

Partial functional correspondence

Michael Bronstein



University of Lugano

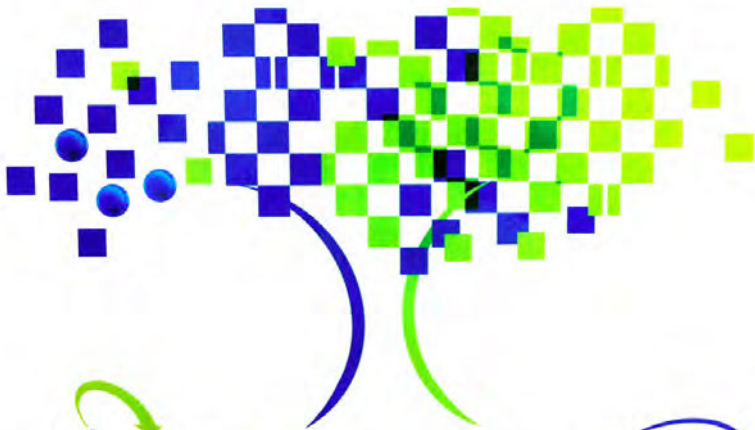


Intel Corporation

Lyon, 7 July 2016



Microsoft Kinect 2010



INVISION *A New Dimension to* **intel**

(Acquired by Intel in 2012)



intel REALSENSE™
TECHNOLOGY



Different form factor computers featuring Intel RealSense 3D camera

Deluge of geometric data



KINECT
for AR, VR, SKIN



SoftKinetic

intel **REALSENSE**

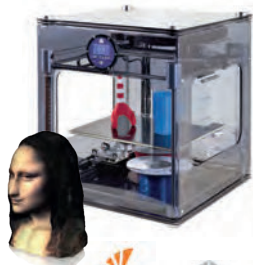
3D sensors



Google 3D warehouse

shapeways

Repositories



Stratasys

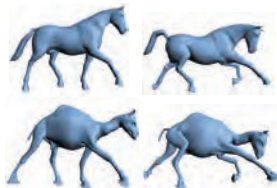


3D printers

Applications



Deformable fusion



Motion transfer



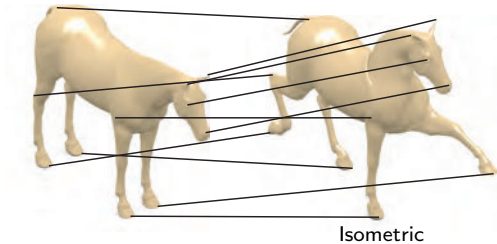
Motion capture



Texture mapping

Dou et al. 2015; Sumner, Popović 2004; Faceshift; Cow image: Moore 2014

Shape correspondence problem



Shape correspondence problem



Isometric



Partial

Shape correspondence problem



Isometric

Partial



Different representation

Shape correspondence problem



Isometric



Partial



Different representation



Non-isometric

Partial Functional Correspondence

E. Rodolà^{1,2}, L. Cozzari³, M. M. Bronstein^{1,2}, A. Torsello², D. Cremers³

¹TU Munich, Germany ²University of Venice, Italy ³University of Lugano, Switzerland

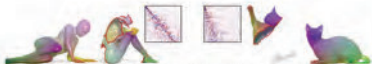


Figure 1: Partial functional correspondence between two pairs of shapes with large missing parts. For each pair we show the matrix C representing the functional map in the spectral domain, and the action of the map by transferring colors from one shape to the other. The spectral domain diagonal structure of C , induced by the partially transformation is first estimated from spectral properties of the two shapes, and then exploited to drive the matching process.

Abstract

In this paper, we propose a method for computing partial functional correspondence between non-rigid shapes. We use perturbation analysis to show how removal of shape parts changes the Laplace-Beltrami eigenfunctions, and exploit it as a prior in the spectral representation of the correspondence. Corresponding parts are optimization variables in our problem and we used to weight the functional correspondences; we are looking for the largest and most regular (in the Mumford-Shah sense) parts that minimize correspondence distortions. We show that our approach can cope with very challenging correspondence settings.

Categories and Subject Descriptors (according to ACM CCS): I.3.5 [Computer Graphics]: Computational Geometry and Object Modeling—Shape Analysis.

Non-Rigid Puzzles

D. Eney^{1,2}, E. Rodolà^{1,2}, A. M. Bronstein^{1,2}, M. M. Bronstein^{1,2}, D. Cremers³

¹ISI Aalto University, Israel ²University of Lugano, Switzerland ³Technion, Israel ⁴Intel Perceptual Computing, Israel ⁵TU Munich, Germany



Figure 1: Example of the non-rigid puzzle problem considered in this paper: given a model human shape (leftmost, first column) and three query shapes (two deformed parts of the human and one unrelated ‘extra’ shape of a cat head), the goal is to find a representation of the model shape (second column, shown in yellow and green), white circles mark without correspondence (two parts corresponding to columns of the query shape). Third column shows the computed correspondence between the parts (corresponding points are marked in similar colors).

Abstract

Shape correspondence is a fundamental problem in computer graphics and vision, with applications in various problems including animation, texture mapping, robotic vision, media of imaging, archeology and many more. In settings where the shapes are allowed to undergo non-rigid deformations and only partial views are available, the problem becomes very challenging. In this work, we present a non-rigid multi-part shape matching algorithm. We assume to be given a reference shape and/or multiple parts undergoing a non-rigid deformation. Each of these query parts can be additionally contaminated by clutter, may overlap with other parts, and there might be missing parts or redundant ones. Our method simultaneously solves for the segmentation of the reference model, and for a dense correspondence to (subset of) the parts. Experimental results on synthetic as well as real world datasets demonstrate the effectiveness of our method in dealing with this challenging matching scenario.

Categories and Subject Descriptors (according to ACM CCS): I.3.5 [Computer Graphics]: Computational Geometry and Object Modeling—Shape Analysis.

Computer Graphics Forum
SGP 2016

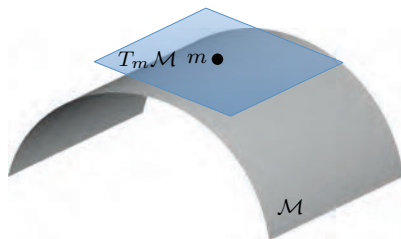
Computer Graphics Forum
SGP 2016 **Best paper award**

Outline

- **Background: Spectral analysis on manifolds**
- Functional correspondence
- Partial functional correspondence
- Non-rigid puzzles

Riemannian geometry in one minute

- **Tangent plane** $T_m\mathcal{M}$ = local Euclidean representation of manifold (surface) \mathcal{M} around m



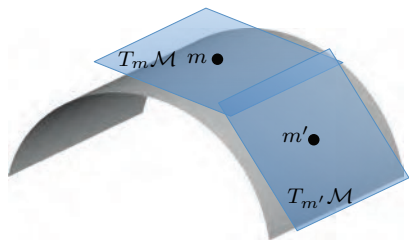
Riemannian geometry in one minute

- **Tangent plane** $T_m\mathcal{M}$ = local Euclidean representation of manifold (surface) \mathcal{M} around m

- **Riemannian metric**

$$\langle \cdot, \cdot \rangle_{T_m\mathcal{M}} : T_m\mathcal{M} \times T_m\mathcal{M} \rightarrow \mathbb{R}$$

depending smoothly on m



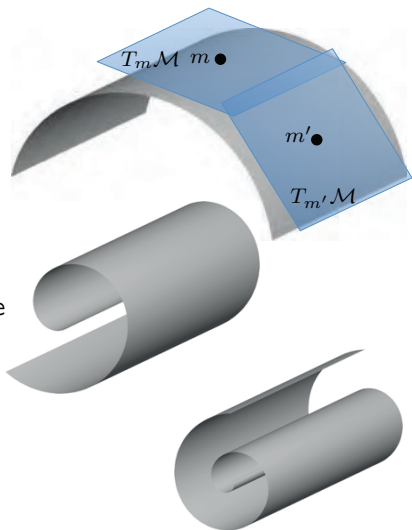
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Isometry = metric-preserving shape deformation



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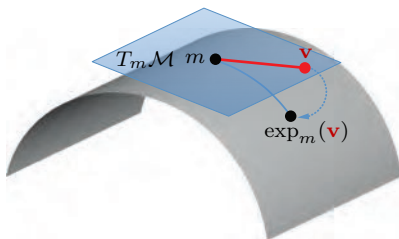
depending smoothly on m

Isometry = metric-preserving shape deformation

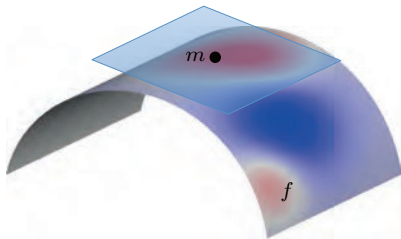
- **Exponential map**

$$\exp_m : T_m\mathcal{M} \rightarrow \mathcal{M}$$

'unit step along geodesic'

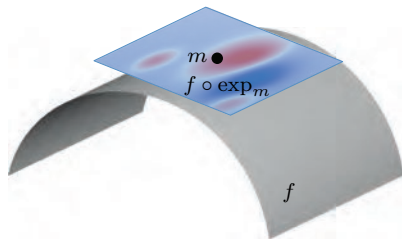


Laplace-Beltrami operator



Smooth field $f : \mathcal{M} \rightarrow \mathbb{R}$

Laplace-Beltrami operator



Smooth field $f \circ \exp_m : T_m \mathcal{M} \rightarrow \mathbb{R}$

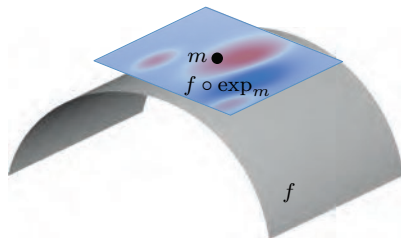
Laplace-Beltrami operator

- Intrinsic gradient

$$\nabla_{\mathcal{M}} f(m) = \nabla(f \circ \exp_m)(\mathbf{0})$$

Taylor expansion

$$(f \circ \exp_m)(\mathbf{v}) \approx f(m) + \langle \nabla_{\mathcal{M}} f(m), \mathbf{v} \rangle_{T_m \mathcal{M}}$$



