the variable u (which has nothing to do with a local coordinate),

$$
K(z_1, z) := \frac{\int_{\sigma(z)}^{z} \omega_{0,2}(z_1, \cdot)}{2(\omega_{0,1}(z) - \omega_{0,1}(\sigma(z)))} \tag{3-5}
$$

and $d_u d_v F_{0,2}(u, v) := \omega_{0,2}(u, v)$. In order to derive this expression, one has used that for $(h, n) \neq (0, 2)$

$$
d_{z_1} F_{h,n}(z_1, \dots, z_n) = \frac{1}{2^{n-1}} \int_{\sigma(z_2)}^{z_2} \cdots \int_{\sigma(z_n)}^{z_n} \omega_{h,n}(z_1, \cdot, \dots, \cdot). \tag{3-6}
$$

The first term of the right hand side is the recursive definition of $d_{z_1} F_{h,n}(z_1,\dots,z_n)$.

The other terms get only contributions from the poles of $\omega_{0,2}$. First of all, one can observe that

$$K(z,z_1)R_{h,n}(z,z_2,\dots,z_n) = K(\sigma(z),z_1)R_{h,n}(\sigma(z),z_2,\dots,z_n) \tag{3-7}$$

meaning that, for $i \in [\![1,n]\!]$,

$$\operatorname*{Res}_{z\to z_i,\sigma(z_i)} K(z_1,z)R_{g,n}(z,\dots,z_n) = 2 \operatorname*{Res}_{z\to z_i} K(z_1,z)R_{g,n}(z,\dots,z_n). \tag{3-8}$$

The same properties imply that

$$\begin{aligned} R_{h,n}(z_1,\dots,z_n) \;:=\; & -d_{u_1}d_{u_2}\Bigg[F_{h-1,n+1}(u_1,u_2,z_2,\dots,z_n) + \\ & + \sum_{\substack{h_1+h_2=h \\ A\sqcup B=\{z_2,\dots,z_n\}}}^{\text{stable}} F_{h_1,|A|+1}(u_1,A)F_{h_2,|B|+1}(u_2,B)\Bigg]\Bigg|_{u_1=u_2=z_1} \\ & - \sum_{j=2}^{n}\int_{\sigma(z_j)}^{z_j}\omega_{0,2}(z_1,\cdot)\; d_{z_1}F_{h,n-1}(z_1,\mathbf{z}_{\{2,\dots,n\}\setminus\{j\}}) \end{aligned} \tag{3-9}$$

One has a simple pole as $z \to z_1$ which gives

$$\begin{aligned} \operatorname*{Res}_{z\to z_1,\sigma(z_1)} K(z_1,z)R_{g,n}(z,\dots,z_n) = & \frac{1}{2\omega_{0,1}(z_1)} d_{u_1}d_{u_2}\Bigg[F_{h-1,n+1}(u_1,u_2,z_2,\dots,z_n) + \\ & + \sum_{\substack{h_1+h_2=h \\ A\sqcup B=\{z_2,\dots,z_n\}}}^{\text{stable}} F_{h_1,|A|+1}(u_1,A)F_{h_2,|B|+1}(u_2,B)\Bigg]\Bigg|_{u_1=u_2=z_1} \\ & + \sum_{j=2}^{n}\frac{\int_{\sigma(z_j)}^{z_j}\omega_{0,2}(z_1,\cdot)}{2\omega_{0,1}(z_1)}\; d_{z_1}F_{h,n-1}(z_1,\mathbf{z}_{\{2,\dots,n\}\setminus\{j\}}) \end{aligned} \tag{3-10}$$

where $\mathbf{z}_{\{2,\dots,n\}\setminus\{j\}} = \{z_2,\dots,z_n\}\setminus\{z_j\}$.

One can further compute

$$\begin{aligned} \operatorname*{Res}_{z\to z_j,\sigma(z_j)} K(z_1,z)R_{h,n}(z,\dots,z_n) \;=\; & 2\operatorname*{Res}_{z\to z_j} K(z_1,z)R_{h,n}(z,\dots,z_n) \\ =\; & -\frac{\int_{\sigma(z_j)}^{z_j}\omega_{0,2}(z_1,\cdot)}{2\omega_{0,1}(z_j)} d_{z_j}F_{h,n-1}(z_j,\mathbf{z}_{\{2,\dots,n\}\setminus\{j\}}) \end{aligned} \tag{3-11}$$

Combining all this, one gets

$$\begin{aligned}
&\frac{1}{2\pi i}\oint_{z\in\delta\mathcal{D}} K(z_1,z)R_{h,n}(z,\dots,z_n) = d_{z_1}F_{h,n}(z_1,\dots,z_n)\\
&+\sum_{j=2}^{n}\int_{\sigma(z_j)}^{z_j}\omega_{0,2}(z_1,\cdot)\left[\frac{d_{z_1}F_{h,n-1}(z_1,\mathbf{z}_{\{2,\dots,n\}\setminus\{j\}})}{2\omega_{0,1}(z_1)}-\frac{d_{z_j}F_{h,n-1}(z_j,\mathbf{z}_{\{2,\dots,n\}\setminus\{j\}})}{2\omega_{0,1}(z_j)}\right]\\
&+\frac{1}{2\omega_{0,1}(z_1)}d_{u_1}d_{u_2}\Bigg[F_{h-1,n+1}(u_1,u_2,z_2,\dots,z_n)\\
&+\sum^{\text{stable}}_{\substack{h_1+h_2=h\\ A\sqcup B=\{z_2,\dots,z_n\}}} F_{h_1,|A|+1}(u_1,\mathbf{z}_A)\,F_{h_2,|B|+1}(u_2,\mathbf{z}_B)\Bigg]_{u_1=u_2=z_1}.
\end{aligned}\tag{3-12}$$

By Riemann bilinear identity, the left hand side is a holomorphic form in z_1, concluding the proof.

Exercise Show that $\oint_{z\in\delta\mathcal{D}} K(z_1,z)R_{h,n}(z,\dots,z_n)$ is indeed holomorphic in z_1.

Exercise Show that

$$\begin{aligned}
&\frac{1}{2\pi i}\oint_{z\in\delta\mathcal{D}} K(z_1,z)R_{0,3}(z,z_2,z_3) = d_{z_1}F_{0,3}(z_1,z_2,z_3)+\frac{\int_{\sigma(z_2)}^{z_2}\omega_{0,2}(z_1,\cdot)\ \int_{\sigma(z_3)}^{z_3}\omega_{0,2}(z_1,\cdot)}{4\omega_{0,1}(z_1)}\\
&-\frac{\int_{\sigma(z_2)}^{z_2}\omega_{0,2}(z_1,\cdot)\ \int_{\sigma(z_3)}^{z_3}\omega_{0,2}(z_2,\cdot)}{4\omega_{0,1}(z_2)}-\frac{\int_{\sigma(z_3)}^{z_3}\omega_{0,2}(z_1,\cdot)\ \int_{\sigma(z_2)}^{z_2}\omega_{0,2}(z_3,\cdot)}{4\omega_{0,1}(z_3)}.
\end{aligned}\tag{3-13}$$

and

$$\frac{1}{2\pi i}\oint_{z\in\delta\mathcal{D}} K(z_1,z)R_{1,1}(z) = d_{z_1}F_{1,1}(z_1)-\frac{\omega_{0,2}(z_1,\sigma(z_1))}{2\omega_{0,1}(z_1)}.\tag{3-14}$$

An interesting property of holomorphic forms on a hyper-elliptic curve is that they can be expressed in terms of residue at the poles of ydx.

