STABILITY THEOREMS

& APPLICATIONS TO PHASE TRANSITIONS

F. MAGGI ICTP TRIESTE

CALC VAR OPT TRAN & GMT

U. LYON 4/7/16

TWO MODELS FOR DROPLETS FORMATION

1 GAUSS FREE ENERGY — BASED ON

SHARP INTERFACE MODEL — CIRAOLO-M. 2015

CLASSICAL CAPILLARITY THEORY — KRUMMEL-M. 2016

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2 GATES-PENROSE-LEBOWITZ (GPL) FREE ENERGY

DIFFUSED INTERFACE MODEL

STATISTICAL MECHANICS

NONLOCAL RELATIVE OF ALLEN-CAHN FREE ENERGY

BASED ON CARLEN-M. 2015

FIGALLI-M.-MOONEY 2016

PART ONE

$E \subseteq \mathbb{R}^{n+1}$ DROPLET REGION

VOLUME OF E = $|E| = m$ FIXED

$$\underbrace{\text{GAUSS FREE ENERGY - NO CONTAINER}} = \text{SURFACE TENSION} + \text{POTENTIAL ENERGY} = \underbrace{P(E)}_{\text{PERIMETER OF } E \,=\, \mathcal{H}^n(\partial E)} + \int_E g(x)\,dx$$

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$\boxed{m \text{ IS SMALL!}}$ $\Rightarrow$ $\boxed{P(E) = O(m^{n/n+1}) \gg \int_E g = O(m)}$

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GLOBAL MINIMIZERS $\Rightarrow$ ALMOST ISOPERIMETRIC

CRITICAL POINTS $\Rightarrow$ ALMOST CONSTANT MEAN CURV.

GLOBAL MINIMIZERS $\Rightarrow$ ALMOST ISOPERIMETRIC

$$1 + O\left(m^{1/(n+1)}\right) \geq \frac{P(E)}{c_{iso}\,|E|^{n/(n+1)}} \geq 1$$

ENERGY COMPARISON WITH BALLS

ISOPERIMETRIC INEQUALITY

GLOBAL MINIMIZERS $\Rightarrow$ ALMOST ISOPERIMETRIC

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IMPROVED ISOPERIMETRY

FUSCO M. PRATELLI ('08)

$$\frac{P(E)}{c_{iso}|E|^{n/(n+1)}} \geq 1 + c(n) \min_{x \in \mathbb{R}^{n+1}} \left[\frac{|E \Delta B^{(m)}(x)|}{m} \right]^2$$

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ADDITIONAL VARIATIONAL ARGUMENTS
FIGALLI M. (10)

∂E IS QUANTITAVELY C^2-CLOSE TO A SPHERE

IN A CONTAINER M.- MIHAILA (15)

CRITICAL POINTS $\Rightarrow$ ALMOST CONSTANT MEAN CURV.

$$\delta_{CMC}(E) = \left\| \frac{H_{\partial E}}{H^0_{\partial E}} - 1 \right\|_{C^0(\partial E)} = O\left(m^{1/n+1}\right)$$

C^0- CONST. MEAN CURV. DEFICIT

$H_{\partial E}$ MEAN CURV. OF ∂E

$$H^0_{\partial E} = \frac{n\,P(E)}{(n+1)\,|E|}$$

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CIRAOLO-VEZZONI (15) $\begin{cases} \partial E \text{ EXT/INT BALL RADIUS } \rho > 0 \\ H^0_{\partial E} = n = H_{\partial B} \quad B = \text{UNIT BALL} \end{cases}$

$$\Rightarrow \quad hd\left(\partial E, \partial B_1(x)\right) \leq C(n, \rho, P(E))\, \delta_{CMC}(E)$$