Laplace-Beltrami operator

- Intrinsic gradient

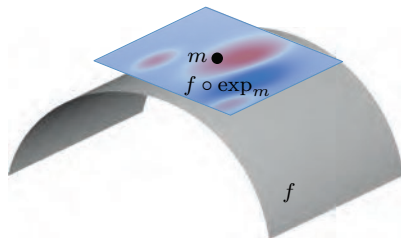
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$$\Delta_{\mathcal{M}}f(m) = \Delta(f \circ \exp_m)(\mathbf{0})$$



Laplace-Beltrami operator

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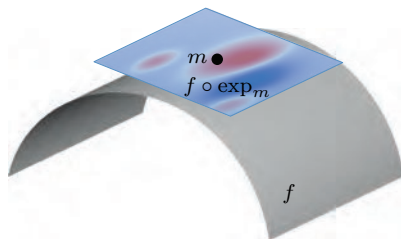
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$$\Delta_{\mathcal{M}} f(m) = \Delta(f \circ \exp_m)(\mathbf{0})$$

- Intrinsic (expressed solely in terms of the Riemannian metric)



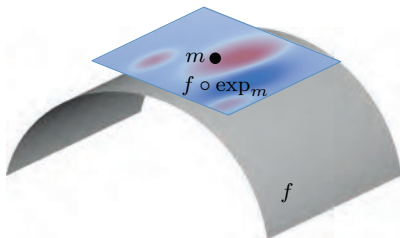
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- Laplace-Beltrami operator

$$\Delta_{\mathcal{M}} f(m) = \Delta(f \circ \exp_m)(\mathbf{0})$$

- Intrinsic (expressed solely in terms of the Riemannian metric)
- Isometry-invariant

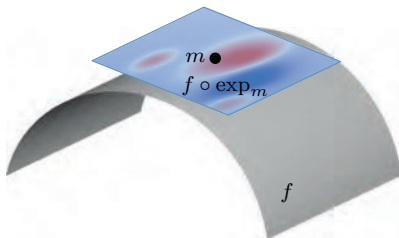
Laplace-Beltrami operator

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Taylor expansion

$$(f \circ \exp_m)(\mathbf{v}) \approx f(m) + \langle \nabla_{\mathcal{M}}f(m), \mathbf{v} \rangle_{T_m\mathcal{M}}$$



- Laplace-Beltrami operator

$$\Delta_{\mathcal{M}}f(m) = \Delta(f \circ \exp_m)(\mathbf{0})$$

- Intrinsic (expressed solely in terms of the Riemannian metric)
- Isometry-invariant
- Self-adjoint $\langle \Delta_{\mathcal{M}}f, g \rangle_{L^2(\mathcal{M})} = \langle f, \Delta_{\mathcal{M}}g \rangle_{L^2(\mathcal{M})}$

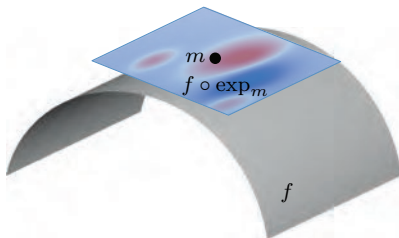
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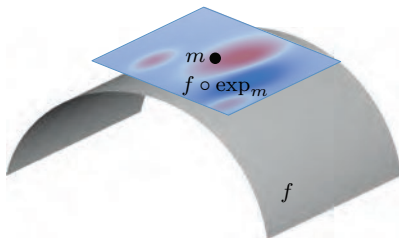
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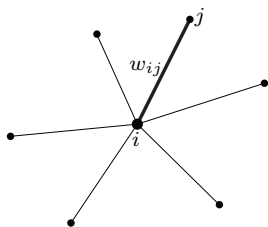


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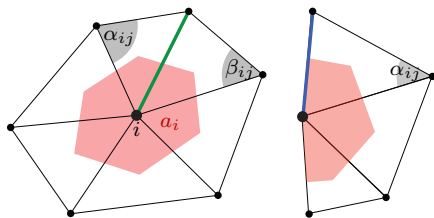
- Intrinsic (expressed solely in terms of the Riemannian metric)
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- Self-adjoint $\langle \Delta_{\mathcal{M}}f, g \rangle_{L^2(\mathcal{M})} = \langle f, \Delta_{\mathcal{M}}g \rangle_{L^2(\mathcal{M})} \Rightarrow$ orthogonal eigenfunctions
- Positive semidefinite \Rightarrow non-negative eigenvalues

Discrete Laplacian



Undirected graph (V, E)

$$(\Delta f)_i \approx \sum_{(i,j) \in E} w_{ij} (f_i - f_j)$$



Triangular mesh (V, E, F)

$$(\Delta f)_i \approx \frac{1}{a_i} \sum_{(i,j) \in E} w_{ij} (f_i - f_j)$$

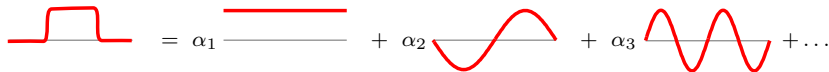
$$w_{ij} = \begin{cases} \frac{\cot \alpha_{ij} + \cot \beta_{ij}}{2} & (i, j) \in E_i \\ \frac{1}{2} \cot \alpha_{ij} & (i, j) \in E_b \\ -\sum_{k \neq i} w_{ik} & i = j \\ 0 & \text{else} \end{cases}$$

a_i = local area element

Fourier analysis (Euclidean spaces)

A function $f : [-\pi, \pi] \rightarrow \mathbb{R}$ can be written as **Fourier series**

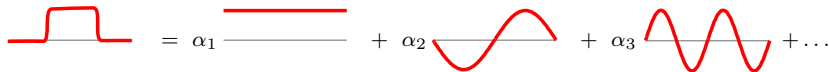
$$f(x) = \sum_{\omega} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\xi) e^{i\omega\xi} d\xi e^{-i\omega x}$$



Fourier analysis (Euclidean spaces)

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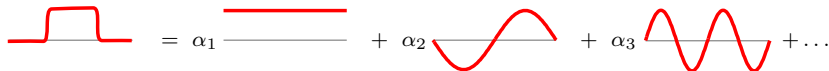
$$f(x) = \sum_{\omega} \underbrace{\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\xi) e^{i\omega\xi} d\xi}_{\hat{f}(\omega) = \langle f, e^{-i\omega x} \rangle_{L^2([-\pi, \pi])}} e^{-i\omega x}$$



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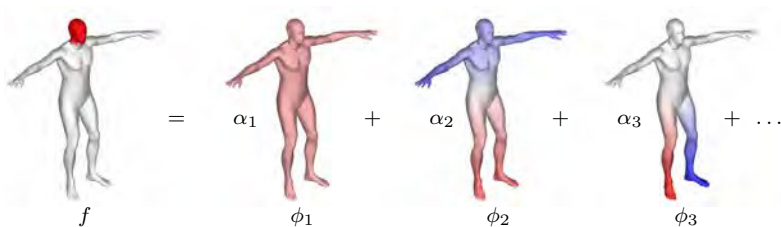


Fourier basis = **Laplacian eigenfunctions**: $\Delta e^{-i\omega x} = \omega^2 e^{-i\omega x}$

Fourier analysis (non-Euclidean spaces)

A function $f : \mathcal{M} \rightarrow \mathbb{R}$ can be written as **Fourier series**

$$f(m) = \sum_{k \geq 1} \underbrace{\int_{\mathcal{M}} f(\xi) \phi_k(\xi) d\xi}_{\hat{f}_k = \langle f, \phi_k \rangle_{L^2(\mathcal{M})}} \phi_k(m)$$

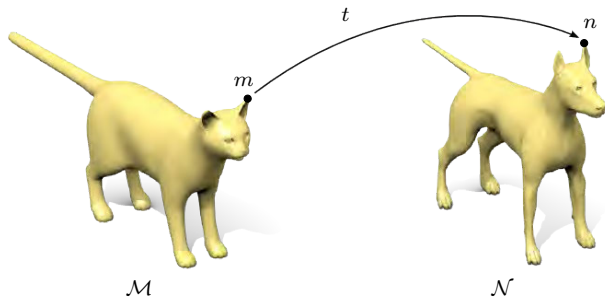


Fourier basis = **Laplacian eigenfunctions**: $\Delta_{\mathcal{M}} \phi_k = \lambda_k \phi_k$

Outline

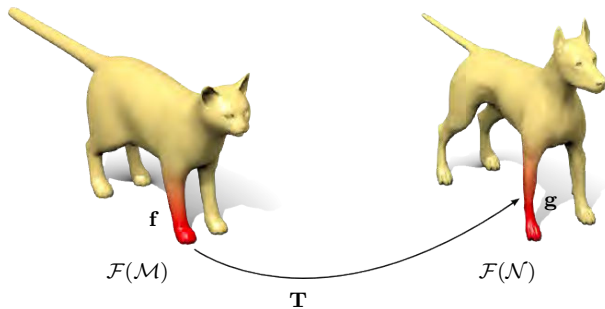
- Background: Spectral analysis on manifolds
- **Functional correspondence**
- Partial functional correspondence
- Non-rigid puzzles

Point-wise correspondence



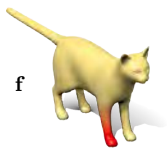
Point-wise maps $t: \mathcal{M} \rightarrow \mathcal{N}$

Functional correspondence

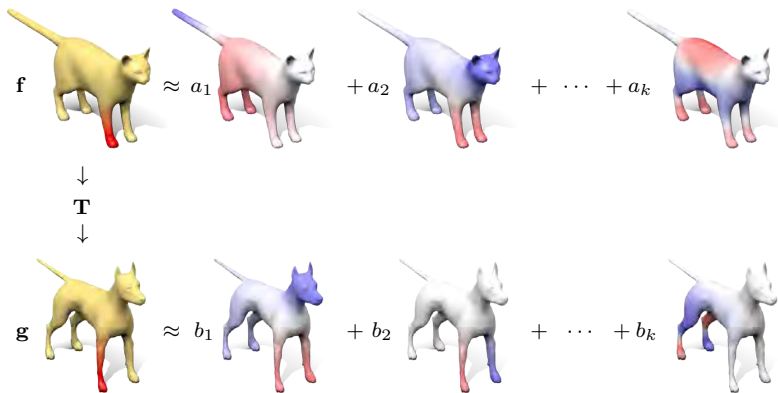


Functional maps $\mathbf{T}: \mathcal{F}(\mathcal{M}) \rightarrow \mathcal{F}(\mathcal{N})$

