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Goodness-of-fit testing strategies from indirect observations

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We consider in this paper a goodness-of-fit testing problem in a density framework. In particular, we deal with an error-in-variables model where each new incoming observation is gathered with a random independent error. It is well known that in such a situation, we are faced with an inverse (deconvolution) problem. Nevertheless, following recent results in the Gaussian white noise model, we prove that using procedures containing a deconvolution step is not always necessary.

Keywords: goodness-of-fit testing; inverse problems

AMS Subject Classifications: 63G05; 62K20

1. Introduction

In this paper, we consider the goodness-of-fit testing problem of an unknown density f from i.i.d random variables $X_1, \ldots, X_n$ observed with measurement errors. Actually let $\mathbf{Y} = (Y_1, \ldots, Y_n)$ be a sample of n i.i.d. observations where

$$Y_i = X_i + \epsilon_i, \quad \forall i \in \{1, \ldots, n\}. \tag{1}$$

The ϵ_i's play the role of measurement errors, which are supposed to be centred and associated to a known density g. In this context, the density of the Y_i corresponds to the convolution product between f and g, denoted by $f * g$.

Inference on the density f is a deconvolution problem and has been widely studied in the statistical literature. Estimation issues have received a great amount of attention. We mention for instance (Fan 1991) for a first study of the optimal rates of convergence related to deconvolution kernel estimators of f or (Delaigle and Gijbels 2004) for a practical calibration of such estimators. More generally, we mention (Meister 2009) for an introduction to nonparametric deconvolution problems or (Engl, Hanke, and Neubauer 1996) and (Cavalier 2008) where precise descriptions on inverse problems and related statistical issues are provided.

Our aim in this paper is not to provide an estimator of the density f but rather to assess optimality of procedures able to decide whether the unknown density f is equal to f_0 where f_0 denotes some given density. For instance, the X_i may correspond to the measures of a quantity of interest (e.g. weight, cholesterol rate, etc.) of n individuals. The purpose is then to compare the distribution

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of this sample to the one of a benchmark population (e.g. diabetic or healthy people). From a statistical point of view, providing an answer to this question corresponds to test

$$H_0^{\mathrm{IP}} : f = f_0, \text{ against } H_1^{\mathrm{IP}} : f \neq f_0, \quad f \in \mathcal{H},$$

where $\mathcal{H}$ denotes a subset of $L^2(\mathbb{R})$. The exponent *IP* traduces the fact that we deal with a contaminated sample with density $f * g$. Hence, inference on f leads to the inverse problem of testing a density in a deconvolution model. This issue has been widely tackled in the statistical literature. We refer for instance to Bissantz, Claeskens, Holzmann, and Munk (2009) or Holzmann, Bissantz, and Munk (2007) in a slightly different setting. Moreover, the papers (Butucea 2007; Butucea and Tsybakov 2007; Butucea, Matias, and Pouet 2009) provide a complete study of this testing problem in many situations, including minimax rates of testing. However, as for any inverse problem, such procedures require to invert the convolution operator and are thus based on a regularisation scheme of the observations (Engl et al. 1996 or Bissantz, Hohage, Munk, and Ryumgaart 2007 for more details). From a practical point of view, such methods are sometimes difficult to implement since one needs to compute a deconvolution kernel in a density framework or the singular value decomposition for more general operators. It is thus natural to ask whether this regularisation step is necessary.

Using reasonable assumptions (which will be made precise later on) on the density g, the convolution operator is injective. Hence, in some sense, the assertions '$f = f_0$' and '$g * f = g * f_0$' are equivalent. In a goodness-of-fit framework, two different points of view may be alternatively considered. The statistician may either perform an *indirect* test

$$H_0^{\mathrm{IP}} : f = f_0, \text{ against } H_1^{\mathrm{IP}} : f \neq f_0, \quad f \in \mathcal{H}, \tag{2}$$

or may use a direct approach directly on the data, without a deconvolution step. In that case he considers the *direct* test

$$H_0^{\mathrm{DP}} : g * f = g * f_0, \text{ against } H_1^{\mathrm{DP}} : g * f \neq g * f_0, \quad g * f \in \mathcal{H}' \tag{3}$$

for some $\mathcal{H}' \subset L^2(\mathbb{R})$.

As proved in Laurent, Loubes, and Marteau (2011) in a Gaussian white noise setting, a surprising outcome is that the inversion is not always necessary in a goodness-of-fit purpose. Indeed, well calibrated direct procedures (i.e. dealing with the observation without regularisation step) can do as well as classical tests based on an inversion of the operator. The aim of this paper is to extend this discussion to the deconvolution framework and to investigate the favourable cases. Actually, depending on the difficulty of the inverse problem defined by the regularity of the density of the noise g, and on the set of assumptions on the density f to be detected, we prove that the specific treatment devoted to an inverse problem which includes an underlying inversion of the operator, may worsen the detection strategy.

This paper is organised as follows. In Section 2, we formalise the testing problem and present the existing results in the literature. In Section 3, we prove that direct testing procedures are often minimax for the inverse testing problem and we construct a test minimax for H_0^{IP} that will fail on H_0^{DP} in some particular situation. Most of the proofs are gathered in Section 5.

2. Two possible settings

Recall that we consider the deconvolution model

$$Y_i = X_i + \epsilon_i, \quad i = 1, \dots, n, \tag{4}$$

where the X_i's and ϵ_i's are i.i.d real random variables with respective density functions (with respect to Lebesgue measure) f and g. Since the X_i's and ϵ_i's are independent, the Y_i's are associated to the density $f * g$ where $*$ denotes the convolution operator. Our aim is to test $f = f_0$ or equivalently $f * g = f_0 * g$ where f_0 denotes a benchmark density function.