Exercise Show that, for any holomorphic differential ω on Σ_ϕ, one has

$$-2\frac{y(z_1)}{dx(z_1)}\omega(z_1)=\sum_{p\in\mathcal{P}}\operatorname*{Res}_{z_2\to p}\frac{\omega(z_2)\,y(z_2)}{x(z_2)-x(z_1)}$$

where $\mathcal{P}=\{\alpha_i^\pm,\infty^\pm\}$.

With this property, one can express the holomorphic functions in terms of residues at the poles of ydx and then express it in terms of variations with respect to the KP times. After evaluating at $z_1=z$ and some simple computations, summing over h and n leads to

$$\frac{\partial^2 S^{+\text{ pert}}_{m-1}(x)}{\partial x^2}+\sum_{m_1+m_2=m-1}\frac{\partial S^{+\text{ pert}}_{m_1}(x)}{\partial x}\,\frac{\partial S^{+\text{ pert}}_{m_2}(x)}{\partial x}=-\sum_{2h-2+n=m}\sum_{p\in\mathcal{P}}\operatorname*{Res}_{z'\to p}\frac{y(z')}{x(z')-x(z)}\frac{d_{z'}F_{h,n}(z',z,\dots,z)}{(n-1)!}.\tag{3-15}$$

Let us now interpret the right hand side in terms of the variational formulas.

Exercise. Prove that 3-15 can be recast into, for $m\geq 2$,

$$0 \;=\; \frac{\partial^2 S^{+\text{ pert}}_{m}(x)}{\partial x^2}+\sum_{m_1+m_2=m-1}\frac{\partial S^{+\text{ pert}}_{m_1}(x)}{\partial x}\,\frac{\partial S^{+\text{ pert}}_{m_2}(x)}{\partial x}$$

$$- \sum_{K=2}^{r_\infty - 2} U_{\infty,K}(x(z)) \frac{\partial S^{+\ \mathrm{pert}}_{m-1}(x)}{\partial T_{\infty,K}} - \sum_\nu \sum_{K=2}^{r_{b_\nu}+1} U_{b_\nu,K}(x(z)) \frac{\partial S^{+\ \mathrm{pert}}_{m-1}(x)}{\partial T_{b_\nu,K}}$$
$$- \sum_{k=0}^{\infty} \delta_{m+1,2k} \left[\sum_{K=2}^{r_\infty - 2} U_{\infty,K}(x(z)) \frac{\partial \omega_{k,0}}{\partial T_{\infty,K}} \sum_\nu \sum_{K=2}^{r_{b_\nu}+1} U_{b_\nu,K}(x(z)) \frac{\partial \omega_{k,0}}{\partial T_{b_\nu,K}} \right]. \tag{3-16}$$

One can obtain similar results for $m \in \{-1,0,1\}$ so that summing over m gives the following result.

Theorem 3.1. *The perturbative wave functions are solutions of the PDE*

$$\left[\hbar^2 \frac{\partial^2}{\partial x^2} - \hbar^2 \sum_{k \in K_\infty} U_{\infty,k}(x) \frac{\partial}{\partial T_{\infty,k}} - \hbar^2 \sum_{\nu=1}^{n} \sum_{k \in K_{X_\nu}} U_{\nu,k}(x) \frac{\partial}{\partial T_{\nu,k}} - H(x) \right] \psi_\pm(x,\hbar) = 0 \tag{3-17}$$

where

$$H(x) = \left[\hbar^2 \sum_{k \in K_\infty} U_{\infty,k}(x) \frac{\partial}{\partial T_{\infty,k}} + \hbar^2 \sum_{\nu=1}^{n} \sum_{k \in K_{X_\nu}} U_{\nu,k}(x) \frac{\partial}{\partial T_{\nu,k}} \right] \left[\log Z^{pert}(\hbar) - \hbar^{-2} \omega_{0,0} \right] + y(x)^2. \tag{3-18}$$

Example

In the preceding example, one gets

$$\left[\hbar^2 \frac{\partial^2}{\partial x^2} - \hbar^2 T_{\infty,4} \frac{\partial}{\partial T_{\infty,2}} - H(x) \right] \psi_\pm(x,\hbar) = 0 \tag{3-19}$$

where

$$H(x) = \hbar^2 T_{\infty,4} \frac{\partial}{\partial T_{\infty,2}} \left[\log Z^{\mathrm{pert}}(\hbar) - \hbar^{-2} \omega_{0,0} \right] + y(x)^2. \tag{3-20}$$

3.3 Monodromies

The perturbative wave functions $\psi_\pm$ satisfy the following properties.

- For $i \in [\![1,g]\!]$, the function $\psi_\pm(x,\hbar,\mathbf{T},\boldsymbol{\epsilon})$ has a formal monodromy along $\mathcal{A}_i$ given by

$$\psi_\pm(x,\hbar,\mathbf{T},\boldsymbol{\epsilon}) \mapsto e^{\pm 2\pi i \frac{\epsilon_i}{\hbar}} \psi_\pm(x,\hbar,\mathbf{T},\boldsymbol{\epsilon}). \tag{3-21}$$

- For $i \in [\![1,g]\!]$, the function $\psi_\pm(x,\hbar,\mathbf{T},\boldsymbol{\epsilon})$ has a formal monodromy along $\mathcal{B}_i$ given by

$$\psi_\pm(x,\hbar,\mathbf{T},\boldsymbol{\epsilon}) \mapsto \frac{Z^{\mathrm{pert}}(\hbar,\mathbf{T},\boldsymbol{\epsilon} \pm \hbar\, \mathbf{e}_i)}{Z^{\mathrm{pert}}(\hbar,\mathbf{T},\boldsymbol{\epsilon})} \psi_\pm(x,\hbar,\mathbf{T},\boldsymbol{\epsilon} \pm \hbar\, \mathbf{e}_i) \tag{3-22}$$

  where $\mathbf{e}_i \in \mathbb{C}^g$ is the vector with the i^{th} component equal to 1 and all others vanishing.

Proof. Reminding that the $\mathcal{A}$-periods of the $\omega_{h,n}$ are vanishing unless for $(h,n) = (0,1)$ where

$$\forall\, j \in [\![1,g]\!] \ : \ \epsilon_j = \oint_{\mathcal{A}_j} \omega_{0,1}, \tag{3-23}$$

one immediately gets the first claim.