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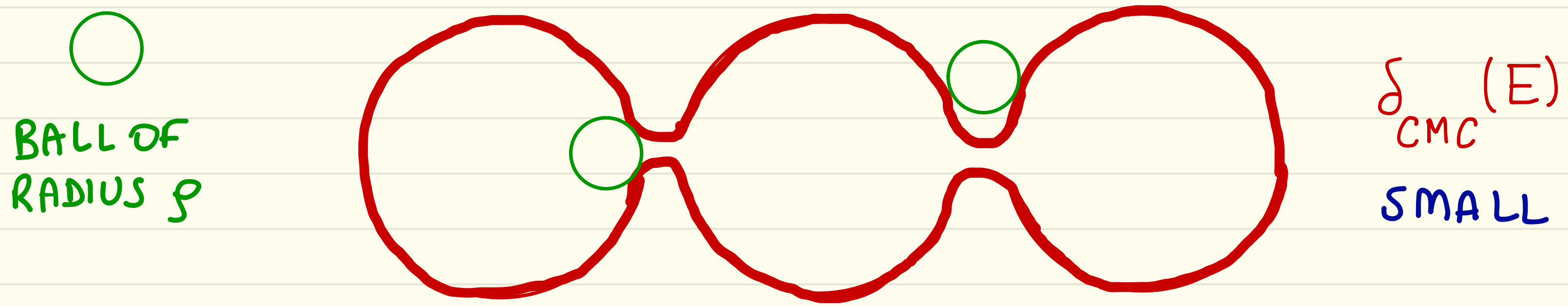
EXT/INT BALL RADIUS $\rho > 0$ NOT TRUE ON ALMOST CMC

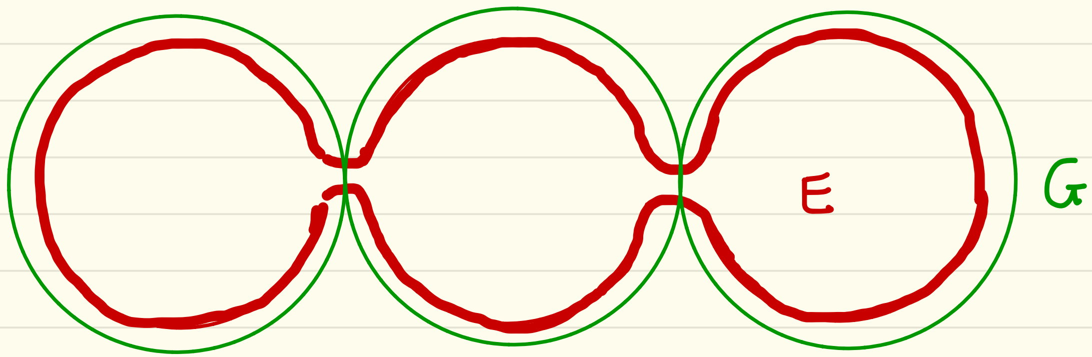


CRITICAL POINTS $\Rightarrow$ ALMOST CONSTANT MEAN CURV.

CIRAOLO M. (15) $\begin{cases} H^{o}_{\partial E} = n & P(E) \leq (L+\tau) P(B) \quad L \in \mathbb{N} \quad 0<\tau<1 \\ \delta_{CMC}(E) \leq \delta_0(n,L,\tau) \end{cases}$

$\Rightarrow$ ∂E QUANTITAVELY $C^{1,\alpha}$ CLOSE TO UNION OF UNIT RADIUS TANGENT BALLS G

E.G. $\dfrac{|E \Delta G|}{|E|} \leq C(n) L^2 \, \delta_{CMC}(E)^{\frac{1}{2n+4}}$ & MANY OTHER ESTIMATES



CRITICAL POINTS $\Longrightarrow$ ALMOST CONSTANT MEAN CURV.

KRUMMEL-M. 16
$$\begin{cases} H^{\circ}_{\partial E} = n & P(E) \le (1+\tau)\,P(B) \qquad 0<\tau<1 \\ \delta_{CMC}(E) \le \delta_0(n,\tau) & \end{cases}$$

$$\Longrightarrow \quad \partial E = \{(1+u(x))\,x : x \in \partial B_1\} \qquad \|u\|_{C^{1,\alpha}} \le C(n,\tau)\,\delta_{CMC}(E)$$