We consider the following general smoothness assumption on the densities given by the decay of their Fourier transform. We assume that both f_0 and f belong to the space $\mathcal{E}(h, L)$ defined as

$$\mathcal{E}(h, L) = \left\{ f : \int_{\mathbb{R}} |\Phi_f(t)|^2 |h(t)|^2 \, dt < 2\pi L \right\},$$

where Φ_f denotes the Fourier transform of a function f, L a positive constant and h an increasing function. Remark that in the particular case where $h(t) = t^s$ for some $s > 0$, $\mathcal{E}(h, L)$ corresponds to the Sobolev space

$$W(s, L) = \left\{ f \in \mathcal{C}^s, \int |\Phi_f(t)|^2 |t|^{2s} \, dt < 2\pi L \right\},$$

while, when $h(t) = \exp(\alpha |t|^r)$ for all $t \in \mathbb{R}$, the set $\mathcal{E}(h, L)$ is identified as the space of supersmooth densities functions

$$S(\alpha, r, L) = \left\{ f \in \mathcal{C}^\infty, \int |\Phi_f(t)|^2 \exp(2\alpha |t|^r) \, dt < 2\pi L \right\}.$$

The development of testing strategies is strongly associated to the notion of separation rates corresponding to the minimax point of view developed in the series of papers due to Ingster (1993). Given the hypothesis H_0, let ρ_n be a decreasing sequence, and H_1 an alternative of the form

$$H_1 : f \neq f_0, \quad f \in \mathcal{H}(h, L, \rho_n),$$

where

$$\mathcal{H}(h, L, \rho_n) = \{\nu \in \mathcal{E}(h, L), \ \|\nu - f_0\| > \rho_n\},$$

and $\| \cdot \|$ denotes the L_2-norm. For a testing procedure Δ_n, we write by convention $\Delta_n = 0$ if we accept H_0 and $\Delta_n = 1$ if we reject the assumption. For a given $0 < \xi < 1$, a testing procedure $\Delta_n^\star$ is said to attain the testing rate ρ_n over $\mathcal{E}(h, L)$ if there exists $C^\star = C^\star(\xi)$ such that

$$\limsup_{n \to +\infty} \left[P_{f_0}(\Delta_n^\star = 1) + \sup_{f \in \mathcal{H}(h, L, C\rho_n)} P_f(\Delta_n^\star = 0) \right] \leq \xi, \tag{5}$$

for all $C > C^\star$. In other words, the procedure Δ_n is able to detect that $f \neq f_0$ with prescribed asymptotic errors as soon as $\|f - f_0\| > C\rho_n$. The rate ρ_n is then said to be the minimax rate of testing on $\mathcal{E}(h, L)$ if there exists $C_\star = C_\star(\xi) > 0$ such that

$$\liminf_{n \to +\infty} \inf_{\Delta_n} \left[P_{f_0}(\Delta_n = 1) + \sup_{\mathcal{H}(h, L, C\rho_n)} P_f(\Delta_n = 0) \right] \geq \xi,$$

for all $0 < C < C_\star$. The infimum in the previous formula is taken over all testing procedures Δ_n. The quality of the testing procedure then relies on the quantity ρ_n, and of course, for a given level and power of test, the smaller the ρ_n, the better.

In the noise-free case (i.e. $\epsilon_i = 0$ for all $i = 1 \cdots n$), minimax rates of testing are described for instance in Ingster (1993). Similar results in the Gaussian sequence space model are obtained in Baraud (2002) or Baraud, Huet, and Laurent (2003) where adaptation with respect to the

Table 1. Minimax rate of testing in the error-in-variables model (4). The term v_* denotes the solution of $2\alpha v_*^{-r} + 2\gamma v_*^{-\beta} = \log(n) - \log\log(n)$.

f/g	$\vert\Phi_g(u)\vert \sim \vert u\vert^{-\beta}$	$\vert\Phi_g(u)\vert \sim \exp^{-\gamma\vert u\vert^{\beta}}$
$W(s,L)$	$n^{-2s/(4s+4\beta+1)}$	$(\log n)^{-s/\beta}$
$S(\alpha,r,L)$	$\log(n)^{(\beta+1/4)/r} n^{-1/2}$	$\exp(-\alpha/v_*^r)$ $(r<\beta)$
		$n^{-1/2} v_*^{(\beta-1)-}$ $\exp(-\gamma/h_*^{\beta})$ $(r>\beta)$

smoothness is discussed. In the inverse problem framework, two different cases are commonly discussed, which are associated to the rate of decay of the Fourier transform of g. One may alternatively assume that

$$0 < |\Phi_g(t)| = \mathcal{O}(|t|^{-\beta}), \quad \text{as } t \to +\infty,$$

which yields a mildly ill-posed inverse problem or that

$$0 < |\Phi_g(t)| = \mathcal{O}(\exp(-\gamma|t|^{\beta})), \quad \text{as } t \to +\infty,$$

which corresponds to a severely ill-posed inverse problem. Note that imposing that the Fourier transform of the noise vanishes nowhere induces that the convolution operator is injective. This warrants the identifiability of the statistical model. In the specific error-in-variables model (4), which is at the heart of the present paper, the most complete reference is up to our knowledge (Butucea 2007). For the sake of convenience, we reproduce the obtained result in Table 1. We mention that similar rates are available in a Gaussian white noise model: see Laurent, Loubes, and Marteau (2012) or Ingster, Sapatinas, and Suslina (2012).

Here, goodness-of-fit tests in a deconvolution model can be built using the two different settings which amount to test either '$f = f_0$', which will be referred to as the *inverse problem setting*, or '$g * f = g * f_0$' which will be named the *direct problem setting*. This problematic has been discussed for the first time in Holzmann et al. (2007). Namely, a *direct procedure* will correspond to a test based on the blurred observations and designed for the detection of differences between $f * g$ and $f_0 * g$ while an *inverse problem procedure* deals with a test based on an estimator of f. More precisely for h and $\tilde{h}$ two functions that will control the respective regularities of f and $f * g$, we define the following set of hypotheses

$$H_0^{\mathrm{DP}} : g * f = g * f_0, \text{ against } H_1^{\mathrm{DP}} : g * f \neq g * f_0, \quad g * f \in \mathcal{H}_1(\tilde{h}, L, \rho_n^{\mathrm{DP}}), \tag{6}$$

while the second one can be considered as an inverse problem, where we test

$$H_0^{\mathrm{IP}} : f = f_0, \text{ against } H_1^{\mathrm{IP}} : f \neq f_0, \quad f \in \mathcal{H}_1(h, L, \rho_n^{\mathrm{IP}}). \tag{7}$$

In this paper, we prove that although the convolution operator is one-to-one, the two testing problems are not equivalent. In particular, we will see that a testing procedure minimax for Equation (6) is also minimax for the problem (7) with a particular choice for $\tilde{h}$. We also investigate cases where the converse is not true: we will exhibit optimal deconvolution testing procedures that are not minimax in the direct case.