The second claim follows a simple computation similar to the one for Painlevé 1 written in [?]. The analytic continuation of the perturbative wave function along the cycle $\mathcal{B}_j$ reads

$$\exp\left[\sum_{h\geq 0}\sum_{n\geq 1}\frac{\hbar^{2h-2}(\pm\hbar)^n}{n!\,2^n}\sum_{n_1+n_2=n}\binom{n}{n_1}2^{n_1}\overbrace{\oint_{\mathcal{B}_j}\cdots\oint_{\mathcal{B}_j}}^{n_1}\overbrace{\int_{\sigma(z)}^{z}\cdots\int_{\sigma(z)}^{z}}^{n_2}\omega_{h,n}\right]$$
$$=\exp\left[\sum_{h\geq 0}\sum_{n\geq 1}\hbar^{2h-2}(\pm\hbar)^n\sum_{n_1+n_2=n}\frac{1}{2^{n_2}n_1!\,n_2!}\frac{\partial^{n_1}}{\partial\epsilon_j^{n_1}}\overbrace{\int_{\sigma(z)}^{z}\cdots\int_{\sigma(z)}^{z}}^{n_2}\omega_{h,n_2}\right]. \tag{3-24}$$

Factoring out the terms with $n_2=0$ gives

$$\exp\left[\sum_{h\geq 0}\sum_{n\geq 1}\frac{\hbar^{2h-2}(\pm\hbar)^n}{n!}\frac{\partial^n}{\partial\epsilon_j^n}\omega_{h,0}\right]\exp\left[\sum_{n_1\geq 0}\frac{(\pm\hbar)^{n_1}}{n_1!}\frac{\partial^{n_1}}{\partial\epsilon_j^{n_1}}\sum_{h\geq 0}\sum_{n_2\geq 1}\hbar^{2h-2}(\pm\hbar)^{n_2}\frac{1}{2^{n_2}\,n_2!}\overbrace{\int_{\sigma(z)}^{z}\cdots\int_{\sigma(z)}^{z}}^{n_2}\omega_{h,n_2}\right] \tag{3-25}$$

leading to the result.

□

4 Non-perturbative quantities

At this stage we have perturbative wave functions $\psi_\pm(x,\hbar,\mathbf{T},\boldsymbol{\epsilon})$ defined as

$$\psi_\pm(x,\hbar,\mathbf{T},\boldsymbol{\epsilon}) = \exp\left(\sum_{k=-1}^{\infty}\hbar^k S_k^\pm(x,\mathbf{T},\boldsymbol{\epsilon})\right) \tag{4-1}$$

where $S_k^\pm$ are directly defined from TR

$$S_k^\pm(x,\mathbf{T},\boldsymbol{\epsilon}) = \sum_{h,n\geq 0\,,\,2h-2+n=k}\frac{(\pm 1)^n}{2^n n!}\int_{\sigma(z)}^{z}\cdots\int_{\sigma(z)}^{z}\omega_{h,n} \tag{4-2}$$

with some regularizations for $S_{-1}^\pm$ and $S_0^\pm$ satisfy some KZ equations:

$$\left[\hbar^2\frac{\partial^2}{\partial x^2}-\hbar^2\sum_{k=2}^{r_\infty-2}U_{\infty,k}(x)\frac{\partial}{\partial T_{\infty,k}}-\hbar^2\sum_{\nu=1}^{n}\sum_{k=2}^{r_\nu+1}U_{\nu,k}(x)\frac{\partial}{\partial T_{\nu,k}}-H(x)\right]\psi_\pm = 0 \tag{4-3}$$

where

$$H(x) = y^2+\left[\hbar^2\sum_{k=2}^{r_\infty-2}U_{\infty,k}(x)\frac{\partial}{\partial T_{\infty,k}}+\hbar^2\sum_{\nu=1}^{n}\sum_{k=2}^{r_\nu+1}U_{\nu,k}(x)\frac{\partial}{\partial T_{\nu,k}}\right]\left(\log Z^{\text{pert}}-h^{-2}\omega_{0,0}\right) \tag{4-4}$$

We recall that

$$U_{\nu,k}(x) \;=\; (k-1)\sum_{l=k-1}^{r_\nu}T_{\nu,l}(x-X_\nu)^{k-l-2}\;,\;\forall\,\nu\in[\![1,n]\!]\;,,\;k\in[\![2,r_\nu+1]\!]$$

$$\begin{aligned} U_{\infty,k}(x) &= (k-1)\sum_{l=k+2}^{r_\infty} T_{\infty,l}x^{l-k-2} \text{ , } \forall\, k\in [\![2,r_\infty-2]\!] \text{ and } n_\infty=0 \\ U_{\infty,k}(x) &= \left(k-\frac{3}{2}\right)\sum_{l=k+2}^{r_\infty} T_{\infty,l}x^{l-k-2} \text{ , } \forall\, k\in [\![2,r_\infty-2]\!] \text{ and } n_\infty=1 \end{aligned} \tag{4-5}$$

and $K_\nu = [\![2, r_\nu + 1]\!]$ for $\nu \in [\![1, n]\!]$ and $K_\infty = [\![2, r_\infty - 2]\!]$.

Terms in green would correspond to the naive quantization (i.e. replacing y by $\hbar\partial_x$). Terms in blue are non-trivial $\hbar$-corrections having only pole singularities at poles $\mathcal{P} = \{X_1, \dots, X_n\} \cup \{\infty\}$ of the initial classical spectral curve . Terms in red are differential terms relatively to KP (spectral) times, making KZ equations a set of PDEs rather than ODEs.

The second important feature is that the perturbative wave functions do not have good monodromies around $(\mathcal{A}_i, \mathcal{B}_i)_{i=1}^g$ cycles:

$$\begin{aligned} \psi_\pm(x,\hbar,\mathbf{T},\boldsymbol{\epsilon}) &\overset{\mathcal{A}_i}{\to} e^{\frac{\pm 2i\pi\epsilon_i}{\hbar}}\psi_\pm(x,\hbar,\mathbf{T},\boldsymbol{\epsilon}) \\ \psi_\pm(x,\hbar,\mathbf{T},\boldsymbol{\epsilon}) &\overset{\mathcal{B}_i}{\to} \frac{Z^{\text{pert}}(\hbar,\mathbf{T},\boldsymbol{\epsilon}+\hbar\mathbf{e}_i)}{Z^{\text{pert}}(\hbar,\mathbf{T},\boldsymbol{\epsilon})}\psi_\pm(x,\hbar,\mathbf{T},\boldsymbol{\epsilon}+\hbar\mathbf{e}_i) \end{aligned} \tag{4-6}$$

In order to obtain better monodromies, we need to "sum over filling fractions", i.e. take formal discrete Fourier transform in order to absorb the shift appearing in the $\mathcal{B}$-cycles.