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IN FACT we can use (AT LEAST IF $n \leq 3$)

$$\delta^{*}_{CMC}(E) = \min \left\{ \left\| \left(\frac{H_{\partial E}}{H^{o}_{\partial E}} - 1 \right)^{+} \right\|_{C^{0}(\partial E)} , \left\| \left(\frac{H_{\partial E}}{H^{o}_{\partial E}} - 1 \right)^{-} \right\|_{L^{2}(\partial E)} \right\}$$

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RELATED TO ALMGREN ISOPERIMETRIC PRINCIPLE

ALMGREN ISOPERIMETRIC PRINCIPLE

(CODIMENSION 1 VERSION)

IF $H_{\partial E} \leq n$ THEN $P(E) \geq P(B_1)$

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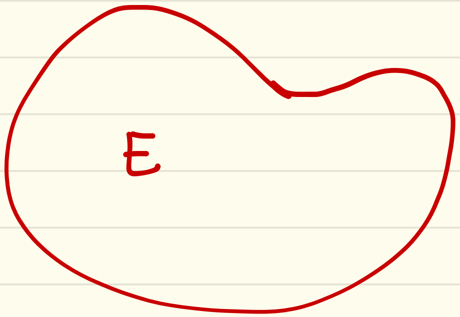

$$P(B_1) = \mathcal{H}^n(S^n) = \int_{\partial A} |\det \nabla v_A|$$

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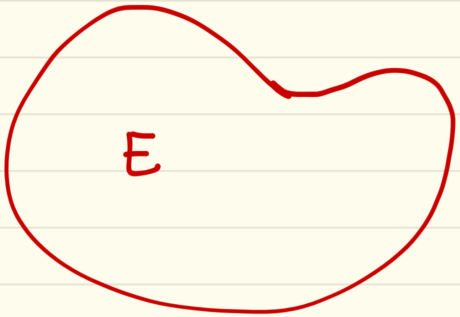

$$P(B_1) = \mathcal{H}^n(S^n) = \int_{\partial A} |\det \nabla \nu_A|$$

$$= \int_{\partial A} K_{\partial A} = \int_{\partial A \cap \partial E} K_{\partial A}$$

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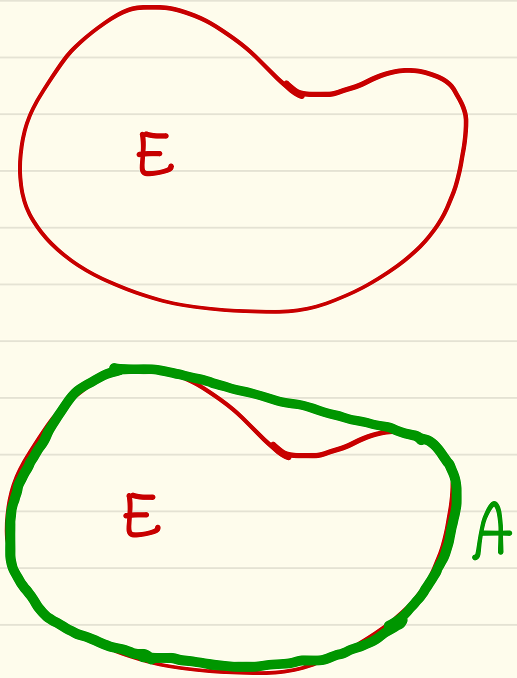

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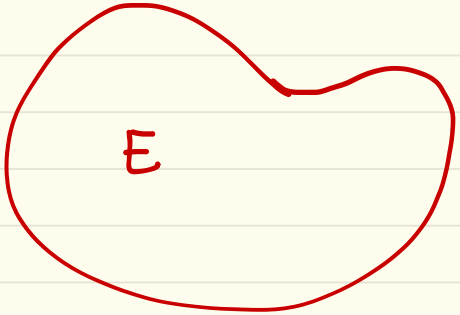

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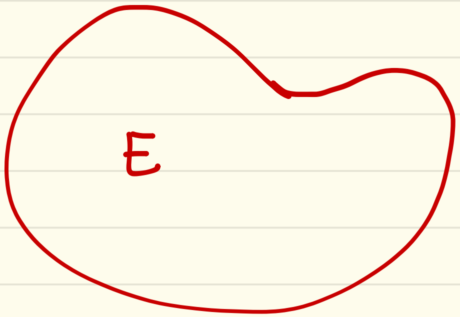