3. Comparison of the inverse and direct approaches

The aim of this section is to compare the testing problems (6) and (7). Our aim is to describe the cases where direct procedures may be minimax for Equation (7) and inverse tests are minimax for Equation (6).

Throughout this section, we will consider a noise having a non null Fourier transform, i.e.

$$\Phi_g(t) \neq 0, \quad \forall t \in \mathbb{R}.$$

This assumption is quite common in the statistical literature, in particular when considering nonparametric deconvolution problems. Actually it allows for instance the construction of a deconvolution kernel, which gives rise to a natural estimator by simply removing the part of the signal due to the noise by a mere division.

More specifically, we deal with the following hypothesis.

ASSUMPTION A1 *There exists a positive constant C_l such that for all $t \in \mathbb{R}$*

$$|\Phi_g(t)| \geq C_l |t|^{-\beta},$$

for mildly ill-posed problems or

$$|\Phi_g(t)| \geq C_l \exp(-\gamma |t|^{\beta}),$$

for severely ill-posed inverse problems.

3.1. *Performances of direct methods*

In this part, we prove that for a well-chosen 'regularity' $\tilde{h}$, (direct) testing procedures that are minimax for Equation (6) are also minimax for Equation (7).

In order to provide a satisfying comparison of the problems (6) and (7), one needs first to have a precise idea of the regularity of $f * g$ provided $f \in \mathcal{E}(\nu, L)$ for some h. Since $\Phi_{f*g} = \Phi_f \cdot \Phi_g$,

$$f \in \mathcal{E}(h, L) \Leftrightarrow \int |\Phi_{f*g}(t)|^2 |\Phi_g(t)|^{-2} |h(t)|^2 \,\mathrm{d}t < L \Leftrightarrow f * g \in \mathcal{E}(\Phi_g^{-1} h, L).$$

In particular, if $|h(t)| = |t|^s$, i.e. $\mathcal{E}(h, L) = W(s, L)$, we get that

$$f \in W(s, L) \Leftrightarrow f * g \in W(s + \beta, L')$$

for some L' provided $|\Phi_g(t)| = \mathcal{O}(|t|^{-\beta})$ and $|\Phi_g(t)| \neq 0$ for all $t \in \mathbb{R}$.

Since the assertions $g * f = g * f_0$ and $f = f_0$ are equivalent provided Assumption A1 is satisfied, it seems clear that both hypotheses H_0^{DP} and H_0^{IP} are equivalent. Consequently, the first kind errors are the same for a given test Δ_n whatever the chosen null hypothesis. The alternatives are therefore at the heart of the comparison between the two testing problems.

The following theorem establishes that in the cases considered in this paper, a direct test performs as well as specific procedures tailored to handle the particular deconvolution issue.

THEOREM 3.1 *Let $\xi \in]0, 1[$ be a fixed level of test. We assume that h is polynomially (Sobolev) or exponentially increasing (super-smooth). The related inverse problem is mildly or severely ill-posed and we assume that Assumption A1 is satisfied. Then every test minimax for H_0^{DP} on $\mathcal{E}(|\Phi_g(\cdot)|^{-1} h, L)$ is also minimax for H_0^{IP} on $\mathcal{E}(h, L)$.*

The proof of this theorem relies on the following lemma which gives a control on $\|\nu * g\|$ from $\|\nu\|$ provided the function of interest ν belongs to $\mathcal{E}(h, L)$ for some h and L.

LEMMA 3.2 *Let γ_n be a positive sequence such that $\gamma_n \to 0$ as $n \to +\infty$. The following embedding holds*:

$$\{\nu \in \mathcal{E}(h,L),\ \|\nu\|^2 \geq \gamma_n\} \subset \{\nu \in \mathcal{E}(h,L),\ \|\nu * g\|^2 \geq \mu_n\},$$

where

$$\mu_n = \tfrac{1}{2}|\Phi_g(\tau_n)|^2\gamma_n,$$

and τ_n is such that $L|h(\tau_n)|^{-2} \leq \gamma_n/2$.

Proof Let $t \in \mathbb{R}^+$ which will be chosen later and $\nu \in \mathcal{E}(h,L)$ such that $\|\nu\|^2 \geq \gamma_n$. For the sake of convenience, we will also suppose that the function $t \mapsto |\Phi_g(t)|$ is non increasing. Then

$$\begin{aligned}\|\nu * g\|^2 &= \int |\Phi_\nu(u)|^2|\Phi_g(u)|^2\,\mathrm{d}u \\ &\geq \int_{|u|\leq t} |\Phi_\nu(u)|^2|\Phi_g(u)|^2\,\mathrm{d}u, \\ &\geq |\Phi_g(t)|^2 \int_{|u|\leq t} |\Phi_\nu(u)|^2\,\mathrm{d}u, \\ &\geq |\Phi_g(t)|^2\left(\|\nu\|^2 - \int_{|u|>t} |\Phi_\nu(u)|^2\,\mathrm{d}u\right).\end{aligned}$$

Note that the assumption about the decay of Φ_g could be relaxed, up to some more complicated algebra in the proofs, replacing $|\Phi_g(t)|^2$ by $\inf_{|u|\leq t}|\Phi_\nu(u)|^2$ in the following. Since $\nu \in \mathcal{E}(h,L)$

$$\int_{|u|>t} |\Phi_\nu(u)|^2\,\mathrm{d}u \leq |h(t)|^{-2}\int_{|u|>t} |h(u)|^2|\Phi_\nu(u)|^2\,\mathrm{d}u \leq L|h(t)|^{-2}.$$

Hence

$$\|\nu * g\|^2 \geq |\Phi_g(t)|^2(\gamma_n - L|h(t)|^{-2}).$$

We conclude the proof choosing $t = \tau_n$ such that $L|h(\tau_n)|^{-2} \leq \gamma_n/2$. ■

Applying Lemma 3.2, we get that for all testing procedures Δ_n,

$$\sup_{f\in\mathcal{E}(h,L),\|f-f_0\|^2\geq\gamma_n} P_f(\Delta_n = 0) \leq \sup_{f\in\mathcal{E}(h,L),\|(f-f_0)*g\|^2\geq\mu_n} P_f(\Delta_n = 0).$$

Consequently, provided γ_n is of order $(\rho_n^{\mathrm{IP}})^2$ and μ_n is of order $(\rho_n^{\mathrm{DP}})^2$, a test minimax for H_0^{DP} will be also minimax for H_0^{IP}. The whole proof of Theorem 3.1 is postponed to the Appendix.