Definition 4.1 (Non-perturbative wave functions)**.** *We define the non-perturbative partition function and wave functions by*

$$\begin{aligned} Z(\hbar,\boldsymbol{T},\boldsymbol{\epsilon},\boldsymbol{\rho}) &= \sum_{\mathbf{k}\in\mathbb{Z}^g} e^{\frac{2i\pi}{\hbar}\sum_{j=1}^g k_j\rho_j} Z^{pert}(\hbar,\mathbf{T},\boldsymbol{\epsilon}+\hbar\mathbf{k}) \\ \Psi_\pm(x,\hbar,\boldsymbol{T},\boldsymbol{\epsilon},\boldsymbol{\rho}) &= \frac{1}{Z(\hbar,\boldsymbol{T},\boldsymbol{\epsilon},\boldsymbol{\rho})}\sum_{\mathbf{k}\in\mathbb{Z}^g} e^{\frac{2i\pi}{\hbar}\sum_{j=1}^g k_j\rho_j} Z^{pert}(\hbar,\mathbf{T},\boldsymbol{\epsilon}+\hbar\mathbf{k})\psi_\pm(x,\hbar,\mathbf{T},\boldsymbol{\epsilon}+\hbar\mathbf{k}) \end{aligned} \tag{4-7}$$

$\boldsymbol{\rho}$ is a given vector. After exchanging the order of the summations $\sum_{\mathbf{k}\in\mathbb{Z}^g}\sum_{k=-1}^{\infty} \to \sum_{k=-1}^{\infty}\sum_{\mathbf{k}\in\mathbb{Z}^g}$, we no longer have formal WKB series but rather trans-series in $\hbar$ of the following form.

$$Z(\hbar,\mathbf{T},\boldsymbol{\epsilon},\boldsymbol{\rho}) = Z^{\text{pert}}(\hbar,\mathbf{T},\boldsymbol{\epsilon})\sum_{m=0}^{\infty}\hbar^m\Theta_m(\hbar,\mathbf{T},\boldsymbol{\epsilon},\boldsymbol{\rho}) \tag{4-8}$$

where Θ_m are finite linear combination of derivatives of Riemann θ-functions:

$$\Theta_m(\hbar,\mathbf{T},\boldsymbol{\epsilon},\boldsymbol{\rho}) = \frac{\partial^m\theta(\mathbf{v},\tau)}{\partial v_{i_1}\dots\partial v_{i_m}}_{|\mathbf{v}=\frac{\boldsymbol{\phi}+\boldsymbol{\rho}}{\hbar}} \tag{4-9}$$

with $\phi_j = \partial_{\epsilon_j}\omega_{0,0}$ and $\tau_{i,j} = \frac{\partial^2\omega_{0,0}}{\partial\epsilon_i\partial\epsilon_j}$. All future equalities are do be understood as formal identifications of the coefficients regarding the trans-monomial expansion.

The main advantage of this formal Fourier transform is that it cures the monodromy issue.

Theorem 4.1. *The non-perturbative wave functions have good monodromies:*

$$\begin{array}{rcl}\Psi_\pm(x,\hbar,\boldsymbol{T},\boldsymbol{\epsilon},\boldsymbol{\rho}) & \overset{\mathcal{A}_i}{\to} & e^{\frac{\pm 2i\pi\epsilon_i}{\hbar}}\psi_\pm(x,\hbar,\boldsymbol{T},\boldsymbol{\epsilon},\boldsymbol{\rho})\\ \Psi_\pm(x,\hbar,\boldsymbol{T},\boldsymbol{\epsilon},\boldsymbol{\rho}) & \overset{\mathcal{B}_i}{\to} & e^{\frac{-\pm 2i\pi\rho_i}{\hbar}}\psi_\pm(x,\hbar,\boldsymbol{T},\boldsymbol{\epsilon},\boldsymbol{\rho})\end{array} \tag{4-10}$$

The proof is straightforward using the monodromies of the perturbative wave functions (Easy exercise for those interested).

The main advantage to deal with wave functions having good monodromy properties is that it will allow to have rational functions of x as soon as we will have quantities with no essential singularities at ∞ later on. Another advantage of the discrete Fourier transform is that it behaves well with the KZ equations that are linear equations. Therefore, former KZ equations may immediately be adapted for non-perturbative wave functions:

Theorem 4.2 (KZ equation for non-perturbative wave functions)**.** *The non-perturbative wave functions satisfies*

$$\left[\hbar^2\frac{\partial^2}{\partial x^2} - \hbar^2\sum_{k=2}^{r_\infty-2}U_{\infty,k}(x)\frac{\partial}{\partial T_{\infty,k}} - \hbar^2\sum_{\nu=1}^{n}\sum_{k=2}^{r_\nu+1}U_{\nu,k}(x)\frac{\partial}{\partial T_{\nu,k}} - \mathcal{H}(x)\right]\Psi_\pm = 0 \tag{4-11}$$

with

$$\mathcal{H}(x) = y^2 + \left[\hbar^2\sum_{k=2}^{r_\infty-2}U_{\infty,k}(x)\frac{\partial}{\partial T_{\infty,k}} + \hbar^2\sum_{\nu=1}^{n}\sum_{k=2}^{r_\nu+1}U_{\nu,k}(x)\frac{\partial}{\partial T_{\nu,k}}\right]\left(\log Z - h^{-2}\omega_{0,0}\right) \tag{4-12}$$

In other words, we only replaced perturbative wave functions and partition function by their non-perturbative counterpart.

5 Quantum curve

5.1 Expression of the quantum curve

So far (4-11) is still a PDE rather than an ODE. We need to find a way to turn it into some ODEs with controlled rational functions. In order to do it, we look at the Wronskians and some associated quantities:

Definition 5.1 (Wronskian and associated quantities)**.** *We define for all $p \in [\![1,n]\!] \cup \{\infty\}$ and $k \in K_p$:*

$$\begin{array}{rcl} W(x) & = & \hbar\left(\Psi_-\partial_x\Psi_+ - \Psi_+\partial_x\Psi_-\right)\\ W_{T_{p,k}}(x) & = & \hbar\left(\Psi_-\partial_{T_{p,k}}\Psi_+ - \Psi_+\partial_{T_{p,k}}\Psi_-\right)\\ R_{p,k}(x) & = & \dfrac{W_{T_{p,k}}(x)}{W(x)}\\ Q_{p,k}(x) & = & \dfrac{\hbar^2}{W(x)}\left((\partial_x\Psi_+)(\partial_{T_{p,k}}\Psi_-) - (\partial_x\Psi_-)(\partial_{T_{p,k}}\Psi_+)\right)\end{array} \tag{5-1}$$

and

$$\begin{array}{rcl} R(x) & = & \displaystyle\sum_{p\in\mathcal{P}}\sum_{p\in K_p}U_{p,k}(x)R_{p,k}(x)\\ Q(x) & = & \displaystyle\sum_{p\in\mathcal{P}}\sum_{p\in K_p}U_{p,k}(x)Q_{p,k}(x)\end{array} \tag{5-2}$$

$R_{p,k}$ and $Q_{p,k}$ are defined so that they transform a derivative relatively to $T_{p,k}$ into a derivative relatively to x:

$$\begin{pmatrix} \Psi_+ & \Psi_- \\ \hbar\partial_{T_{p,k}}\Psi_+ & \hbar\partial_{T_{p,k}}\Psi_- \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ Q_{p,k}(x) & R_{p,k}(x) \end{pmatrix} \begin{pmatrix} \Psi_+ & \Psi_- \\ \hbar\partial_x\Psi_+ & \hbar\partial_x\Psi_- \end{pmatrix} \tag{5-3}$$