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RMK 1 EQUALITY HOLDS

$\iff$ $E = B_1(x)$

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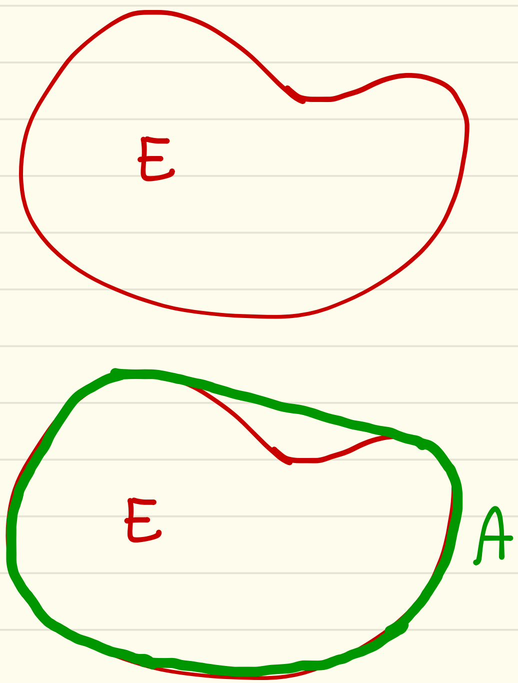

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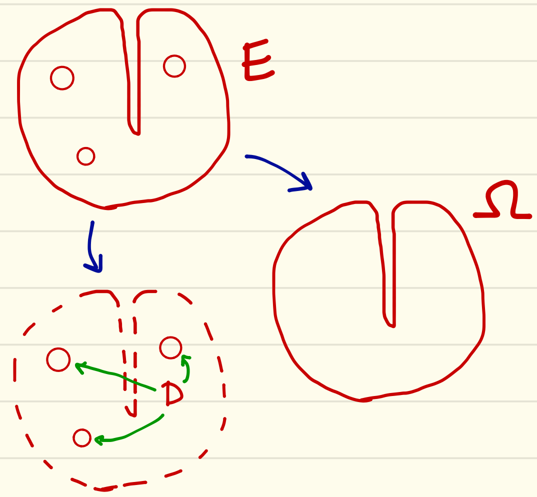
RMK 1 EQUALITY HOLDS

$\iff$ $E = B_1(x)$

RMK 2 YES! IT REMINDS A LOT ABP!

KRUMMEL-M. 16 IF $H_{\partial E} \leq n$ & $\delta(E) = P(E) - P(B_1) \leq \delta_0(n)$ SMALL



THEN $\partial E = \partial D \cup \partial \Omega$

D "DUST SET" $P(D) \leq C(n)\, \delta(E)$

$\partial\Omega$ CONNECTED

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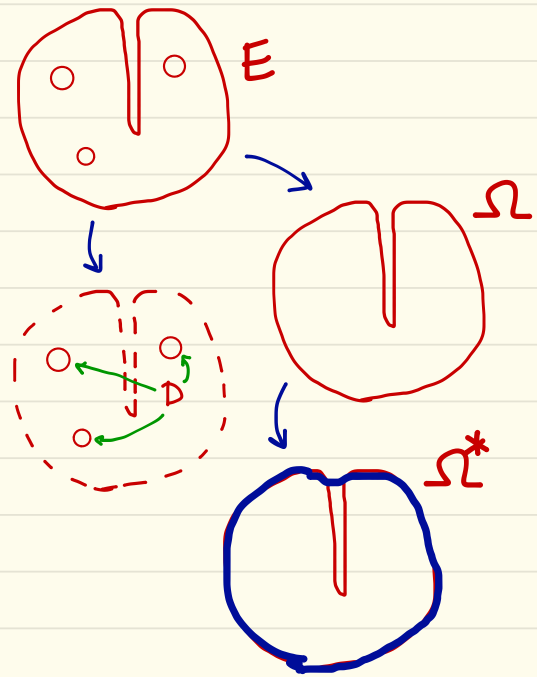