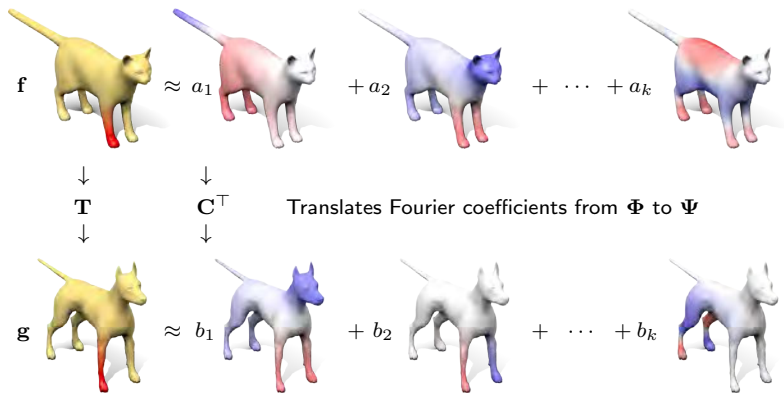
Functional correspondence



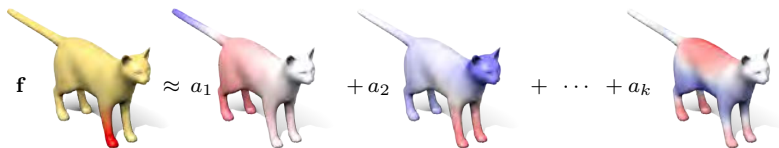
Functional correspondence



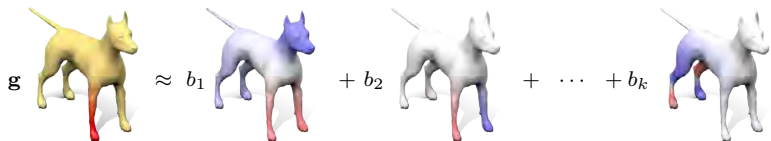
Functional correspondence



Functional correspondence



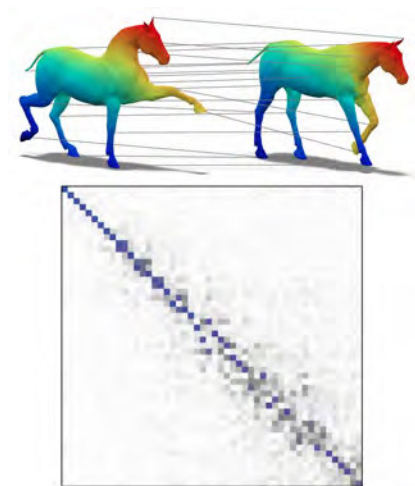
Translates Fourier coefficients from Φ to Ψ



$$\mathbf{g}^T \Psi_k = \mathbf{f}^T \Phi_k \mathbf{C}$$

where $\Phi_k = (\phi_1, \dots, \phi_k)$, $\Psi_k = (\psi_1, \dots, \psi_k)$ are truncated Laplace-Beltrami eigenbases

Functional correspondence in Laplacian eigenbases

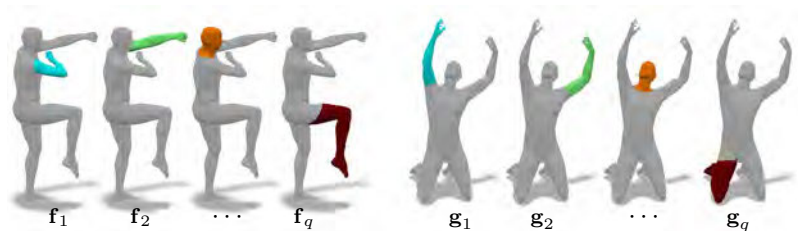


For **isometric simple spectrum** shapes \mathbf{C} is diagonal since $\psi_i = \pm \mathbf{T}\phi_i$

Computing functional correspondence

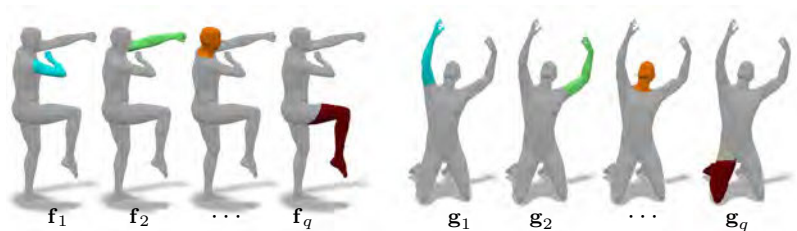


Computing functional correspondence



- Given **ordered** set of functions f_1, \dots, f_q on \mathcal{M} and corresponding functions g_1, \dots, g_q on \mathcal{N} ($g_i \approx \mathbf{T}f_i$)

Computing functional correspondence



- Given **ordered** set of functions $\mathbf{f}_1, \dots, \mathbf{f}_q$ on \mathcal{M} and corresponding functions $\mathbf{g}_1, \dots, \mathbf{g}_q$ on \mathcal{N} ($\mathbf{g}_i \approx \mathbf{T}\mathbf{f}_i$)
- \mathbf{C} found by solving a system of qk equations with k^2 variables

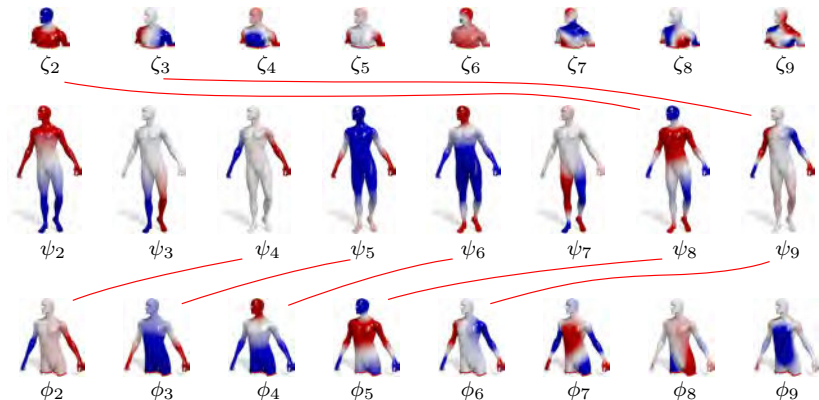
$$\mathbf{G}^\top \Psi_k = \mathbf{F}^\top \Phi_k \mathbf{C}$$

where $\mathbf{F} = (\mathbf{f}_1, \dots, \mathbf{f}_q)$ and $\mathbf{G} = (\mathbf{g}_1, \dots, \mathbf{g}_q)$ are $n \times q$ and $m \times q$ matrices

Key issues

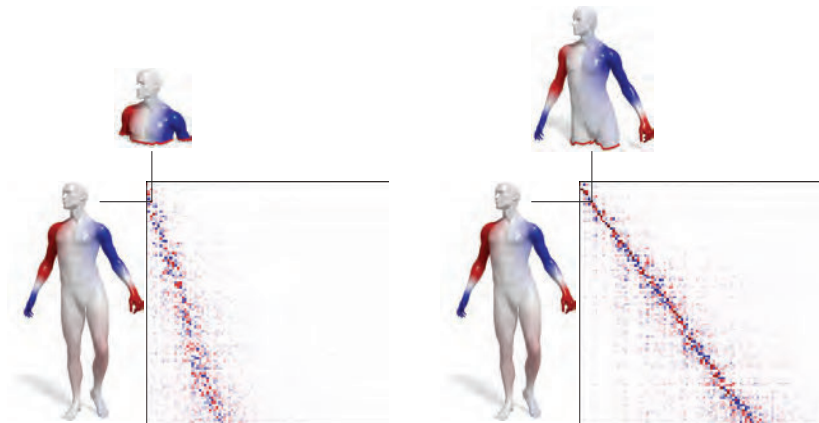
- How to recover point-wise correspondence with some guarantees (e.g. bijectivity)?
- How to automatically find corresponding functions \mathbf{F} , \mathbf{G} ?
- Near isometric shapes: easy (a lot of structure!)
- Non-isometric shapes: hard
- Does not work well in case of **missing parts** and topological noise

Partial Laplacian eigenvectors



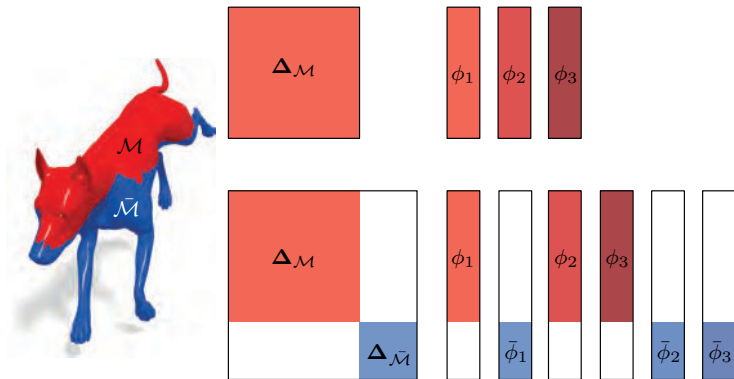
Laplacian eigenvectors of a shape with missing parts
(Neumann boundary conditions)