For example, it is trivial to check that

$$Q_{p,k}\Psi_\pm + R_{p,k}\hbar\partial_x\Psi_\pm = \hbar\partial_{T_{p,k}}\Psi_\pm \tag{5-4}$$

Using these quantities, the KZ equations turns into an ODE:

Theorem 5.1 (Quantum curve)**.** *We have*

$$\left[\hbar^2\frac{\partial^2}{\partial x^2} - \hbar^2 R(x)\frac{\partial}{\partial x} - \hbar Q(x) - \mathcal{H}(x)\right]\Psi_\pm = 0 \tag{5-5}$$

The proof is trivial because the PDE part of the KZ equation reads:

$$\left[\hbar^2\sum_{k=2}^{r_\infty-2} U_{\infty,k}(x)\frac{\partial}{\partial T_{\infty,k}} - \hbar^2\sum_{\nu=1}^{n}\sum_{k=2}^{r_\nu+1} U_{\nu,k}(x)\frac{\partial}{\partial T_{\nu,k}}\right]\Psi_\pm = Q(x)\Psi_\pm + R(x)\hbar\partial_x\Psi_\pm \tag{5-6}$$

If the quantum curve looks interesting, it does not contain any interesting information if one cannot control the singularity structure of the coefficients involved. This requires to study the singularity structure of $R(x)$ and $Q(x)$ (since $\mathcal{H}(x)$ has a x-dependence given by $(U_{p,k}(x))_{p\in[\![1,n]\!]\cup\{\infty\},k\in K_p}$ that only have poles in $\mathcal{P}$). We first get that they are rational functions of x because

- Wronskians $W(x)$ and $W_{T_{p,k}}(x)$ are rational functions of x since they have no monodromies and no essential singularities at ∞.
- Consequently $R_{p,k}(x)$ are rational functions of x
- Alternative expression (left in exercise) for $Q_{p,k}$ given by

  $$Q_{p,k}(x) = \frac{\hbar}{2}\sum_{p\in\mathcal{P}}\sum_{k\in K_p}\frac{U_{p,k}(x)\left(\frac{\partial W(x)}{\partial T_{p,k}} - \frac{\partial W_{T_{p,k}}(x)}{\partial x}\right)}{W(x)} \tag{5-7}$$

  implies that $Q_{p,k}$ is also a rational function of x.
- Since $U_{p,k}(x)$ are rational functions, $R(x)$ and $Q(x)$ are finally rational functions.

5.2 Location of the poles

By definition, $\Psi_\pm$ are constructed by TR so that they may be singular at critical values and at $x\in\mathcal{P}$. This implies that the Wronskians and all associated quantities have a priori the same singularity structure. What we need is to exclude singularities at the critical values. This can be achieved using the following results

1. $R_{p,k}$ and $Q_{p,k}$ have no pole at the ramification points and thus $R(x)$ and $Q(x)$ have no pole at ramification points
   This result is very technical and no proof will be given here. Let us quickly say that it follows from the identification of $\partial_x^2 \partial_{T_{p,k}} \Psi_\pm = \partial_{T_{p,k}} \partial_x^2 \Psi_\pm$ and identification of the coefficients in the trans-series. The main tool is to observe that derivation relatively to x preserve the $\hbar$-grading while derivation relatively to $T_{p,k}$ decreases the $\hbar$-grading by 1. The proof follows by contradiction at leading order in $\hbar$ and then from a technical induction for higher orders in the $\hbar$-trans-series.

2. We have $R(x) = \frac{1}{W(x)} \frac{\partial}{\partial x} W(x)$. This is a classical result of second order linear ODE. It also follows directly from $\partial_x W(x) = \hbar \left((\partial_x^2 \Psi_+) \Psi_- - \Psi_+ (\partial_x^2 \Psi_-) \right)$ and using the ODE (5-5) to replace the second derivative.

3. $W(x)$ and $W_{T_{p,k}}(x)$ may only have poles at $x \in \mathcal{P}$. Indeed, we know that the only possible poles are $x \in \mathcal{P}$ and critical values. However a pole at a critical value would imply that $R(x) = \frac{1}{W(x)} \frac{\partial}{\partial x} W(x)$ would be singular at that point which contradicts the previous result. Identity $W_{T_{p,k}}(x) = R_{p,k}(x) W(x)$ provides the result for $W_{T_{p,k}}(x)$.

5.3 Control of the order of the poles

We know that $R(x)$ and $Q(x)$ are rational functions with only poles in $\mathcal{P}$ so that the locations of the poles in the coefficients of the quantum curve are indeed the same as the initial classical spectral curve. However in order to have a better understanding of the quantum curve, we need to control the order of the poles and hopefully prove that they remain similar to the initial classical spectral curve. This can be achieved by the study of the asymptotics of the wave functions (which follows from the TR correlators). Indeed, we know that $S_\pm(x) = \log \Psi_\pm$:

$$\begin{aligned}
S_\pm(x) &= \mp \hbar^{-1} \sum_{k=2}^{r_\nu} \frac{T_{\nu,k}}{k-1} (x - X_\nu)^{-(k-1)} \pm \hbar^{-1} T_{\nu,1} \log(x - X_\nu) + \sum_{k=0}^{\infty} A^\pm_{\nu,k} (x - X_\nu)^k \\
S_\pm(x) &= \mp \hbar^{-1} \sum_{k=2}^{r_\nu} \frac{T_{\nu,k}}{k-1} x^{k-1} \mp \hbar^{-1} T_{\nu,1} \log x - \frac{1}{2} \log x + \sum_{k=0}^{\infty} A^\pm_{\infty,k} x^{-k} \,,\ (n_\infty = 0) \\
S_\pm(x) &= \mp \hbar^{-1} \sum_{k=2}^{r_\nu} \frac{T_{\nu,k}}{2k-3} x^{\frac{2k-3}{2}} \mp \hbar^{-1} T_{\nu,1} \log x - \frac{1}{4} \log x + \sum_{k=0}^{\infty} A^\pm_{\infty,k} x^{-\frac{k}{2}} \,,\ (n_\infty = 1)
\end{aligned} \tag{5-8}$$

Inserting this into the Wronskians gives:
At $x = X_\nu$:

$$\begin{aligned}
W(x) &\overset{x \to X_\nu}{\sim} \frac{2 T_{\nu, r_\nu}}{(x - X_\nu)^{r_\nu}} c_\nu \\
W_{T_{\nu,k}}(x) &\overset{x \to X_\nu}{\sim} - \frac{2 c_\nu}{(k-1)(x - X_\nu)^{k-1}} \\
W_{T_{\nu',k}}(x) &\overset{x \to X_\nu}{\sim} O(1)
\end{aligned} \tag{5-9}$$

At $x = \infty$ when $n_\infty = 0$:

$$W(x) \overset{x \to \infty}{\sim} -2 T_{\infty, r_\infty} c_\infty x^{r_\infty - 3} \ (n_\nu = 0)$$

$$\begin{aligned} W_{T_{\nu,k}}(x) &= O\left(x^{-1}\right) \\ W_{T_{\infty,k}}(x) &= -\frac{2}{k-1} c_\infty x^{k-2} \end{aligned} \tag{5-10}$$