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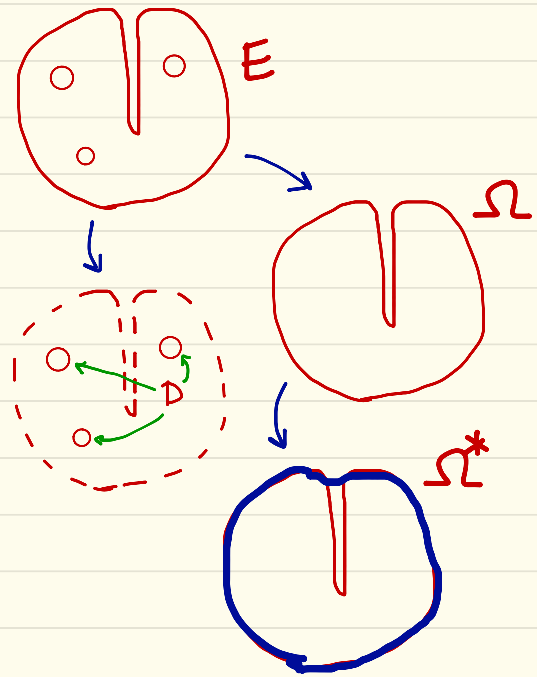
$\partial\Omega$ CONNECTED

$\Omega \subseteq \Omega^*$ $\quad |H_{\partial\Omega^*}| \leq n$

$$|\Omega^* \setminus \Omega| + \mathcal{H}^n(\partial\Omega^* \setminus \partial\Omega) \leq C(n)\, \delta(E)$$

KRUMMEL-M. 16 IF $H_{\partial E} \le n$ & $\delta(E) = P(E) - P(B_1) \le \delta_0(n)$ SMALL



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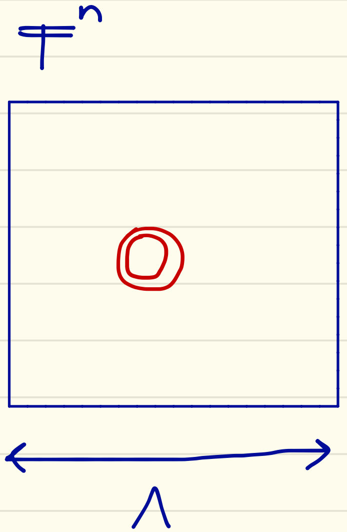
$\partial\Omega^* = \{(1+u(x))\,x : x \in \partial B_1\}$

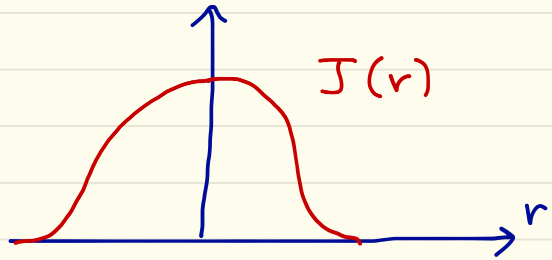
$\|u\|_{W^{1,1}} + \|u^+\|_{C^0} \le C(n)\,\delta(E)$ $\qquad \|u\|_{C^0} \le C(n) \begin{cases} \delta(E) & n=1 \\ \delta(E)\log\left(C(2)/\delta(E)\right) & n=2 \\ \delta(E)^{1/n-1} & n \ge 3 \end{cases}$

PART TWO GPL FREE ENERGY



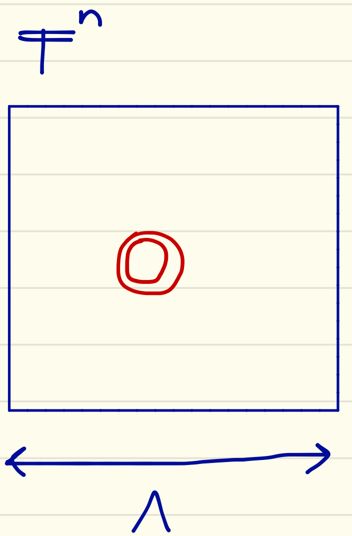
$$u : T^n \longrightarrow (-1,1)$$

$$\int_{T^n} u = m \in (-1,1)$$



$$\mathcal{F}(u) = \iint_{T^n \times T^n} |u(x) - u(y)|^2 J(|x-y|)\,dx\,dy + \int_{T^n} W(u(x))\,dx$$