The consequence of the previous result is that classical testing procedures designed for the direct problem (6) can be directly used on our noisy observations without performing any regularisation step. No need for a special treatment is required since direct testing procedures are proved to be minimax. Hence, the following direct testing procedure allows, following the results presented in Theorem 3.1, to tackle the testing issue of a wide variety of inverse problems.

Let K denote a kernel i.e. a function $K : \mathbb{R} \to \mathbb{R}$ such that

$$\int_{\mathbb{R}} K(x)\,\mathrm{d}x = 1 \quad \text{and} \quad \int_{\mathbb{R}} K^2(x)\,\mathrm{d}x < \infty.$$

Define the test statistics as

$$\tilde{T}_{n,\lambda} = \frac{1}{n(n-1)} \sum_{k \neq j} \langle K_\lambda(\cdot - Y_k) - f_0, K_\lambda(\cdot - Y_j) - f_0 \rangle, \tag{8}$$

where $K_\lambda := \lambda^{-1} K_g(\cdot/\lambda)$ for some bandwidth λ. Remark that the term $\tilde{T}_{n,\lambda}$ is an estimator of $\|(f - f_0) * g\|^2$. Then, we set

$$\tilde{\Delta}_{n,\lambda} = \mathbf{1}_{\{|\tilde{T}_{n,\lambda}| > \tilde{C} u_n^2\}}, \tag{9}$$

where $\tilde{C}$ denotes a sufficiently large constant and $(u_n)_{n \in \mathbb{N}}$ a positive real sequence. This decision rule means that we reject H_0^{DP} if the estimator of $\|(f - f_0) * g\|^2$ takes large values. Otherwise, if $\tilde{T}_{n,\lambda}$ is small w.r.t. the sequence $(u_n)_{n \in \mathbb{N}}$, we accept H_0^{DP}. This kind of estimator has been widely studied in the literature and appropriate values for λ and u_n have been proposed in several different setting. In particular, it has been proved (we refer to Ingster 1993 or Butucea 2007 among others) that the test $\tilde{\Delta}_{n,\lambda}$ is minimax for H_0^{DP} on both Sobolev and super-smooth function classes.

3.2. *Limitation of inverse approaches*

In this section, we prove that the direct and inverse testing problems for goodness-of-fit in a deconvolution model are not equivalent in the sense that a testing procedure optimal for the inverse problem (7) may fail when dealing with the direct observation model (6). This property is essentially due to the fact that a testing procedure designed for an inverse problem (i.e. containing a regularisation step) is calibrated in order to take into account a possible large variance for the errors. But the drawback is that such a procedure may be too conservative or unable to detect alternatives with small intensity.

The key of the proof is to exhibit a testing procedure minimax in the inverse setting (7) that may fail in certain situations for the direct testing issue. We deal in this paper with a procedure proposed by Butucea (2007) in the error-in-variable model. In a first time, we need to construct a preliminary estimator of f using a kernel type estimator.

First, consider K a kernel. The associated deconvolution kernel K_g is defined as

$$\Phi_{K_g} = \frac{\Phi_K(t)}{\Phi_g(t/\lambda)} \tag{10}$$

for some bandwidth λ. We refer to Meister (2009) for more details on the construction of such quantities and discussion on the Assumption A1. The testing procedure is then based on an estimation of $\|f - f_0\|^2$. A candidate is given by

$$T_n^\star = \frac{1}{n(n-1)} \sum_{k \neq j} \langle K_{g,\lambda}(\cdot - Y_k) - f_0, K_{g,\lambda}(\cdot - Y_j) - f_0 \rangle,$$

where $K_{g,\lambda} := \lambda^{-1} K_g(\cdot/\lambda)$. Then, we set

$$\Delta_n^\star = \mathbf{1}_{\{|T_n^\star| > C^\star t_n^2\}}, \tag{11}$$

where $C^\star$ and t_n are positive parameters that should be properly chosen. For an appropriate choice of t_n and h_n (see, e.g. Butucea 2007) this test is known to be minimax for H_0^{IP} on the different smoothness classes considered in this paper. In particular,

$$\lim_{n\to+\infty}\left[P_{f_0}(\Delta_n^\star=1)+\sup_{f\in\mathcal{H}(h,L,C\rho_n)}P_f(\Delta_n^\star=0)\right]\le\xi.$$

Nevertheless, we prove that this procedure fails in the direct case, i.e. that there exists $f_{1,n}\in\mathcal{E}(\Phi_g(\cdot)^{-1}h,L)$ such that

$$\|(f_{1,n}-f_0)*g\|\ge\mathcal{C}_1\rho_n(\mathcal{E}(\Phi_g(\cdot)^{-1}h,L))\text{ but }\lim_{n\to+\infty}[P_{f_0}(\Delta_n^\star=1)+P_{f_{1,n}}(\Delta_n^\star=0)]>\xi,$$

whatever the value of $\mathcal{C}_1$.

Theorem 3.3 *Let $0<\xi<1$ be a test level and a function h to be fixed. Assume that Assumption A1 holds. For both mildly and severely ill-posed problems, there exist level-α tests minimax for H_0^{IP} on $\mathcal{E}(h,L)$ but not for H_0^{DP} on $\mathcal{E}(h\cdot\Phi_g^{-1},L')$.*

Clearly, both testing problems (6) and (7) are not equivalent. In some sense, direct methods are more robust with respect to measurement errors. Inverse methods are too conservative in the direct case since they are designed for high variance testing problems.