At $x = \infty$ when $n_\infty = 1$:

$$\begin{aligned} W(x) &\overset{x\to\infty}{\sim} -T_{\infty,r_\infty} c_\infty x^{r_\infty - 3}\ (n_\nu = 0) \\ W_{T_{\nu,k}}(x) &= O\left(x^{-\frac{1}{2}}\right) \\ W_{T_{\infty,k}}(x) &= -\frac{2}{2k-3} c_\infty x^{k-2} \end{aligned} \tag{5-11}$$

Using $g = r_\infty + \sum_{\nu=1}^{n} r_\nu - 3$, we finally obtain some formal expression for $W(x)$, $R(x)$ and $Q(x)$:

Theorem 5.2 (Pole structure of the quantum curve)**.** *We have*

$$W(x) = w\, \frac{\prod_{j=1}^{g} (x - q_j)}{\prod_{\nu=1}^{n} (x - X_\nu)^{r_\nu}} \tag{5-12}$$

so that

$$\begin{aligned} R(x) &= \sum_{j=1}^{g} \frac{1}{x - q_j} - \sum_{\nu=1}^{n} \frac{r_\nu}{x - X_\nu} \\ Q(x) &= \sum_{i=1}^{g} \frac{p_i}{x - q_i} + \sum_{k=0}^{r_\infty - 4} Q_{\infty,k} x^k + \sum_{\nu=1}^{n} \sum_{k=1}^{r_\nu + 1} \frac{Q_{\nu,k}}{(x - X_\nu)^k} \end{aligned} \tag{5-13}$$

with $p_i = -\hbar \frac{\partial \log \Psi_\pm}{\partial x}_{|x=q_i}$.

At this stage, the x-structure of the quantum curve is fully determined and is the same as the initial classical spectral curve (even the order of the poles). Pairs $(q_i, p_i)_{i=1}^{g}$ appearing in the expressions shall be used as Darboux coordinates in relation with integrable systems. Moreover, inserting the definition $p_i = -\hbar \frac{\partial \log \Psi_\pm}{\partial x}_{|x=q_i}$ into the quantum curve (5-5) gives:

$$p_i^2 = \mathcal{H}(q_i) - \hbar p_i \left[\sum_{j\neq i} \frac{1}{q_i - q_j} - \sum_{\nu=1}^{n} \frac{r_\nu}{q_i - X_\nu} \right] + \hbar \left(\sum_{j\neq i} \frac{p_j}{q_i - q_j} + \sum_{k=0}^{r_\infty - 4} Q_{\infty,k} q_i^k + \sum_{\nu=1}^{n} \sum_{k=1}^{r_\nu+1} \frac{Q_{\nu,k}}{(q_i - X_\nu)^k} \right) \tag{5-14}$$

which provides g relations between all the coefficients.

6 Sl_2 connection with no apparent singularities

As for any linear differential equations of degree $d = 2$, the quantum curve is equivalent to a companion-like matrix (size $d \times d$) differential system of degree 1:

$$\hbar \frac{\partial}{\partial x} \begin{pmatrix} \Psi_+ & \Psi_- \\ \hbar \frac{\partial}{\partial x} \Psi_+ & \hbar \frac{\partial}{\partial x} \Psi_- \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \mathcal{H}(x) + \hbar Q(x) & R(x) \end{pmatrix} \begin{pmatrix} \Psi_+ & \Psi_- \\ \hbar \frac{\partial}{\partial x} \Psi_+ & \hbar \frac{\partial}{\partial x} \Psi_- \end{pmatrix}$$

$$\overset{\text{def}}{=} \hat{L}(x) \begin{pmatrix} \Psi_+ & \Psi_- \\ \hbar\frac{\partial}{\partial x}\Psi_+ & \hbar\frac{\partial}{\partial x}\Psi_- \end{pmatrix} \tag{6-1}$$

However, there are two issues with the companion-like matrix form. First, $\hat{L}$ does not belong to Sl_2 while we would expect a quantization procedure that should preserve this initial symmetry. Moreover, the Lax matrix $\hat{L}$ exhibits some **apparent singularities** at $x \in \{q_i\}_{i=1}^g$, i.e. some entries have singularities that are not singularities of the wave functions. This happens because the choice $(\Psi_+, \hbar\partial_x\Psi_+)$ is not a good choice of basis. Indeed, one may choose any linear combinations (possibly depending on x) of Ψ_+ and $\hbar\partial_x\Psi_+$ as a possible second line, and such a choice would modify the Lax matrix $\hat{L}$ without changing the quantum curve itself. In order to remove these apparent singularities, we perform a change of basis:

$$\begin{pmatrix} \Psi_\pm \\ \hbar\partial_x\Psi_\pm \end{pmatrix} \to \begin{pmatrix} \frac{1}{W(x)}(P(x)\Psi_\pm + \hbar\partial_x\Psi_\pm) \\ \Psi_\pm \end{pmatrix} \tag{6-2}$$

for some (at this time) unknown rational function $P(x)$. This corresponds to a gauge change:

$$\Psi(x) = \begin{pmatrix} \frac{1}{W(x)}(P(x)\Psi_+ + \hbar\partial_x\Psi_+) & \frac{1}{W(x)}(P(x)\Psi_- + \hbar\partial_x\Psi_-) \\ \Psi_+ & \Psi_- \end{pmatrix} = \begin{pmatrix} \frac{P(x)}{W(x)} & \frac{\hbar}{W(x)} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \Psi_\pm \\ \hbar\partial_x\Psi_\pm \end{pmatrix} \overset{\text{def}}{=} G(x)\hat{\Psi}(x) \tag{6-3}$$

In this gauge, the differential system turns into

$$\hbar\partial_x\Psi(x) = \begin{pmatrix} P(x) & M(x) \\ W(x) & -P(x) \end{pmatrix} \Psi(x) \overset{\text{def}}{=} L(x)\Psi(x) \tag{6-4}$$

with

$$M(x) = \frac{1}{W(x)} \left[\mathcal{H}(x) - P^2(x) + \hbar Q(x) + \hbar P'(x) - P(x)\frac{W'(x)}{W(x)} \right] \tag{6-5}$$

Equation (5-14) implies that M is regular at $x = q_i$ if and only if $P(q_i) = p_i$. However, we need to be careful because we need that the pole of $M(x)$ remains as low as possible. And this is not obvious because of the $P^2(x)$ term. Even with this condition, some degree of freedom to choose $P(x)$ remain. In order to fix them it is conventional to ask for the leading order at infinity of $L(x)$ to be of standard form $\begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}$ for $n_\infty = 0$ or $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ for $n_\infty = 1$ (plus an additional condition for the subleading term that should be of the form $\begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix}$ in this degenerate case). This can be achieved by setting:

For $n_\infty = 0$:

$$P(x) = \frac{T_{\infty,r_\infty}x^{g+1} + \left(T_{\infty,r_\infty-1} + \frac{\hbar}{2}\right)x^g + \sum_{l=0}^{g-1}\alpha_l x^l}{\prod_{\nu=1}^{n}(x - X_\nu)^{r_{b_\nu}}} \tag{6-6}$$

with the g coefficients $(\alpha_l)_{l=1}^g$ determined by interpolation using $P(q_i) = p_i$ for all $i \in [\![1, g]\!]$.