PART TWO GPL FREE ENERGY



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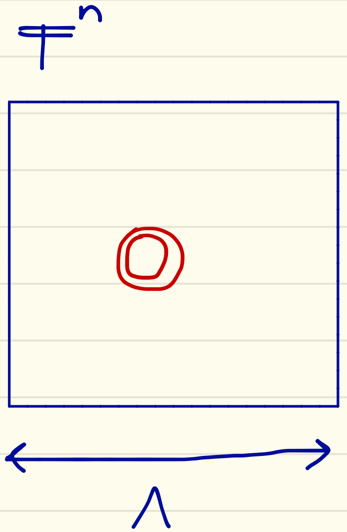
IF m/Λ SMALL* & u LOW ENERGY

THEN $u(x) \approx$ SHARP TRANSITION ALONG A SMALL SPHERE

PART TWO GPL FREE ENERGY



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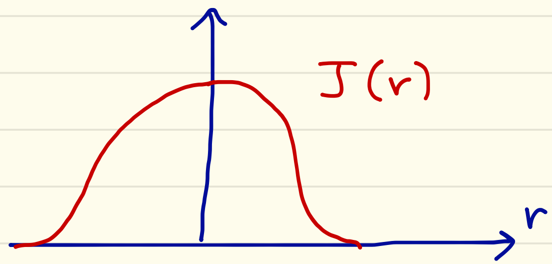

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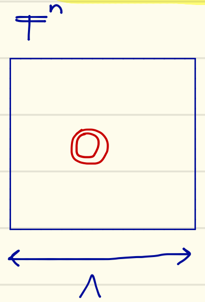
IF m/Λ SMALL* & u LOW ENERGY

THEN $u(x) \approx$ SHARP TRANSITION ALONG A SMALL SPHERE

LOW ENERGY = LIKELY TO BE OBSERVED

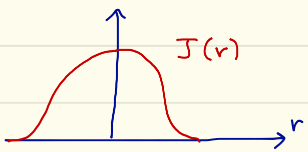
* BUT NOT TOO MUCH... OTHERWISE UNIFORM STATE ; SEE CARLEN SURVEY

PART TWO GPL FREE ENERGY

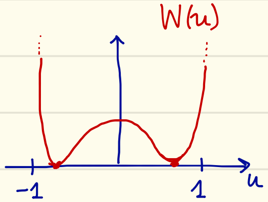


$u: T^n \to (-1,1)$ $\quad \int_{T^n} u = m \in (-1,1)$ $\quad \int_{T^n} J = 1$

$$\mathcal{F}(u) = \iint_{T^n \times T^n} |u(x)-u(y)|^2 J(|x-y|)\,dx\,dy + \int_{T^n} W(u(x))\,dx$$



PART TWO GPL FREE ENERGY ($\mathbb{T}^n \to \mathbb{R}^n$)



$u : \mathbb{R}^n \longrightarrow (-1,1) \qquad \int_{\mathbb{R}^n} u = m \in (-1,1) \qquad \int_{\mathbb{R}^n} J = 1$

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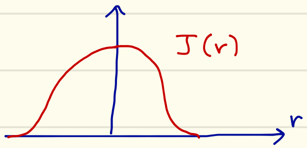
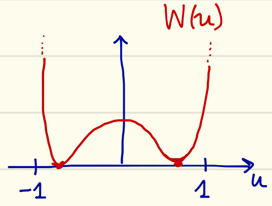