3.3. *Conclusion*

The main contribution of this paper is that the regularisation is not necessary in the minimax sense when considering goodness-of-fit testing problems. This allows to propose direct procedures that are not based on a deconvolution scheme, i.e. that require the inversion of the underlying operator.

This point of view may allow to consider specific problems for which this deconvolution might be difficult or even impossible. In this sense, the case of uniform measurement errors is of first interest. Indeed, in this specific case, $\Phi_g(t)=0$ for some $t\in\mathbb{R}$ which does not allow the construction of Equation (10). Nevertheless, Lemma 3.2 can not be directly generalised in this situation or at least up to some additional constraint on the signal of interest (for instance source conditions, see Engl et al. 1996 or Loubes and Rivoirard 2009 for instance). Yet a lower bound would be necessary for the testing problem in such a case, which still remains an open problem.

4. Proof of the main results

In this section, the quantities C and c will denote generic constants that may vary from line to line, and even in the same line. Given two real sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$, we write $a_n\sim b_n$ if there exists c,C positive constants such that $c\le a_n/b_n\le C$ for all $n\in\mathbb{N}$.

4.1. *Proof of Theorem 3.1*

For the sake of convenience, we recall in Table 2 the different testing rates in the direct case, according to the regularity of the considered densities.

Table 2. Minimax rate of testing for the problem (6).

$f * g \in W(s', L)$	$n^{-2s'/(4s'+1)}$
$f * g \in S(\alpha', r, L)$	$\log(n)^{1/4r} n^{-1/2}$

Let $\xi \in]0,1[$ be fixed and Δ_n a testing procedure minimax for H_0^{DP} on $\mathcal{E}(|\Phi_g^{-1}|h, L)$. Then, according to Equation (5)

$$\lim_{n\to+\infty}\left[P_{f_0}(\Delta_n = 1) + \sup_{f*g\in\mathcal{H}(|\Phi_g|^{-1}h,L,C\rho_n^{\mathrm{DP}})} P_f(\Delta_n = 0)\right] \le \xi,$$

for some positive constant C. The proof is then based on the following scheme. We first remark that $f \in \mathcal{E}(h, L) \Rightarrow g * f \in \mathcal{E}(|\Phi_g|^{-1}h, L')$ for some L'. Hence, using Lemma 3.2, we get

$$\{f \in \mathcal{E}(h, L),\ \|f - f_0\|^2 \ge C_2(\rho_n^{\mathrm{IP}})^2\} \subset \{g * f \in \mathcal{E}(|\Phi_g|^{-1}h, L'),$$
$$\|(f - f_0) * g\|^2 \ge C_2(|\Phi_g(\tau_n)|\rho_n^{\mathrm{IP}})^2\}.$$

Consequently, the test Δ_n will be minimax as soon as

$$|\Phi_g(\tau_n)|\rho_n^{\mathrm{IP}} > C\rho_n^{\mathrm{DP}}, \quad \text{where } L|h(\tau_n)|^{-2} \le 1/2(\rho_n^{\mathrm{IP}})^2, \tag{12}$$

and C denotes a positive constant independent of n. We consider 4 different cases according to the possible regularities of the unknown density f_0 (Sobolev or super-smooth) and the degree of ill-posedness of the problem (mildly or severely ill-posed). For each considered case, we prove that the bound (12) holds. For the sake of convenience, the minimax rates of convergence in the direct and inverse setting are written respectively ρ_n^{DP} and ρ_n^{IP}. If we do not write explicitly the dependency with respect to the functional spaces, it will yet be recalled for each different setting.

Case 1: The problem is mildly ill-posed i.e. $|\Phi_g(t)| = \mathcal{O}(|t|^{-\beta})$ for some $\beta > 0$ and $f \in W(s, L)$ for some $s > 0$, i.e. $|h(t)| = t^s$. In this case, $f * g \in \mathcal{E}(\tilde{h}, L) \subset W(s + \beta, L')$ for some L'. The term τ_n is chosen as

$$L|\tau_n|^{-2s} = 1/2\gamma_n \Leftrightarrow \tau_n \sim [\rho_n^{\mathrm{IP}}]^{-1/s} \sim n^{2/(4s+4\beta+1)}.$$

Then

$$|\Phi_g(\tau_n)|^2(\rho_n^{\mathrm{IP}})^2 \sim \tau_n^{-2\beta} n^{-4s/(4s+4\beta+1)} \sim n^{-(4s+4\beta)/(4s+4\beta+1)} \sim (\rho_n^{\mathrm{DP}})^2.$$

Case 2: The problem is mildly ill-posed and $f \in S(\alpha, r, L)$. In this setting, remark that the function $f * g \in S(\alpha, r, L')$ for some L'. Then, according to the rates obtained in Butucea (2007),

$$(\rho_n^{\mathrm{IP}})^2 \sim \frac{\log(n)^{(2\beta+1/2)/r}}{n} \quad \text{and} \quad (\rho_n^{\mathrm{DP}})^2 \sim \frac{\log(n)^{1/2r}}{n}.$$

We choose τ_n as

$$\tau_n = \left\lceil \left(\frac{1}{2\alpha}\ln(n)\right)^{1/r} \right\rceil,$$

where for any $x \in \mathbb{R}$, $\lceil x \rceil$ denotes the smaller integer greater than x. Then for n large enough,

$$L^2|h(\tau_n)|^{-2} \le L^2 n^{-1} \le L^2 n^{-1}\log(n)^{(2\beta+1/2)/r} \le C(\rho_n^{\mathrm{IP}})^2$$

for some $C > 0$. Moreover

$$|\Phi_g(\tau_n)|^2(\rho_n^{\mathrm{IP}})^2 \sim \log(n)^{1/2r}\frac{1}{n} \sim (\rho_n^{\mathrm{DP}})^2.$$

Case 3: We assume that $f \in W(s, L)$ for some $s > 0$ and that the problem is severely ill-posed: $|\Phi_g(t)| = \mathcal{O}(\mathrm{e}^{-\gamma|t|^\beta})$ as $t \to +\infty$, for some $\gamma > 0$. In this setting, remark that $c\,\mathrm{e}^{\gamma|t|^\beta} \le c(t) :=$