For $n_\infty = 1$:

$$P(x) = \frac{\sum_{l=0}^{g-1}\alpha_l x^l}{\prod_{\nu=1}^{n}(x - X_\nu)^{r_{b_\nu}}} \tag{6-7}$$

with the g coefficients $(\alpha_l)_{l=1}^{g}$ determined by interpolation using $P(q_i) = p_i$ for all $i \in [\![1, g]\!]$.

The proof is straightforward by direct computation of $L(x)$ and in particular computing the orders of the poles of each entry.

The existence of a Sl_2 connection allows to define a $\hbar$-deformed spectral curve by taking the characteristic polynomial of $L(x)$. It reads:

$$0 = \det\left(ydxI_2 - L(x)dx\right) = y^2(dx)^2 - \phi_\hbar \tag{6-8}$$

with

$$\phi_\hbar = \left(\hbar P'(x) - \hbar\frac{W'(x)}{W(x)}P(x) + \hbar Q(x) + \mathcal{H}(x)\right)(dx)^2 \tag{6-9}$$

Note that the y^2 coefficients in $\mathcal{H}(x)$ is the only surviving in the limit $\hbar \to 0$, thus recovering the initial spectral curve. All other terms are $\hbar$-corrections that are not obvious from the naive quantization perspective.

7 Lax pairs

In addition to the Lax matrix $L(x)$, we may also look at spectral times derivatives of the wave functions. However spectral times are not well-suited for the study, and it is better to trade them to isomonodromic times $(t_{\nu,l})_{l=1}^{r_\nu-1}$, $(t_{\infty,l})_{l=1}^{r_\infty-3}$. These are obtained by the conditions:

$$\begin{aligned}
\forall\, l \in [\![1, r_\nu - 1]\!] \,:\, \frac{\partial}{\partial t_{\nu,l}} &= \sum_{k=2}^{r_\nu-l+1} T_{\nu,k+l-1}\frac{\partial}{\partial T_{\nu,k}} \\
\forall\, l \in [\![1, r_\infty - 3]\!] \,:\, \frac{\partial}{\partial t_{\infty,l}} &= \sum_{k=2}^{r_\nu-l+1} (k-1)T_{\infty,k+l+1}\frac{\partial}{\partial T_{\infty,k}} \,,\ \text{if } n_\infty = 0 \\
\forall\, l \in [\![1, r_\infty - 3]\!] \,:\, \frac{\partial}{\partial t_{\infty,l}} &= \sum_{k=2}^{r_\nu-l+1} \frac{2k-3}{2}T_{\infty,k+l+1}\frac{\partial}{\partial T_{\infty,k}} \,,\ \text{if } n_\infty = 1
\end{aligned} \tag{7-1}$$

This provides a one-to-one map between spectral times and isomonodromic times. Moreover, they are defined in such a way that the time differential terms in the KZ equations read:

$$\begin{aligned}
\sum_{k\in K_p} U_{\infty,k}(x)\frac{\partial}{\partial T_{\infty,k}} &= \sum_{l=1}^{r_\infty-3} x^{l-1}\frac{\partial}{\partial t_{\infty,l}} \\
\sum_{k\in K_p} U_{\nu,k}(x)\frac{\partial}{\partial T_{\nu,k}} &= \sum_{l=1}^{r_\nu-1}(x - X_\nu)^{-(l+1)}\frac{\partial}{\partial t_{\nu,l}}
\end{aligned} \tag{7-2}$$

These isomonodromic times allow to construct Lax pairs (one of the main building blocks of integrable systems) for all $p \in \{\infty\} \cup [\![1, n]\!]$ and $k \in K_\nu$:

$$\begin{aligned}
\hbar\frac{\partial \Psi}{\partial x} &= L(x)\Psi \\
\hbar\frac{\partial \Psi}{\partial t_{p,k}} &= A_{t_{p,k}}(x)\Psi
\end{aligned} \tag{7-3}$$

for which the compatibility equations read

$$\hbar\partial_{t_{p,k}} L - \hbar\frac{\partial A_{t_{p,k}}}{\partial x} + \left[L, A_{t_{p,k}}\right] = 0 \tag{7-4}$$

In order to determine these matrices $A_{t_{p,k}}$, there are two options:

- Analyze the asymptotics of the Wronskian $W_{t_{p,k}}$ at $x = X_{\nu'}$ and $x = \infty$ and obtain the general dependence in x of the matrix $A_{t_{p,k}}$. The remaining unknown coefficients are then given by solving the compatibility equations (7-4).
- Define $L_{t_{\infty,k}} = \left[x^{-k}L(x)\right]_{\infty,+}$ and $L_{t_{\nu,k}} = \left[(x-X_\nu)^k L(x)\right]_{X_\nu,-}$. If $\partial_x L_{t_{p,k}} = \delta_{t_{p,k}} L$ (explicit derivation relatively to $t_{p,k}$ only) then $A_{t_{p,k}}(x) = L_{t_{p,k}}$.

Compatibility equations (7-4) provides all $\frac{\partial T_{p',k'}}{\partial t_{p,k}}$ as well as all $\frac{\partial q_i}{\partial t_{p,k}}$ and $\frac{\partial p_i}{\partial t_{p,k}}$. In general, the evolution (q_i, p_i) relatively to isomonodromic times is Hamiltonian, i.e. there exists some Hamiltonians $\text{Ham}_{p,k}$ such that for all $i \in [\![1, g]\!]$:

$$\begin{aligned} \hbar\frac{\partial q_i}{\partial t_{p,k}} &= -\frac{\partial \text{Ham}_{p,k}}{\partial p_i} \\ \hbar\frac{\partial p_i}{\partial t_{p,k}} &= \frac{\partial \text{Ham}_{p,k}}{\partial q_i} \end{aligned} \tag{7-5}$$

8 Application to Painlevé 2

The Painlevé 2 system corresponds to the case $n = 0$, $r_\infty = 4$ and $n_\infty = 0$, i.e.

$$y^2 = H_{\infty,4}x^4 + H_{\infty,3}x^3 + H_{\infty,2}x^2 + H_{\infty,1}x + H_{\infty,0} \tag{8-1}$$

with (infinity is not ramified)

$$ydx = \pm\left(T_{\infty,4}x^2 + T_{\infty,3}x + T_{\infty,2} + \frac{T_{\infty,1}}{x}\right)dx + O\left(\frac{dx}{x^2}\right) \tag{8-2}$$

The coefficients $(H_{\infty,k})_{k=1}^4$ are Casimirs and are expressed in terms of the spectral times:

$$H_{\infty,4} = T_{\infty,4}^2,\ H_{\infty,3} = 2T_{\infty,4}T_{\infty,3},\ H_{\infty,1} = 2(T_{\infty,1}T_{\infty,4} + T_{\infty,2}T_{\infty,3}) \tag{8-3}$$