Partial Laplacian eigenvectors



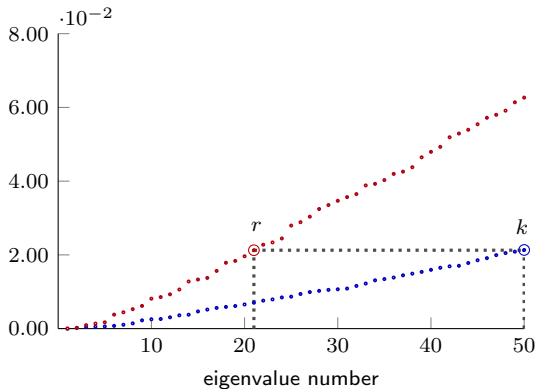
Functional correspondence matrix C

Perturbation analysis: intuition



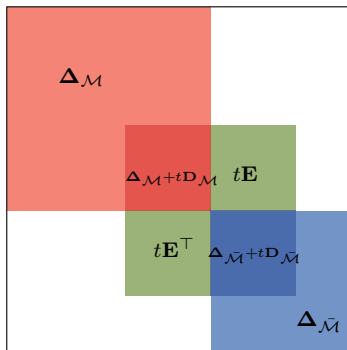
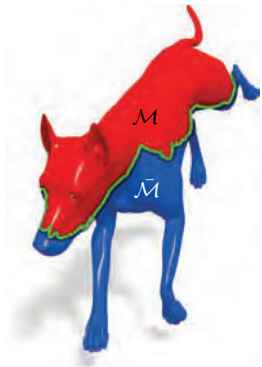
- Ignoring boundary interaction: disjoint parts (block-diagonal matrix)
- Eigenvectors = Mixture of eigenvectors of the parts

Perturbation analysis: eigenvalues

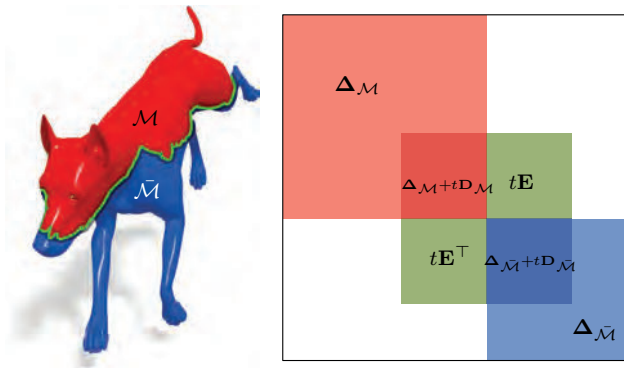


- Slope $\frac{r}{k} \approx \frac{|\mathcal{M}|}{|\mathcal{N}|}$ (depends on the **area** of the cut)
- Consistent with **Weil's law**

Perturbation analysis: details

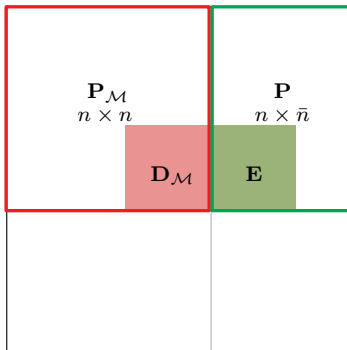
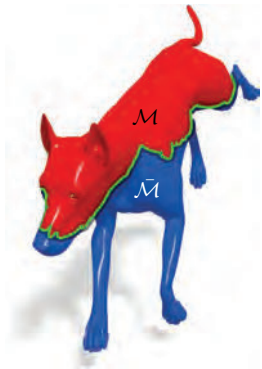


Perturbation analysis: details



“How would the Laplacian eigenvalues and eigenvectors of the red part change if we attached a blue part to it?”

Perturbation analysis: details



“How would the Laplacian eigenvalues and eigenvectors of the **red** part change if we attached a **blue** part to it?”

Perturbation analysis: details

Denote $\Delta_{\mathcal{M}} + t\mathbf{P}_{\mathcal{M}} = \Phi(t)\Lambda(t)\Phi(t)^\top$, $\Delta_{\bar{\mathcal{M}}} = \bar{\Phi}\bar{\Lambda}\bar{\Phi}^\top$, $\Phi = \Phi(0)$, and $\Lambda = \Lambda(0)$.

Theorem 1 (eigenvalues) The derivative of the non-trivial eigenvalues is given by

$$\frac{d}{dt}\lambda_i = \phi_i^\top \mathbf{P}_{\mathcal{M}} \phi_i \quad \mathbf{P}_{\mathcal{M}} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{\mathcal{M}} \end{pmatrix}$$

Perturbation analysis: details

Denote $\Delta_{\mathcal{M}} + t\mathbf{P}_{\mathcal{M}} = \Phi(t)\Lambda(t)\Phi(t)^\top$, $\Delta_{\bar{\mathcal{M}}} = \bar{\Phi}\bar{\Lambda}\bar{\Phi}^\top$, $\Phi = \Phi(0)$, and $\Lambda = \Lambda(0)$.

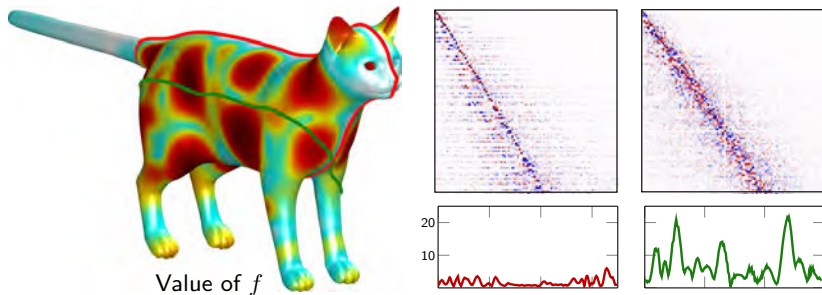
Theorem 1 (eigenvalues) The derivative of the non-trivial eigenvalues is given by

$$\frac{d}{dt}\lambda_i = \phi_i^\top \mathbf{P}_{\mathcal{M}} \phi_i \quad \mathbf{P}_{\mathcal{M}} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{\mathcal{M}} \end{pmatrix}$$

Theorem 2 (eigenvectors) Assuming $\lambda_i \neq \lambda_j$ for $i \neq j$ and $\lambda_i \neq \bar{\lambda}_j$ for all i, j , the derivative of the non-trivial eigenvectors is given by

$$\frac{d}{dt}\phi_i = \sum_{\substack{j=1 \\ j \neq i}}^n \frac{\phi_i^\top \mathbf{P}_{\mathcal{M}} \phi_j}{\lambda_i - \lambda_j} \phi_j + \sum_{j=1}^{\bar{n}} \frac{\phi_i^\top \mathbf{P} \bar{\phi}_j}{\lambda_i - \bar{\lambda}_j} \bar{\phi}_j \quad \mathbf{P} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{E} & \mathbf{0} \end{pmatrix}$$

Perturbation analysis: boundary interaction strength



- Eigenvector perturbation depends on **length** and **position** of the boundary
- Perturbation strength $\|\frac{d}{dt}\Phi\|_F \leq c \int_{\partial\mathcal{M}} f(m)dm$, where

$$f(m) = \sum_{\substack{i,j=1 \\ j \neq i}}^n \left(\frac{\phi_i(m)\phi_j(m)}{\lambda_i - \lambda_j} \right)^2$$

Laplacian perturbation: typical picture

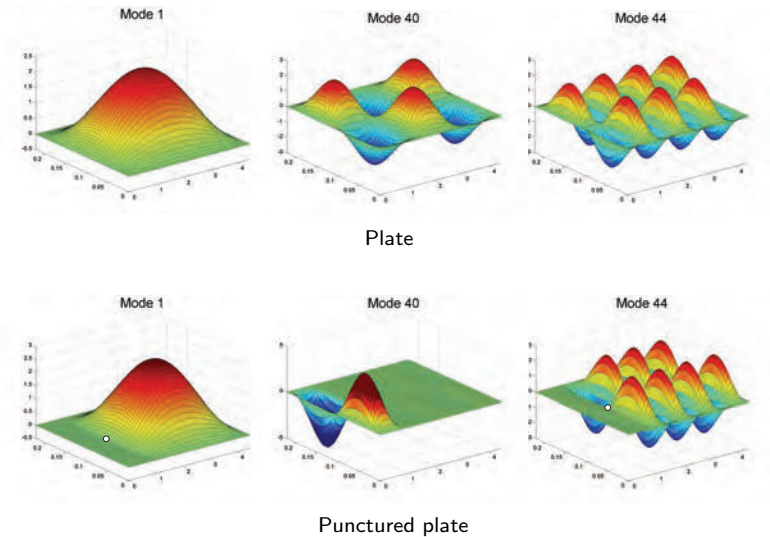
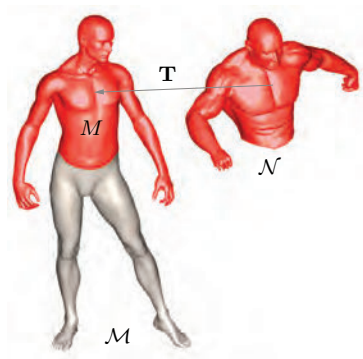


Figure: Filoche, Mayboroda 2009

Partial functional maps

- Model shape \mathcal{M}
- Query shape \mathcal{N}
- Part $M \subseteq \mathcal{M} \approx$ isometric to \mathcal{N}
- Data \mathbf{F}, \mathbf{G}
- Partial functional map

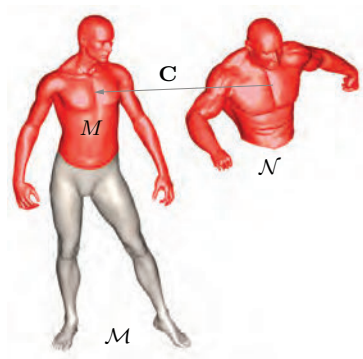
$$\mathbf{T}\mathbf{G} \approx \mathbf{F}(M)$$



Partial functional maps

- Model shape \mathcal{M}
- Query shape \mathcal{N}
- Part $M \subseteq \mathcal{M} \approx$ isometric to \mathcal{N}
- Data \mathbf{F}, \mathbf{G}
- Partial functional map

$$\mathbf{G}^T \Psi \mathbf{C} \approx \mathbf{F}(M)^T \Phi$$

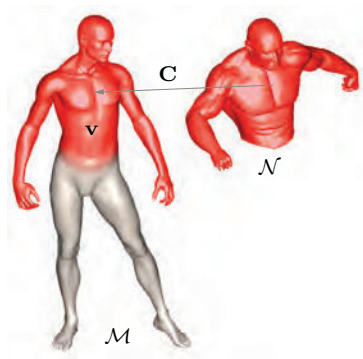


Partial functional maps

- Model shape \mathcal{M}
- Query shape \mathcal{N}
- Part $M \subseteq \mathcal{M} \approx$ isometric to \mathcal{N}
- Data \mathbf{F}, \mathbf{G}
- Partial functional map

$$\mathbf{G}^T \Psi \mathbf{C} \approx \mathbf{F}^T \text{diag}(\mathbf{v}) \Phi$$