$\underbrace{\geq}_{} \mathcal{F}(u^*) \qquad u^*$ SHWARTZ REARRANG. u

BY **RIESZ REARRANGEMENT INQ.**

PART TWO GPL FREE ENERGY



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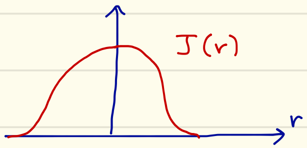

$$\geq \mathcal{F}(u^*) \qquad u^* \text{ SHWARTZ REARRANG. } u$$

BY **RIESZ REARRANGEMENT INQ.**

1. RADIALLY DECREASING — CARLEN - CARVALHO - ESPOSITO
   LOW ENERGY STATES — LEBOWITZ - MARRA (09)

PART TWO GPL FREE ENERGY

$u:\mathbb{R}^n \longrightarrow (-1,1)$ $\quad \int_{\mathbb{R}^n} u = m \in (-1,1)$ $\quad \int_{\mathbb{R}^n} J = 1$

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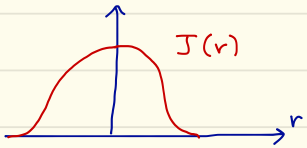

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BY RIESZ REARRANGEMENT INQ.

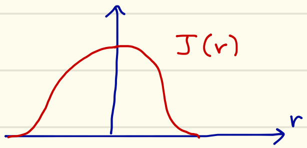
(1) RADIALLY DECREASING LOW ENERGY STATES — CARLEN - CARVALHO-ESPOSITO LEBOWITZ - MARRA (09)

(2) QUANTITATIVE RIESZ REARRANGEMENT INQ — CARLEN - M. (15) BASED ON A QUANTITATIVE BRUNN-MINKOWSKI INEQUALITY

PART TWO GPL FREE ENERGY

$$u: \mathbb{R}^n \longrightarrow (-1,1) \qquad \int_{\mathbb{R}^n} u = m \in (-1,1) \qquad \int_{\mathbb{R}^n} J = 1$$

$$\mathcal{F}(u) = \iint_{\mathbb{R}^n \times \mathbb{R}^n} |u(x)-u(y)|^2 J(|x-y|)\,dx\,dy + \int_{\mathbb{R}^n} W(u(x))\,dx$$



$\underbrace{\geq}_{} \mathcal{F}(u^*)$ $\qquad$ u^* SHWARTZ REARRANG. u

BY RIESZ REARRANGEMENT INQ.

CARLEN'S SURVEY FOR MORE GPL

① RADIALLY DECREASING LOW ENERGY STATES — CARLEN - CARVALHO-ESPOSITO LEBOWITZ - MARRA (09)

② QUANTITATIVE RIESZ REARRANGEMENT INQ — CARLEN - M. (15) BASED ON A QUANTITATIVE BRUNN-MINKOWSKI INEQUALITY

BM INEQUALITY $|E+F|^{1/n} \geq |E|^{1/n} + |F|^{1/n} \quad \forall E,F \subseteq \mathbb{R}^n$

$=$ E, F HOMOTHETIC CONVEX SETS

BM INEQUALITY $\qquad |E+F|^{1/n} \geq |E|^{1/n} + |F|^{1/n} \quad \forall E, F \subseteq \mathbb{R}^n$

$=$ E, F HOMOTHETIC CONVEX SETS

QUANTITATIVE BM

$$\delta(E,F) = \max\left\{\frac{|F|}{|E|}, \frac{|E|}{|F|}\right\}\left(\frac{|E+F|^{1/n}}{|E|^{1/n}+|F|^{1/n}} - 1\right)$$

$$\alpha(E,F) = \inf\left\{\frac{|E \,\Delta\, (x+\lambda F)|}{|E|} \text{ s.t. } |\lambda F| = |E|\right\}$$

BM INEQUALITY $|E+F|^{1/n} \geq |E|^{1/n} + |F|^{1/n} \quad \forall E,F \subseteq \mathbb{R}^n$

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1. FIGALLI M. PRATELLI $\quad$ E, F CONVEX $\quad \delta(E,F) \geq c(n)\,\alpha(E,F)^2$

BM INEQUALITY $|E+F|^{1/n} \geq |E|^{1/n} + |F|^{1/n} \quad \forall E,F \subseteq \mathbb{R}^n$

$=$ E,F HOMOTHETIC CONVEX SETS

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① FIGALLI M. PRATELLI $\quad$ E,F CONVEX $\quad$ $\delta(E,F) \geq c(n)\, \alpha(E,F)^2$