We have:

$$U_{\infty,2}(x) = T_{\infty,4} \tag{8-4}$$

The genus of the curve is 1 so that we may write (p, q) instead of (p_1, q_1) to lighten notations. The coefficients of the quantum curve reads:

$$\begin{aligned} W(x) &= w(x-q) \\ R(x) &= \frac{1}{x-q} \\ Q(x) &= \frac{p}{x-q} + Q_{\infty,0} \\ \mathcal{H}(x) &= S_4(x) + H_{\infty,0} + \hbar^2 T_{\infty,4}\alpha \end{aligned} \tag{8-5}$$

where

$$S_4(x) = T_{\infty,4}^2 x^4 + 2T_{\infty,4}T_{\infty,3}x^3 + (T_{\infty,3}^2 + 2T_{\infty,2}T_{\infty,4})x^2 + 2(T_{\infty,1}T_{\infty,4} + T_{\infty,2}T_{\infty,3})x \tag{8-6}$$

and

$$\alpha = \frac{\partial}{\partial T_{\infty,4}}(\log Z - \hbar^{-2}\omega_{0,0}) \tag{8-7}$$

Conditions (5-14) is equivalent to $p^2 = \mathcal{H}(q) + \hbar Q_{\infty,0}$ which provides

$$H_{\infty,0} + \hbar^2 T_{\infty,4}\alpha + \hbar Q_{\infty,0} = p^2 - S_4(q) \tag{8-8}$$

Therefore the quantum curve reads

$$\left[\hbar^2 \frac{\partial^2}{\partial x^2} - \frac{\hbar^2}{x-q}\frac{\partial}{\partial x} - \frac{\hbar p}{x-q} + (S_4(q) - S_4(x) - p^2)\right]\Psi_\pm = 0 \tag{8-9}$$

In particular it only depends on $(T_{\infty,4}, T_{\infty,3}, T_{\infty,2}, T_{\infty,1}, p, q)$ but no longer on α, $Q_{\infty,0}$ or $H_{\infty,0}$.

The construction of the Sl_2 connection implies the gauge transformation $G(x) = \begin{pmatrix} \frac{P(x)}{W(x)} & \frac{\hbar}{W(x)} \\ 1 & 0 \end{pmatrix}$ and provides a Lax matrix:

$$L(x) = \begin{pmatrix} P(x) & M(x) \\ W(x) & -P(x) \end{pmatrix} \tag{8-10}$$

with

$$\begin{aligned} W(x) &= w(x-q) \\ P(x) &= T_{\infty,4}x^2 + T_{\infty,3}x + p - T_{\infty,4}q^2 - T_{\infty,3}q \\ M(x) &= \frac{1}{w}\Big[2T_{\infty,4}(T_{\infty,4}q^2 + T_{\infty,3}q + T_{\infty,2} - p)x \\ &\quad +2T_{\infty,4}^2 q^3 + 4T_{\infty,4}T_{\infty,3}q^2 + 2(T_{\infty,3}^2 + T_{\infty,2}T_{\infty,4} - T_{\infty,4}p)q \\ &\quad +2T_{\infty,1}T_{\infty,4} + 2T_{\infty,2}T_{\infty,3} - 2T_{\infty,3}p + \hbar T_{\infty,4}\Big] \end{aligned} \tag{8-11}$$

The associated $\hbar$-deformed spectral curve is

$$\phi_\hbar = (S_4(x) - S_4(q) + p^2)(dx)^2 \tag{8-12}$$

We define

$$L_t = \left[x^{-1}L\right]_{\infty,+} = \begin{pmatrix} T_{\infty,4}x + T_{\infty,3} & \frac{2T_{\infty,4}}{w}\left[T_{\infty,4}q^2 + T_{\infty,3}q + T_{\infty,2} - p\right] \\ w & -(T_{\infty,4}x + T_{\infty,3}) \end{pmatrix} \tag{8-13}$$

Compatibility of the equations:

$$\begin{aligned} \hbar\frac{\partial}{\partial x}\Psi &= L\Psi \\ \hbar\frac{\partial}{\partial t}\Psi &= L_t\Psi \end{aligned} \tag{8-14}$$

i.e. $\hbar\partial_t L - \hbar\partial_x L_t + [L, L_t] = 0$, provides:

$$\frac{\partial T_{\infty,4}}{\partial t} = \frac{\partial T_{\infty,3}}{\partial t} = \frac{\partial T_{\infty,1}}{\partial t} = 0\,,\; \frac{\partial T_{\infty,2}}{\partial t} = T_{\infty,4} \tag{8-15}$$

and

$$\begin{aligned}
\hbar\frac{\partial w}{\partial t} &= 2wqT_{\infty,4} \\
\hbar\frac{\partial q}{\partial t} &= -2p \\
\hbar\frac{\partial p}{\partial t} &= -4T_{\infty,4}^2 q^3 - 6T_{\infty,4}T_{\infty,3}q^2 - 2(T_{\infty,3}^2 + T_{\infty,2}T_{\infty,4})q - 2(T_{\infty,1}T_{\infty,4} + T_{\infty,2}T_{\infty,3}) - \hbar T_{\infty,4}
\end{aligned} \tag{8-16}$$

Thus, we may choose $T_{\infty,2} = T_{\infty,4}\,t$ so that

$$L_t(x) = \begin{pmatrix} T_{\infty,4}x + T_{\infty,3} & \frac{2T_{\infty,4}}{w}\left(-p + T_{\infty,4}q^2 + T_{\infty,3}q + T_{\infty,4}t - p\right) \\ w & -(T_{\infty,4}x + T_{\infty,3}) \end{pmatrix} \tag{8-17}$$

The evolution of (p, q) is Hamiltonian:

$$\hbar\frac{\partial q}{\partial t} = -\frac{\partial \mathrm{Ham}}{\partial p}\,,\; \hbar\frac{\partial p}{\partial t} = \frac{\partial \mathrm{Ham}}{\partial q} \tag{8-18}$$

with

$$\mathrm{Ham}(p,q,t) = p^2 - T_{\infty,4}^2 q^4 - 2T_{\infty,4}T_{\infty,3}q^3 - (T_{\infty,3}^2 + 2T_{\infty,4}^2 t)q^2 - 2(T_{\infty,1}T_{\infty,4} + T_{\infty,4}T_{\infty,3}t)q \tag{8-19}$$

It provides a Painlevé 2 like equation for q:

$$\hbar^2\frac{\partial^2}{\partial t^2}q = 8T_{\infty,4}^2 q^3 + 12T_{\infty,4}T_{\infty,3}q^2 + 4(T_{\infty,3}^2 + 2T_{\infty,4}^2 t)q + 4T_{\infty,4}(T_{\infty,1} + tT_{\infty,3})q \tag{8-20}$$

Quantities may be rescaled to obtain a standard Painlevé 2 equation:

$$\hbar^2\frac{\partial^2}{\partial s^2}u = 2u^3 + tu - \theta \tag{8-21}$$