$\mathbf{v} \in \mathcal{F}(\mathcal{M})$ indicator function of M



Partial functional maps

- **Model** shape \mathcal{M}
- **Query** shape \mathcal{N}
- **Part** $M \subseteq \mathcal{M} \approx$ isometric to \mathcal{N}
- Data \mathbf{F}, \mathbf{G}
- **Partial functional map**

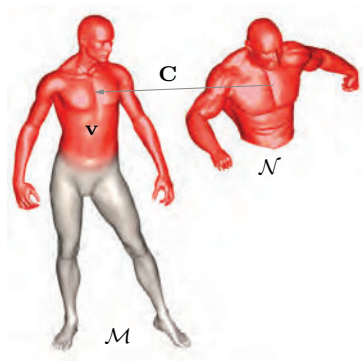
$$\mathbf{G}^\top \Psi \mathbf{C} \approx \mathbf{F}^\top \text{diag}(\eta(\mathbf{v})) \Phi$$

$\mathbf{v} \in \mathcal{F}(\mathcal{M})$ indicator function of M

$$\eta(t) = \frac{1}{2}(\tanh(2t) + 1)$$

Optimization problem w.r.t. correspondence \mathbf{C} and part \mathbf{v}

$$\min_{\mathbf{C}, \mathbf{v}} \|\mathbf{G}^\top \Psi \mathbf{C} - \mathbf{F}^\top \text{diag}(\eta(\mathbf{v})) \Phi\|_{2,1} + \rho_{\text{corr}}(\mathbf{C}) + \rho_{\text{part}}(\mathbf{v})$$



Partial functional maps

$$\min_{\mathbf{C}, \mathbf{v}} \|\mathbf{G}^\top \Psi \mathbf{C} - \mathbf{F}^\top \text{diag}(\eta(\mathbf{v})) \Phi\|_{2,1} + \rho_{\text{corr}}(\mathbf{C}) + \rho_{\text{part}}(\mathbf{v})$$

Partial functional maps

$$\min_{\mathbf{C}, \mathbf{v}} \|\mathbf{G}^\top \Psi \mathbf{C} - \mathbf{F}^\top \text{diag}(\eta(\mathbf{v})) \Phi\|_{2,1} + \rho_{\text{corr}}(\mathbf{C}) + \rho_{\text{part}}(\mathbf{v})$$

- **Part regularization**

$$\rho_{\text{part}}(\mathbf{v}) = \mu_1 \left(|\mathcal{N}| - \int_{\mathcal{M}} \eta(\mathbf{v}) dm \right)^2 + \mu_2 \int_{\mathcal{M}} \xi(\mathbf{v}) \|\nabla_{\mathcal{M}} \mathbf{v}\| dm$$

where $\xi(t) \approx \delta(\eta(t) - \frac{1}{2})$

Partial functional maps

$$\min_{\mathbf{C}, \mathbf{v}} \|\mathbf{G}^\top \Psi \mathbf{C} - \mathbf{F}^\top \text{diag}(\eta(\mathbf{v})) \Phi\|_{2,1} + \rho_{\text{corr}}(\mathbf{C}) + \rho_{\text{part}}(\mathbf{v})$$

- **Part regularization**

$$\rho_{\text{part}}(\mathbf{v}) = \underbrace{\mu_1 \left(|\mathcal{N}| - \int_{\mathcal{M}} \eta(\mathbf{v}) dm \right)^2}_{\text{area preservation}} + \underbrace{\mu_2 \int_{\mathcal{M}} \xi(\mathbf{v}) \|\nabla_{\mathcal{M}} \mathbf{v}\| dm}_{\text{Mumford-Shah}}$$

where $\xi(t) \approx \delta \left(\eta(t) - \frac{1}{2} \right)$

Partial functional maps

$$\min_{\mathbf{C}, \mathbf{v}} \|\mathbf{G}^\top \Psi \mathbf{C} - \mathbf{F}^\top \text{diag}(\eta(\mathbf{v})) \Phi\|_{2,1} + \rho_{\text{corr}}(\mathbf{C}) + \rho_{\text{part}}(\mathbf{v})$$

- **Part regularization**

$$\rho_{\text{part}}(\mathbf{v}) = \underbrace{\mu_1 \left(|\mathcal{N}| - \int_{\mathcal{M}} \eta(\mathbf{v}) dm \right)^2}_{\text{area preservation}} + \underbrace{\mu_2 \int_{\mathcal{M}} \xi(\mathbf{v}) \|\nabla_{\mathcal{M}} \mathbf{v}\| dm}_{\text{Mumford-Shah}}$$

where $\xi(t) \approx \delta\left(\eta(t) - \frac{1}{2}\right)$

- **Correspondence regularization**

$$\rho_{\text{corr}}(\mathbf{C}) = \mu_3 \|\mathbf{C} \circ \mathbf{W}\|_{\text{F}}^2 + \mu_4 \sum_{i \neq j} (\mathbf{C}^\top \mathbf{C})_{ij}^2 + \mu_5 \sum_i ((\mathbf{C}^\top \mathbf{C})_{ii} - d_i)^2$$

Partial functional maps

$$\min_{\mathbf{C}, \mathbf{v}} \|\mathbf{G}^\top \Psi \mathbf{C} - \mathbf{F}^\top \text{diag}(\eta(\mathbf{v})) \Phi\|_{2,1} + \rho_{\text{corr}}(\mathbf{C}) + \rho_{\text{part}}(\mathbf{v})$$

- **Part regularization**

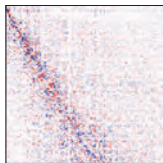
$$\rho_{\text{part}}(\mathbf{v}) = \underbrace{\mu_1 \left(|\mathcal{N}| - \int_{\mathcal{M}} \eta(\mathbf{v}) dm \right)^2}_{\text{area preservation}} + \underbrace{\mu_2 \int_{\mathcal{M}} \xi(\mathbf{v}) \|\nabla_{\mathcal{M}} \mathbf{v}\| dm}_{\text{Mumford-Shah}}$$

where $\xi(t) \approx \delta \left(\eta(t) - \frac{1}{2} \right)$

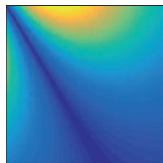
- **Correspondence regularization**

$$\rho_{\text{corr}}(\mathbf{C}) = \underbrace{\mu_3 \|\mathbf{C} \circ \mathbf{W}\|_{\text{F}}^2}_{\text{slant}} + \underbrace{\mu_4 \sum_{i \neq j} (\mathbf{C}^\top \mathbf{C})_{ij}^2}_{\approx \text{orthogonality}} + \underbrace{\mu_5 \sum_i ((\mathbf{C}^\top \mathbf{C})_{ii} - d_i)^2}_{\text{rank} \approx r}$$

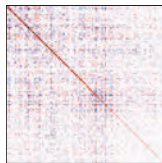
Structure of partial functional correspondence



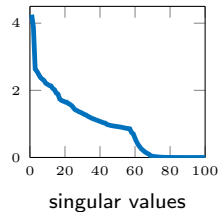
C



W



$C^T C$



Alternating minimization

- **C-step:** fix \mathbf{v}^* , solve for correspondence \mathbf{C}

$$\min_{\mathbf{C}} \|\mathbf{G}^\top \Psi \mathbf{C} - \mathbf{F}^\top \text{diag}(\eta(\mathbf{v}^*)) \Phi\|_{2,1} + \rho_{\text{corr}}(\mathbf{C})$$

- **v-step:** fix \mathbf{C}^* , solve for part \mathbf{v}

$$\min_{\mathbf{v}} \|\mathbf{G}^\top \Psi \mathbf{C}^* - \mathbf{F}^\top \text{diag}(\eta(\mathbf{v})) \Phi\|_{2,1} + \rho_{\text{part}}(\mathbf{v})$$

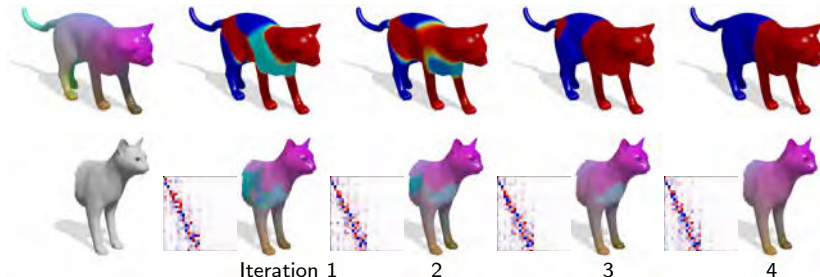
Alternating minimization

- **C-step:** fix \mathbf{v}^* , solve for correspondence \mathbf{C}

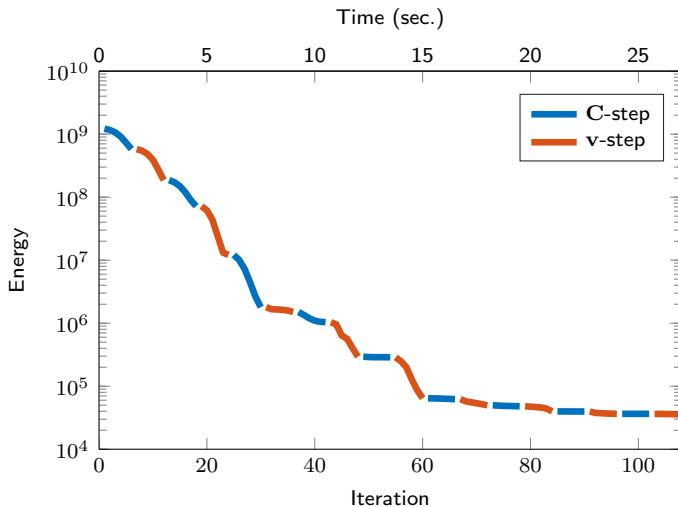
$$\min_{\mathbf{C}} \|\mathbf{G}^T \Psi \mathbf{C} - \mathbf{F}^T \text{diag}(\eta(\mathbf{v}^*)) \Phi\|_{2,1} + \rho_{\text{corr}}(\mathbf{C})$$