$h(t)\Phi_g^{-1}(t) \leq C\,\mathrm{e}^{2\gamma|t|^\beta}$ for all t. Hence, using the results of Butucea (2007), we get

$$\rho_n^{\mathrm{IP}} \sim (\log(n))^{-s/\beta} \quad \text{and} \quad \rho_n^{\mathrm{DP}} \sim \frac{1}{\sqrt{n}}(\log(n))^{1/4\beta}.$$

We set

$$\tau_n = \left\lfloor \frac{1}{2\gamma}\log(n(\log n)^{-1/2\beta}) + \frac{1}{2\gamma}\log((\log n)^{-2s/\beta}) \right\rfloor^{1/\beta},$$

where for any $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the greater integer smaller than x. Then

$$L|h(\tau_n)|^{-2} \leq CL|\tau_n|^{-2s} \leq C(\log(n))^{-2s/\beta} \leq C(\rho_n^{\mathrm{IP}})^2,$$

where $C_2 > 0$, $0 < c < 1$, and

$$|\Phi_g(\tau_n)|^2(\rho_n^{\mathrm{IP}})^2 \geq C\,\mathrm{e}^{-2\gamma|\tau_n|^\beta}(\log n)^{-2s/\beta} \geq C\frac{1}{n}(\log(n))^{1/2\beta} \simeq (\rho_n^{\mathrm{DP}})^2.$$

Case 4: Assume that $f \in S(\alpha, r, L)$ for some $\alpha, r > 0$ and that the problem is severely ill-posed. Consider in a first time the case where $\beta > r$. In this setting, remark that $\tilde{c_1}\,\mathrm{e}^{\gamma|t|^\beta} \leq |h(t)||\Phi_g(t)|^{-1} \leq \tilde{c_2}\,\mathrm{e}^{2\gamma|t|^\beta}$ for all $t \in \mathbb{R}$. Hence, from Butucea (2007), we get

$$(\rho_n^{\mathrm{IP}})^2 \simeq \mathrm{e}^{-4\alpha/\tilde{t}^\star} \quad \text{and} \quad (\rho_n^{\mathrm{DP}})^2 \simeq \frac{1}{n}(\log(n))^{1/2\beta},$$

where $t^\star$ denotes the integer part of the solution of $2\alpha(t^\star)^r + 2\gamma(t^\star)^\beta = \log(n)$. Then, we set

$$\tau_n = \left\lfloor \frac{1}{2\gamma}\log(n(\log n)^{1/2\beta}) - \frac{\alpha}{\gamma}(t^\star)^r \right\rfloor.$$

Remark that for σ small enough, $\tau_n \geq t^\star$ since $r < \beta$. Therefore

$$|h(\tau_n)|^{-2} = R^2\,\mathrm{e}^{-2\alpha\tau_n^r} \leq cC_2\,\mathrm{e}^{-2\alpha(t^\star)^r},$$

where $C_2 > 0$ and $0 < c < 1$. Then

$$|\Phi_g(\tau_n)|^2(\rho_n^{\mathrm{IP}})^2 \geq C\,\mathrm{e}^{-2\gamma\tau_n^\beta - 2\alpha t_\star^s} \geq C\frac{1}{n}(\log n)^{1/2'\beta} \sim (\rho_n^{\mathrm{DP}})^2.$$

The proof of the case $r < \beta$ follows essentially the same lines.

4.2. *Proof of Theorem 3.3*

As explained in Section 3.2, we deal with the deconvolution testing procedure $\Delta_n^\star$ introduced in Equation (11). We prove that this procedure fails in the direct case, i.e. that there exists $f_1 \in \mathcal{E}(\Phi_g(.)^{-1}h, L)$ such that

$$\|(f_1 - f_0) * g\| \geq \mathcal{C}_1\rho_n(\mathcal{E}(\Phi_g(\cdot)^{-1}h, L)) \text{ but } \lim_{n\to+\infty}[P_{f_0}(\Delta_n^\star = 1) + P_{f_1}(\Delta_n^\star = 0)] > \xi,$$

whatever the value of $\mathcal{C}_1$.

4.2.1. *Construction of a particular $f_1 = f_{1,n}$*

To this end, we consider the function $f_{1,n}$ defined as

$$f_{1,n} = f_0 + r_n\omega, \tag{13}$$

where

$$f_0(x) \geq \frac{C}{1+x^2}, \quad \omega(x) = \cos(\sigma x)\cdot\frac{1-\cos(x)}{\pi x^2},$$

C, σ are positive constants such that $f_{1,n}(x) \geq 0$ for all $x \in \mathbb{R}$ and r_n is a parameter which is allowed to depend on n. The term $r_n\omega$ corresponds to a perturbation of the density f_0. It is possible to see that

$$\Phi_\omega(t) = (1 - |t-\sigma|)_+, \quad \forall t \in \mathbb{R}.$$

In particular, remark that $\mathrm{supp}(\Phi_\omega) \subset [1-\sigma; 1+\sigma] \subset [-2; 2]$ for $0 < \sigma < 1$. Hence, the perturbation $r_n\omega$ only concerns low frequencies. We can also verify that

$$\Phi_{f_{1,n}}(0) = \Phi_{f_0}(0) + r_n\Phi_\omega(0) = 1,$$

which ensures that $f_{1,n}$ is a density w.r.t. the Lebesgue measure. Moreover,

$$\|g*(f_{1,n}-f_0)\|^2 = r_n^2\int_{-2}^{2}|\Phi_\omega(t)|^2|\Phi_g(t)|^2 \geq Cr_n^2 \inf_{t\in[-2;2]}|\Phi_g(t)|^2. \tag{14}$$

As in the proof of Theorem 3.1, we consider four different cases according to possible regularity of the function f and the degree of ill-posedness of the problem. In each case, choosing $r_n = \mathcal{C}_1\rho_n^{\mathrm{DP}}$, we get

$$\|g*(f_{1,n}-f_0)\| > C\rho_n^{\mathrm{DP}} \text{ but } \|f-f_0\| = r_n^2\|\omega\|^2 \leq C\rho_n^{\mathrm{DP}} \ll \rho_n^{\mathrm{IP}}, \quad \text{as } n \to +\infty,$$

for some positive constant $C > 0$. The inequality in the right-hand side of the previous equation is often incompatible with the optimality of $\Delta_n^\star = \mathbf{1}_{\{|T_n^\star|^2 \geq \mathcal{C}^\star t_n^2\}}$.