② FIGALLI JERISON CHRIST $\quad$ E,F GENERIC $\quad$ NON-SHARP ESTIMATES

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$=$ E, F HOMOTHETIC CONVEX SETS

QUANTITATIVE BM

$$\delta(E,F) = \max\left\{\frac{|F|}{|E|}, \frac{|E|}{|F|}\right\}\left(\frac{|E+F|^{1/n}}{|E|^{1/n}+|F|^{1/n}} - 1\right)$$

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① FIGALLI M. PRATELLI   E, F CONVEX $\delta(E,F) \geq c(n)\,\alpha(E,F)^2$

② FIGALLI JERISON
CHRIST   E, F GENERIC   NON-SHARP ESTIMATES

③ CARLEN-M.   E GENERIC
F CONVEX $\delta(E,F)\max\left\{1, \frac{|F|}{|E|}\right\}^{4+\frac{2}{n}} \geq c(n)\,\alpha(E,F)^4$

E GENERIC $F = B_r$ $r > 0$ $\Rightarrow$ BM BECOMES

EUCLIDEAN CONCENTRATION INEQUALITY $|I_r(E)| \geq |I_r(B_{r_E})|$ $\forall r > 0$

$I_r(E) = \{ \text{DIST FROM } E < r \}$

E GENERIC $\quad F = B_r \quad r > 0 \Rightarrow$ BM BECOMES

EUCLIDEAN CONCENTRATION INEQUALITY

$$|I_r(E)| \geq |I_r(B_{r_E})| \quad \forall r > 0$$

$$I_r(E) = \{\text{DIST FROM } E < r\}$$

SHARP QUANTITATIVE EUCLIDEAN CONCENTRATION INEQUALITY

FIGALLI-M.-MOONEY ('16)

$$|I_r(E)| \geq |I_r(B_{r_E})| \left\{ 1 + c(n) \min\left\{ \frac{r}{r_E}, \frac{r_E}{r} \right\} \left[\frac{|E \Delta B_{r_E}(x)|}{|E|} \right]^2 \right\}$$

E GENERIC $\quad F = B_r \quad r > 0 \implies$ BM BECOMES

EUCLIDEAN CONCENTRATION INEQUALITY

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* NO SYMMETRIZATION

(FUSCO M PRATELLI)

E GENERIC $\quad F = B_r \quad r>0 \implies$ BM BECOMES

EUCLIDEAN CONCENTRATION INEQUALITY $\quad |I_r(E)| \geq |I_r(B_{r_E})| \quad \forall r>0$

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SHARP QUANTITATIVE EUCLIDEAN CONCENTRATION INEQUALITY $\quad$ FIGALLI-M.-MOONEY (16)

$$|I_r(E)| \geq |I_r(B_{r_E})| \left\{ 1 + c(n) \min\left\{ \frac{r}{r_E}, \frac{r_E}{r} \right\} \left[\frac{|E \Delta B_{r_E}(x)|}{|E|} \right]^2 \right\}$$

* NO SYMMETRIZATION (FUSCO M PRATELLI)

* NO MASS TRANSP. (FIGALLI M PRATELLI)

E GENERIC $\quad F = B_r \quad r > 0 \quad \Rightarrow$ BM BECOMES

EUCLIDEAN CONCENTRATION INEQUALITY $\quad |I_r(E)| \geq |I_r(B_{r_E})| \quad \forall r > 0$

$I_r(E) = \{\text{DIST FROM } E < r\}$

SHARP QUANTITATIVE EUCLIDEAN CONCENTRATION INEQUALITY

FIGALLI-M.-MOONEY ('16)

$$|I_r(E)| \geq |I_r(B_{r_E})| \left\{ 1 + c(n) \min\left\{ \frac{r}{r_E}, \frac{r_E}{r} \right\} \left[\frac{|E \Delta B_{r_E}(x)|}{|E|} \right]^2 \right\}$$

* NO SYMMETRIZATION (FUSCO M PRATELLI)
* NO MASS TRANSP. (FIGALLI M PRATELLI)
* NO REGULARITY (CICALESE LEONARDI)