- **v-step:** fix \mathbf{C}^* , solve for part \mathbf{v}

$$\min_{\mathbf{v}} \|\mathbf{G}^T \Psi \mathbf{C}^* - \mathbf{F}^T \text{diag}(\eta(\mathbf{v})) \Phi\|_{2,1} + \rho_{\text{part}}(\mathbf{v})$$



Example of convergence

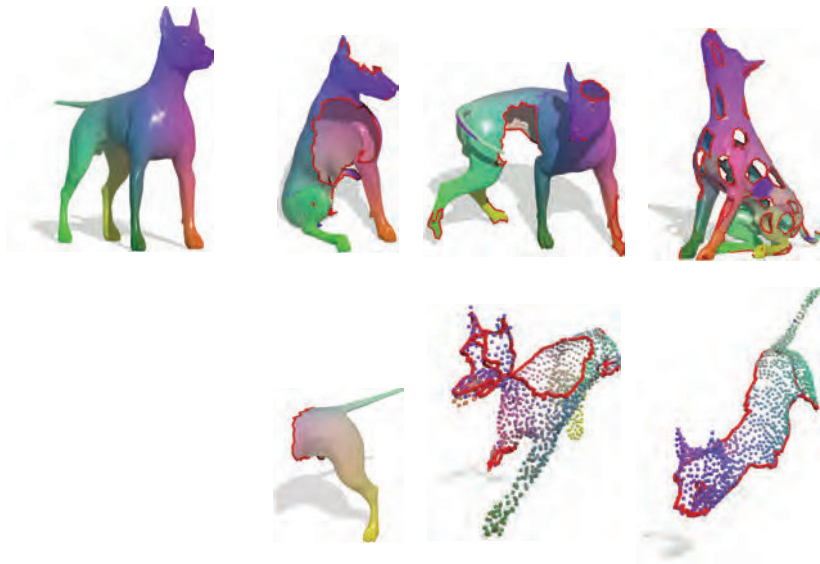


Examples of partial functional maps



Rodolà, Cosmo, B, Torsello, Cremers 2016

Examples of partial functional maps



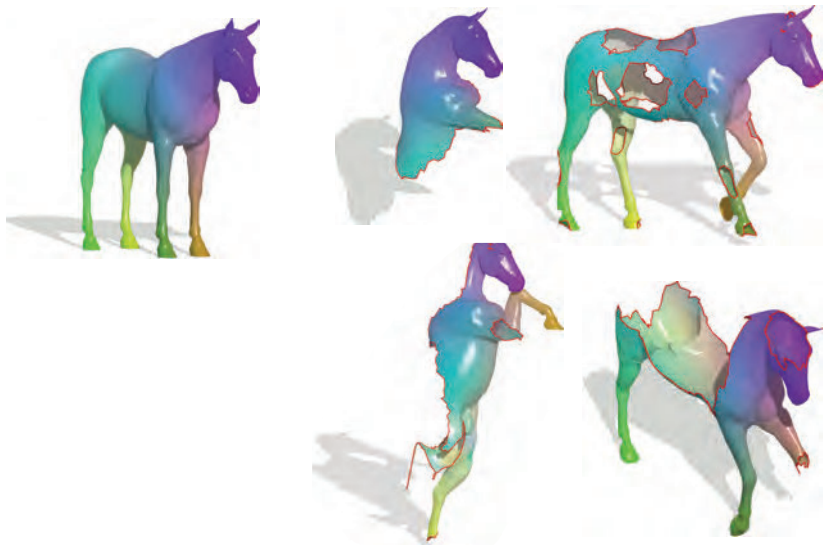
Rodolà, Cosmo, B, Torsello, Cremers 2016

Examples of partial functional maps



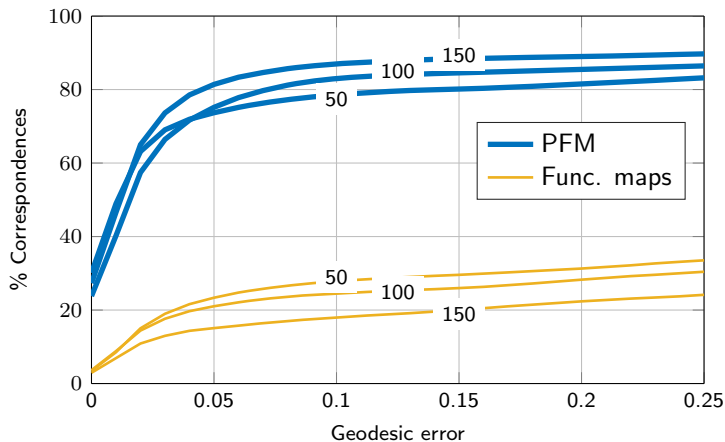
Rodolà, Cosmo, B, Torsello, Cremers 2016

Examples of partial functional maps



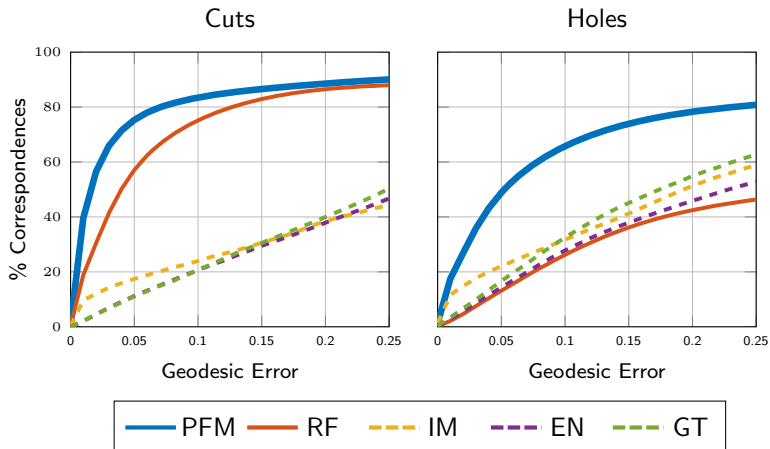
Rodolà, Cosmo, B, Torsello, Cremers 2016

Partial functional maps vs Functional maps



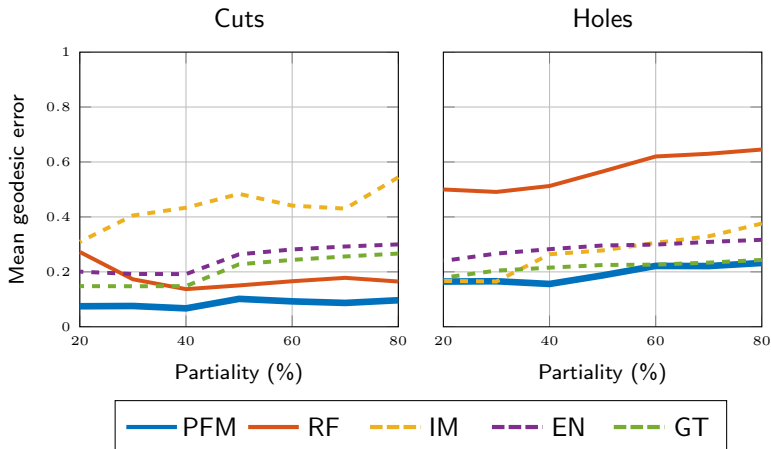
Correspondence performance for different rank values k

Partial correspondence performance



SHREC'16 Partial Matching benchmark Rodolà et al. 2016; Methods: Rodolà, Cosmo, B, Torsello, Cremers 2016 (PFM); Sahillioglu, Yemez 2012 (IM); Rodolà, Bronstein, Albarelli, Bergamasco, Torsello 2012 (GT); Rodolà et al. 2013 (EN); Rodolà et al. 2014 (RF)

Partial correspondence performance



SHREC'16 Partial Matching benchmark Rodolà et al. 2016; Methods: Rodolà, Cosmo, B, Torsello, Cremers 2016 (PFM); Sahillioglu, Yemez 2012 (IM); Rodolà, Bronstein, Albarelli, Bergamasco, Torsello 2012 (GT); Rodolà et al. 2013 (EN); Rodolà et al. 2014 (RF)

Deep learning + Partial functional maps



Correspondence



Correspondence error

Deep learning + Partial functional maps



Correspondence



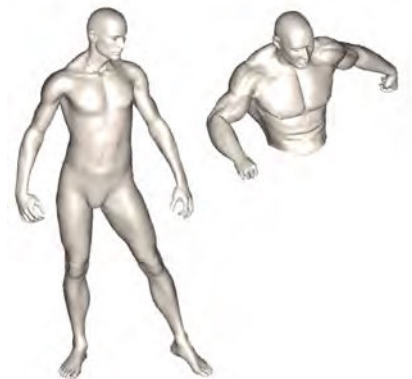
Correspondence error

Outline

- Background: Spectral analysis on manifolds
- Functional correspondence
- Partial functional correspondence
- **Non-rigid puzzles**