4.2.2. *Lower bounds for the second kind error*

In the following, we prove that for all fixed ξ, with a good choice of $C^\star$, we have

$$\|g*(f_{1,n}-f_0)\| > \mathcal{C}_1\rho_n^{\mathrm{DP}} \text{ but } \lim_{n\to+\infty}[P_{f_0}(\Delta_n^\star = 1) + P_{f_{1,n}}(\Delta_n^\star = 0)] > \xi,$$

whatever the value of $\mathcal{C}_1$.

1st case: In a first time, we assume that the density of the noise is polynomially smooth, i.e. $|\Phi_g(t)| = \mathcal{O}(|t|^{-\beta})$ as $t \to +\infty$ for some $\beta > 0$. Provided

$$t_n = n^{-2s/(4s+4\beta+1)} \quad \text{and} \quad h = n^{-2/(4s+4\beta+1)},$$

the test $\Delta_n^\star$ is minimax for H_0^{IP} on $W(s, L)$ for $\mathcal{C}^\star$ large enough. Nevertheless, we prove that this test is not minimax for H_0^{DP} on $W(s+\beta, L')$. To this end, we consider the function $f_{1,n}$ defined in

Equation (13) with

$$r_n = \mathcal{C}_1 n^{-(2s+2\beta)/(4s+4\beta+1)},$$

for some positive constant $\mathcal{C}_1$. With such a construction, we have

$$\|g * f_{1,n} - g * f_0\|^2 > \mathcal{C}_1 c_g n^{-(4s+4\beta)/(4s+4\beta+1)} \quad \text{and} \quad \lim_{n\to+\infty} \inf P_{f_{1,n}}(\Delta_n^\star = 0) > \xi \tag{15}$$

for some constant c_g depending on the density g. The left-hand side of Equation (15) is a direct consequence of Equation (14). We only have to prove the right-hand side. First, introduce

$$V_{f_{1,n}}(T_n^\star) = \mathrm{Var}(T_n^\star) \quad \text{and} \quad B_f(T_n^\star) = |\mathbb{E}_{f_{1,n}}[T_n^\star] - \|f_{1,n} - f_0\|^2|.$$

The, using simple algebra, we get

$$\begin{aligned} P_{f_{1,n}}(\Delta_n^\star = 1) &= P_{f_{1,n}}(|T_n^\star| > C^\star t_n^2), \\ &\le P_{f_{1,n}}(|T_n^\star - \mathbb{E}_{f_{1,n}}(T_n^\star)| + \|f_{1,n} - f_0\|^2 + B(T_n^\star) > C^\star t_n^2), \\ &\le P_{f_{1,n}}(|T_n^\star - \mathbb{E}_{f_{1,n}}(T_n^\star)| > C^\star t_n^2 - \|f_{1,n} - f_0\|^2 - B(T_n^\star)), \\ &\le P_{f_{1,n}}\left(\frac{|T_n^\star - \mathbb{E}_{f_{1,n}}(T_n^\star)|}{\sqrt{V_f(T_n^\star)}} > \frac{C^\star t_n^2 - \|f_{1,n} - f_0\|^2 - B(T_n^\star)}{\sqrt{V_{f_{1,n}}(T_n^\star)}}\right), \\ &\le \left[\frac{\sqrt{V_{f_{1,n}}(T_n^\star)}}{C^\star t_n^2 - \|f_{1,n} - f_0\|^2 - B(T_n^\star)}\right]^2. \end{aligned}$$

In Butucea (2007), it is proved that $B_{f_{1,n}}(T_n^\star) \le L h_n^{2s}(1 + o(1))$ as $n \to +\infty$. Hence

$$C^\star t_n^2 - \|f - f_0\|^2 - B(T_n^\star) \ge (C^\star - L) t_n^2 (1 + o(1)), \quad \text{as } n \to +\infty.$$

Since $\sqrt{V_f(T_n^\star)} \le O(t_n^2)$ as $n \to +\infty$, we obtain that

$$P_{f_{1,n}}(\Delta_n^\star = 1) \le \left[\frac{C}{(C^\star - L)(1 + o(1))}\right]^2, \quad \text{as } n \to +\infty.$$

Provided $C^\star = C_\xi^\star$ is large enough, we obtain that

$$\lim_{n\to+\infty} \sup P_{f_{1,n}}(\Delta_n^\star = 1) \le 1 - \xi \Rightarrow \lim_{n\to+\infty} \inf P_{f_{1,n}}(\Delta_n^\star = 0) \ge \xi.$$

This result holds whatever the value of $\mathcal{C}_1$ in the construction of r_n. Hence Equation (15) is proved which entails that the test $\Delta_n^\star$ is not minimax for H_0^{DP} on $W(s + t, L')$.

Case 2: The function f belongs to $S(\alpha, r, L)$ and the noise is polynomially smooth. In this case, following Butucea (2007), the test $\Delta_n^\star$ is minimax provided t_n and h_n are chosen as follows

$$t_n = \frac{1}{\sqrt{n}}\left(\frac{\log(n)}{2\alpha}\right)^{(\beta+1/4)/r} \quad \text{and} \quad h = \left(\frac{\log(n)}{2\alpha} - \frac{2\beta + 1/2}{2\alpha r_n} \log\log n\right)^{-1/r},$$

for $\mathcal{C}^\star$ large enough. Once again, we consider the function $f_{1,n}$ defined in Equation (13) with

$$r_n = \frac{\mathcal{C}_1}{\sqrt{n}}\left(\frac{\log(n)}{2\alpha}\right)^{1/4r} \ge C\rho_n^{\mathrm{DP}}$$

for some fixed constant C. This entails

$$\|g*(f_{1,n}-f_0)\|^2 > c_g\frac{\mathcal{C}_1}{n}\left(\frac{\log(n)}{2\alpha}\right)^{1/2r} \quad\text{and}\quad \|f_{1,n}-f_0\|^2 = \mathcal{C}_1 r_n^2 = \frac{1}{n}\left(\frac{\log(n)}{2\alpha}\right)^{1/2r}.$$