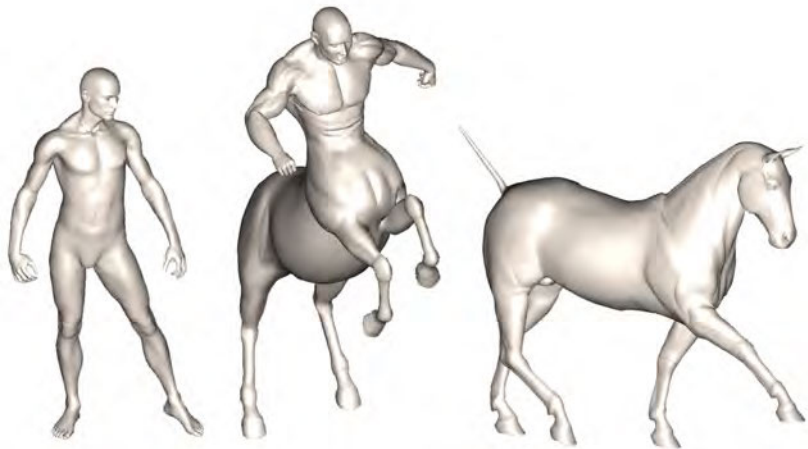


Partial correspondence



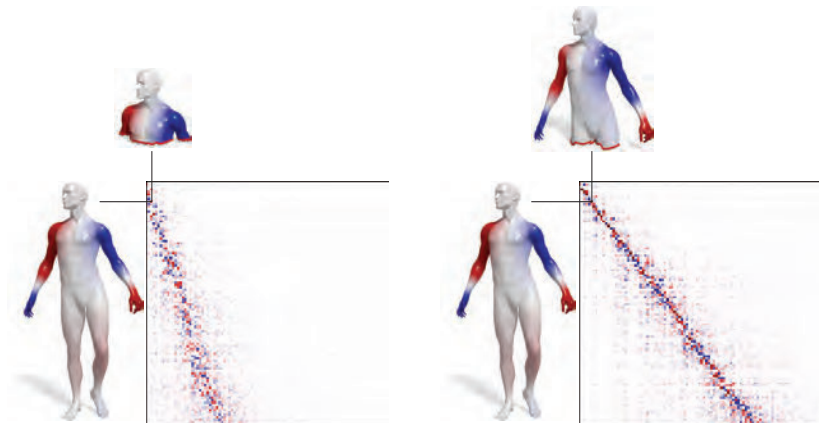
Rodolà, Cosmo, B, Torsello, Cremers 2016

Non-rigid puzzle



Litani, Rodolà, BB, Cremers 2016

Partial Laplacian eigenvectors



Functional correspondence matrix C

Key observation



C_{NN}

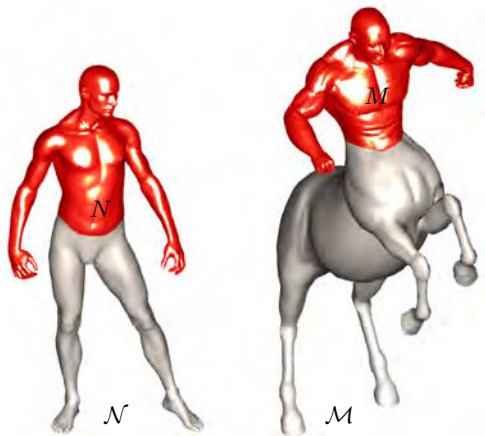
$$\text{slant} \propto \frac{|N|}{|\mathcal{N}|}$$



C_{MM}

$$\text{slant} \propto \frac{|M|}{|\mathcal{M}|}$$

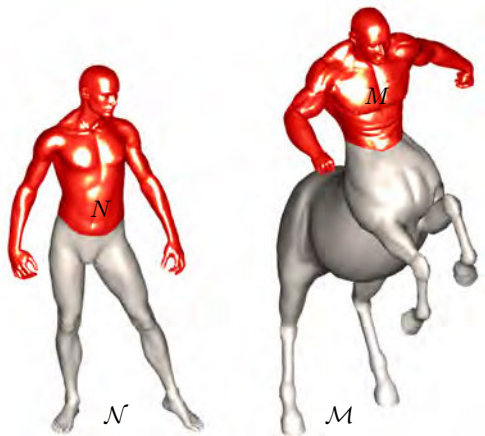
Key observation



$$\mathbf{C}_{NM} = \mathbf{C}_{NN} \mathbf{C}_{NM} \mathbf{C}_{MM}$$

$$\text{slant} \propto \frac{|N|}{|\mathcal{N}|} \frac{|\mathcal{M}|}{|M|}$$

Key observation



$$\mathbf{C}_{NM} = \mathbf{C}_{NN}\mathbf{C}_{NM}\mathbf{C}_{MM}$$

$$\text{slant} \propto \frac{|N|}{|N|} \frac{|M|}{|M|} = \frac{|N|}{|M|}$$

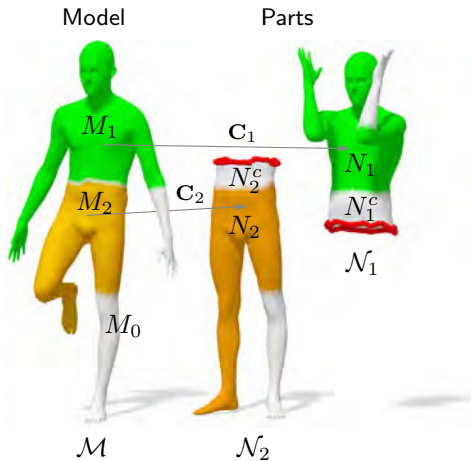
Non-rigid puzzles problem formulation

Input

- Model \mathcal{M}
- Parts $\mathcal{N}_1, \dots, \mathcal{N}_p$

Output

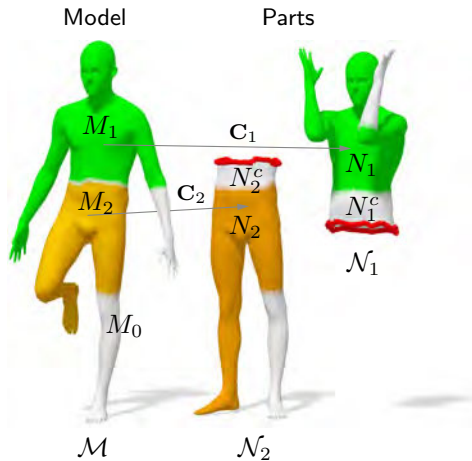
- Segmentation $M_i \subseteq \mathcal{M}$
- Located parts $N_i \subseteq \mathcal{N}_i$
- Correspondences \mathbf{C}_i
- Clutter N_i^c
- Missing parts M_0



Non-rigid puzzles problem formulation

- Data $\mathbf{F}_i, \mathbf{G}_i$
- Model basis $\Phi, \Phi(M_i)$
- Part bases $\Psi_i, \Psi_i(N_i)$
- Data term

$$\mathbf{F}_i^\top \Phi(M_i) \approx \mathbf{G}_i^\top \Psi_i(N_i) \mathbf{C}_i$$



Non-rigid puzzles problem formulation

$$\begin{aligned} \min_{\substack{\mathbf{C}_i \\ M_i \subseteq \mathcal{M}, N_i \subseteq \mathcal{N}_i}} \quad & \sum_{i=1}^p \|\mathbf{G}_i^\top \boldsymbol{\Psi}_i(N_i) \mathbf{C}_i - \mathbf{F}_i^\top \boldsymbol{\Phi}(M_i)\|_{2,1} \\ & + \lambda_{\mathcal{M}} \sum_{i=0}^p \rho_{\text{part}}(M_i) + \lambda_{\mathcal{N}} \sum_{i=1}^p \rho_{\text{part}}(N_i) \\ & + \lambda_{\text{corr}} \sum_{i=1}^p \rho_{\text{corr}}(\mathbf{C}_i) \\ \text{s.t.} \quad & M_i \cap M_j = \emptyset \quad \forall i \neq j \\ & M_0 \cup M_1 \cup \dots \cup M_p = \mathcal{M} \\ & |M_i| = |N_i| \geq \alpha |\mathcal{N}_i|, \end{aligned}$$

Non-rigid puzzles problem formulation

$$\begin{aligned} \min_{\substack{\mathbf{C}_i \\ \mathbf{u}_i, \mathbf{v}_i}} \quad & \sum_{i=1}^p \|\mathbf{G}_i^\top \text{diag}(\eta(\mathbf{u}_i)) \Psi_i \mathbf{C}_i - \mathbf{F}_i^\top \text{diag}(\eta(\mathbf{v}_i)) \Phi\|_{2,1} \\ & + \lambda_{\mathcal{M}} \sum_{i=0}^p \rho_{\text{part}}(\eta(\mathbf{v}_i)) + \lambda_{\mathcal{N}} \sum_{i=1}^p \rho_{\text{part}}(\eta(\mathbf{u}_i)) \\ & + \lambda_{\text{corr}} \sum_{i=1}^p \rho_{\text{corr}}(\mathbf{C}_i) \\ \text{s.t.} \quad & \sum_{i=1}^p \eta(\mathbf{u}_i) = 1 \\ & \mathbf{a}_{\mathcal{M}}^\top \mathbf{u}_i = \mathbf{a}_{\mathcal{N}}^\top \mathbf{v}_i \geq \alpha \mathbf{a}_{\mathcal{N}_i}^\top \mathbf{1} \end{aligned}$$

Convergence example



Outer iteration 1

Convergence example



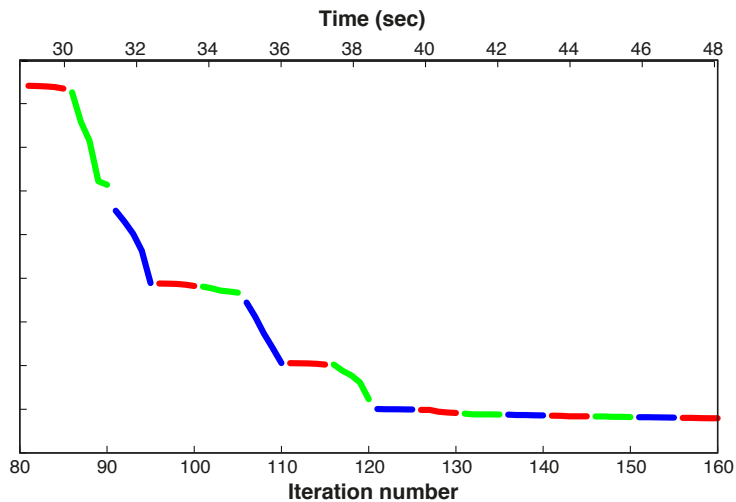
Outer iteration 2

Convergence example



Outer iteration 3

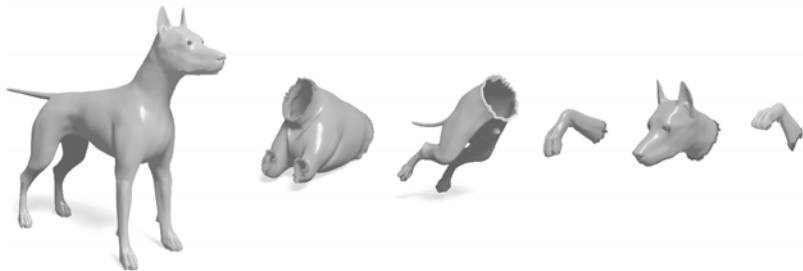
Convergence example



Litani, Rodolà, BB, Cremers 2016

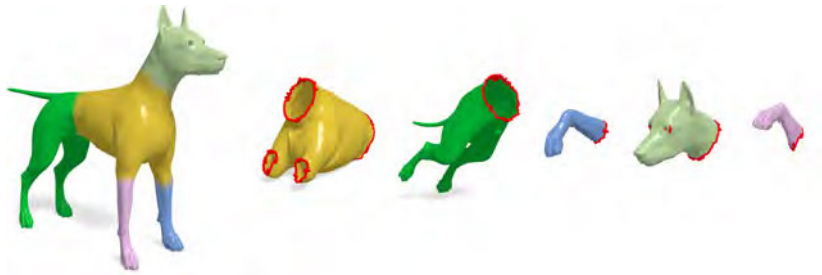
“Perfect puzzle” example

Model/Part	Synthetic (TOSCA)
Transformation	Isometric
Clutter	No
Missing part	No
Data term	Dense (SHOT)