We prove that the test $\Delta_n^\star$ is not minimax in this situation. Using a similar algebra compared to the first case, we obtain

$$P_{f_{1,n}}(\Delta_n^\star = 1) \le \left[\frac{\sqrt{V_{f_{1,n}}(T_n^\star)}}{C^\star t_n^2 - \|f_{1,n}-f_0\|^2 - B(T_n^\star)}\right]^2.$$

Using the upper bound obtained in Butucea (2007), we get

$$V_{f_{1,n}}(T_n^\star) \le C_1\|f_{1,n}-f_0\|_2^2\left(\log\frac{2\alpha}{\|f-f_0\|_2^2}\right)^{2\beta/r}.$$

Hence

$$\begin{aligned}
P_{f_{1,n}}(\Delta_n^\star = 1) &\le \left[\frac{\|f_{1,n}-f_0\|_2^2}{C^\star t_n^2(1+o(1))}\left(\log\frac{2\alpha}{\|f_{1,n}-f_0\|_2^2}\right)^{2\beta/r}\right]^2,\\
&= \frac{C_1}{C^\star}\frac{1}{\log(n)^{2\beta/r}}(\log(nC_1^2))^{2\beta/r} \le \frac{C_1}{C^\star},
\end{aligned}$$

which yields

$$\lim_{n\to+\infty} P_{f_{1,n}}(\Delta_n = 1) \le 1-\xi,$$

whatever the value of $\mathcal{C}_1$, provided $\mathcal{C}^\star$ is large enough.

Case 3: The noise is exponentially smooth and the function f belongs to $W(s,L)$. In this case, the test $\Delta_n^\star$ is known to be minimax provided

$$t_n = \sqrt{L}\left(\frac{\log(n)}{2\gamma}\right)^{-s/\beta} \quad\text{and}\quad h_n = \left(\frac{\log(n)}{2\gamma} - \frac{2s+1}{2\gamma\beta}\log\frac{\log n}{2\gamma}\right)^{-1/\beta}.$$

We consider the function $f_{1,n}$ defined in Equation (13) with

$$r_n = \frac{1}{\sqrt{n}}\left(\frac{\log(n)}{\gamma}\right)^{1/4\beta}.$$

This entails

$$\|g*f_{1,n}-g*f_0\|^2 > \frac{1}{n}\left(\frac{\log(n)}{\gamma}\right)^{1/2\beta} \quad\text{and}\quad \|f_{1,n}-f_0\|^2 = \mathcal{C}_1 r_n = \frac{1}{n}\left(\frac{\log(n)}{\gamma}\right)^{1/2\beta}.$$

Then

$$\begin{aligned}
P_{f_{1,n}}(\Delta_n^\star = 1) &\le \left[\frac{\sqrt{V_f(T_n^\star)}}{C^\star t_n^2 - \|f-f_0\|^2 - B(T_n^\star)}\right]^2,\\
&\le \left[\frac{h_n^{2s+\beta/2}}{(C^\star-1)t_n^2(1+o(1))}\right]^2,\\
&\le \frac{C\log(n)^{-2s/\beta-1/2}}{C^\star\log(n)^{-2s/\beta}} \le \frac{C}{C^\star\log^{1/2}(n)}.
\end{aligned}$$

Once again, the limit of the above term is bounded from above by $1-\xi$ for a large enough $C^\star$.

Case 4: The noise is exponentially smooth and the function f belongs to $S(\alpha, r, L)$. For the sake of brevity, we only consider the case where $r < \beta$. In this case, the test $\Delta_n^\star$ is known to be minimax provided

$$t_n = h_n = \exp\left(-\frac{\alpha}{h_\star^r}\right).$$

Remark that $f_{1,n} * g \in S(\gamma, \beta, L')$. Moreover, choosing

$$\tau_n = \frac{\log(n)^{1/4\beta}}{\sqrt{n}},$$

we get

$$\|f_{1,n} - f_0\|^2 = \tau_n^2 = \frac{1}{n}\log(n)^{1/2\beta} \geq C(\rho_n^{\mathrm{DP}})^2$$

for some positive constant C. Then recalling that

$$\frac{\alpha}{h_\star^r} + \frac{\gamma}{h_\star^\beta} = \frac{1}{2}[\log(n) - (\log\log n)^2],$$

we get that

$$t_n = \exp\left(-\frac{\alpha}{h_\star^r}\right) = \exp\left(-\frac{1}{2}\log(n) + \frac{(\log\log n)^2}{2} + \frac{\gamma}{h_\star^r}\right) \geq \frac{C}{\sqrt{n}}\exp(\log\log n)^2).$$

Moreover, we get from Butucea (2007) that

$$V_f(T_n^\star) \leq \frac{h_\star^{\beta-1}}{n}\exp\left(\frac{2\gamma}{h_\star^\beta} - \frac{2\alpha}{h_\star^r}\right) + \frac{h_\star^{\beta-1}}{n^2}\exp\left(\frac{4\gamma}{h_\star^s}\right).$$

Hence

$$\begin{aligned}
\frac{\sqrt{V_f(T_n^\star)}}{C^\star t_n^2} &= \frac{h_\star^{(\beta-1)/2}/\sqrt{n}\exp(\gamma/h_\star^\beta - \alpha/h_\star^r)}{\exp(-2\alpha/h_\star^r)}, \\
&= \frac{h_\star^{(\beta-1)/2}}{\sqrt{n}}\exp\left(\frac{\alpha}{h_\star^r} + \frac{\gamma}{h_\star^\beta}\right), \\
&= \frac{h_\star^{(\beta-1)/2}}{\sqrt{n}}\exp(\log(\sqrt{n}) - (\log\log n)^2), \\
&= \frac{h_\star^{(\beta-1)/2}}{n}\exp(-(\log\log n)^2).
\end{aligned}$$

We can conclude the proof remarking that $h_\star^\beta$ is of order $(\log(n))^{-1}$ provided $\beta > r$. Hence, the previous term tends to 0 as $n \to +\infty$.

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