“Perfect puzzle” example

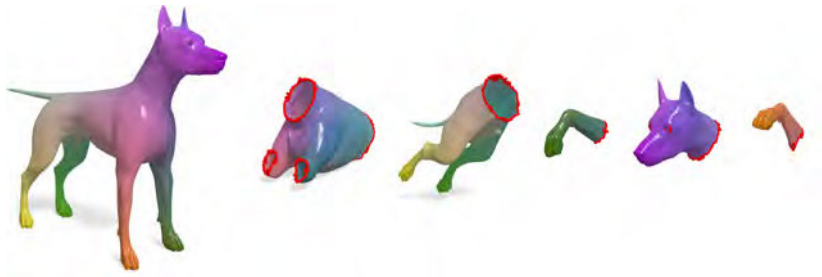
Model/Part	Synthetic (TOSCA)
Transformation	Isometric
Clutter	No
Missing part	No
Data term	Dense (SHOT)



Segmentation

“Perfect puzzle” example

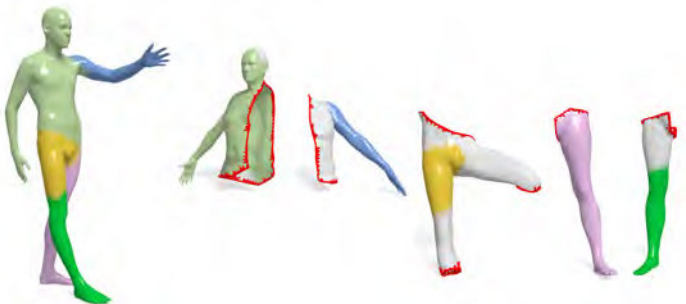
Model/Part	Synthetic (TOSCA)
Transformation	Isometric
Clutter	No
Missing part	No
Data term	Dense (SHOT)



Correspondence

Overlapping parts example

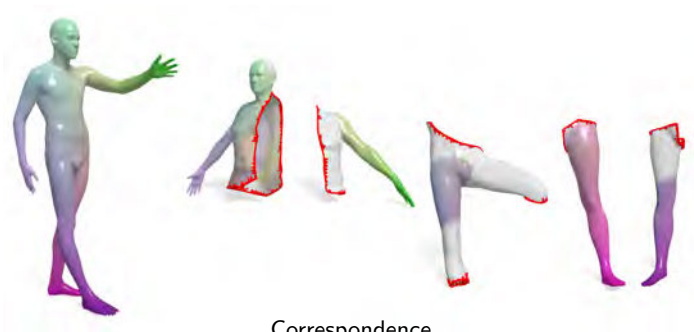
Model/Part	Synthetic (FAUST)
Transformation	Near-isometric
Clutter	Yes (overlap)
Missing part	No
Data term	Dense (SHOT)



Segmentation

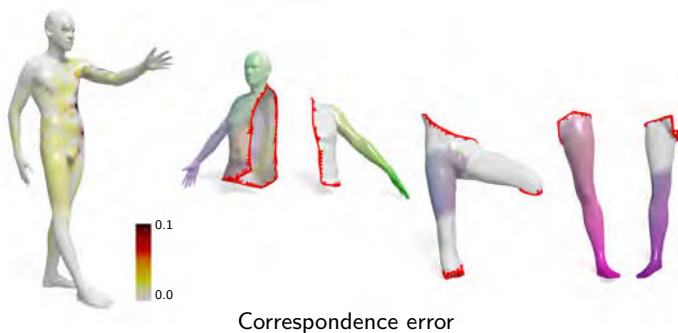
Overlapping parts example

Model/Part	Synthetic (FAUST)
Transformation	Near-isometric
Clutter	Yes (overlap)
Missing part	No
Data term	Dense (SHOT)



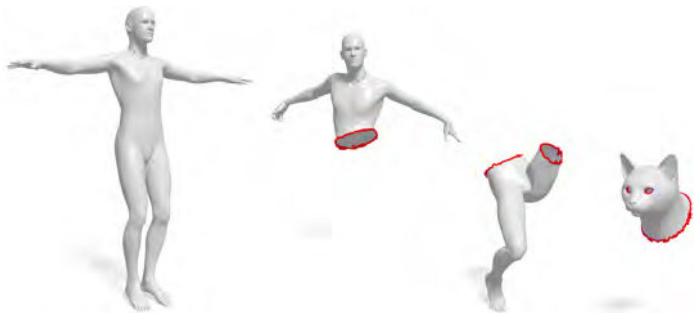
Overlapping parts example

Model/Part	Synthetic (FAUST)
Transformation	Near-isometric
Clutter	Yes (overlap)
Missing part	No
Data term	Dense (SHOT)



Missing parts example

Model/Part	Synthetic (TOSCA)
Transformation	Isometric
Clutter	Yes (extra part)
Missing part	Yes
Data term	Dense (SHOT)



Missing parts example

Model/Part	Synthetic (TOSCA)
Transformation	Isometric
Clutter	Yes (extra part)
Missing part	Yes
Data term	Dense (SHOT)



Segmentation

Missing parts example

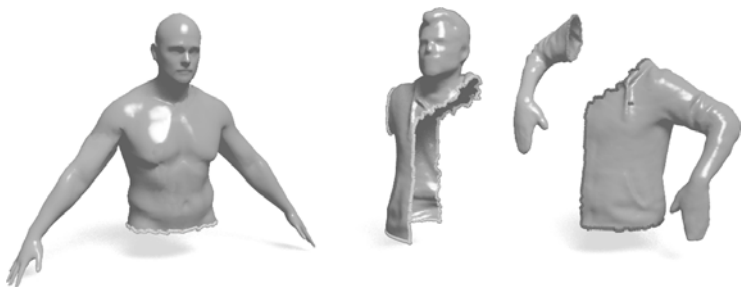
Model/Part	Synthetic (TOSCA)
Transformation	Isometric
Clutter	Yes (extra part)
Missing part	Yes
Data term	Dense (SHOT)



Correspondence

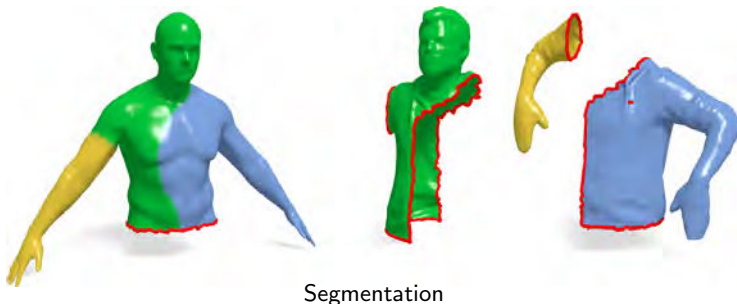
Scanned data example

Model/Part	Synthetic (TOSCA) / Scan
Transformation	Non-Isometric
Clutter	No
Missing part	No
Data term	Sparse deltas

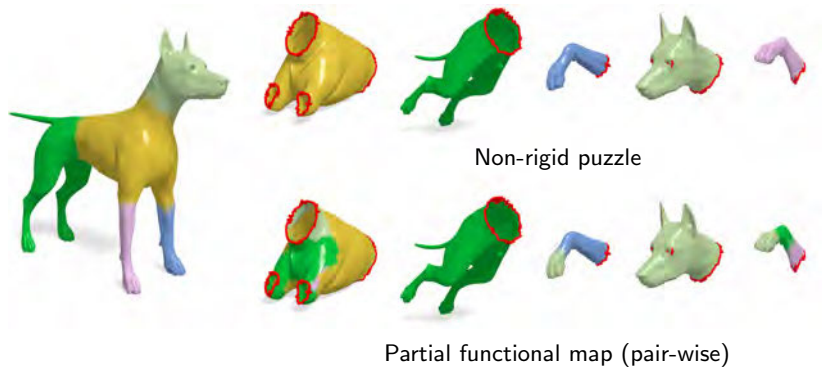


Scanned data example

Model/Part	Synthetic (TOSCA) / Scan
Transformation	Non-Isometric
Clutter	No
Missing part	No
Data term	Sparse deltas



Non-rigid puzzle vs Partial functional map



Rodolà, Cosmo, B, Torsello, Cremers 2016; Litani, Rodolà, BB, Cremers 2016

Summary

- New insights about spectral properties of Laplacians
- Extension of functional correspondence framework to the partial setting
- Practically working methods for challenging shape correspondence settings
- Code available (SGP Reproducibility Stamp)
- Some over-engineering - can be done simpler! (stay tuned...)



E. Rodolà



A. Bronstein



O. Litany



L. Cosmo



A. Torsello



D. Cremers



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THE INNOVATION PROMOTION AGENCY

HASLERSTIFTUNG



Thank you!