Volume reconstruction from slices

Elie Bretin[*] François Dayrens[†] Simon Masnou[‡]

May 10, 2017

Abstract

We propose a variational approach for the reconstruction of a volume from slices. The reconstructed set is obtained as a minimizer of a geometric regularity criterion, either the perimeter or the Willmore energy, with inclusion-exclusion constraints associated with the cross-sections. We propose a phase field approximation of this model, and we analyze it when the regularity criterion is the perimeter. We derive simple and accurate numerical schemes for both the perimeter-based and the Willmore-based formulations, and we illustrate with several numerical examples the performances of our approach, which proves to be effective for a large category of constraints.

The aim of this work is to develop and justify a new approach to reconstruct a surface or a volume from a collection of given cross-sections. It is motivated by the many applications in medical imaging (CT or MRI scans) and computer graphics [8, 33].

There is a rich literature on surface or volume reconstruction from cross-sections. Roughly speaking, methods can be divided into two categories. In the first category, a first rough approximation of the surface is found based on topological assumptions, then a more accurate surface is interpolated using either a parametric [8, 9, 25, 29, 31, 35, 33] or an implicit representation [1, 3, 13, 14, 17]. The second category of approaches involves a variational viewpoint: the surface is obtained as the result of an optimization problem with constraints [15, 20, 22, 26, 32, 40] but, in general, without topological assumptions.

We propose in this paper a geometric variational method for the reconstruction of the best possible set E^* fitting (exactly or approximately, see below) a given collection of cross-sections and minimizing a geometric criterion as the perimeter or the Willmore energy. For simplicity, we formulate the problem for planar cross-sections, but as will be seen with the numerical experiments, our approach is general enough to handle also non-planar cross-sections, or even less structured partial data as point clouds

Let us now describe informally our model (a more rigorous formulation will be given later). We assume that we are given in $\mathbb{R}^d$ a finite family of hyperplans $\{\Pi_n\}_n$ on each of which inner and outer constraints $\omega_n^{in}, \omega_n^{out} \subset \Pi_n$ are prescribed. We impose for every reconstruction candidate $E \subset \mathbb{R}^d$ that $\omega_n^{in} \subset E \cap \Pi_n \subset \Pi_n \setminus \omega_n^{out}$. Obviously, $\omega^{in} \cap \omega^{out} = \emptyset$ but we do not require $\omega_n^{in} \cup \omega_n^{out}$ to cover Π_n. This is an important point: our method is not restricted to spanning-type situations (i.e., the surface boundary ∂E must contain a given collection of curves) in the sense that it can handle also loose constraints where only parts of the interior volume E and the exterior volume E^c are prescribed. Among all shapes satisfying the constraints, which ones may be considered as "natural"? Since area and curvature are key ingredients in many models which consider real life shapes as optimal shapes, e.g. bubbles, red cells, etc., it is quite natural to look for shapes which minimize either the perimeter or the Willmore energy (see below). Therefore, a natural model for the reconstruction of a volume from few slices is variational and consists in finding a minimizer E^* such that

$$E^* = \underset{\substack{\omega^{in} \subset E \\ E \cap \omega^{out} = \emptyset}}{\operatorname{argmin}} \; J(E), \tag{1}$$

[*]Univ Lyon, INSA de Lyon, CNRS UMR 5208, Institut Camille Jordan, 20 avenue Albert Einstein, F-69621 Villeurbanne Cedex, France (elie.bretin@insa-lyon.fr, http://math.univ-lyon1.fr/~ebretin).

[†]UMPA, ENS de Lyon, CNRS UMR 5669, 46 Allée d'Italie, F-69364 Lyon Cedex 07, France (francois.dayrens@ens-lyon.fr, http://perso.ens-lyon.fr/francois.dayrens).

[‡]Univ Lyon, Université Claude Bernard Lyon 1, CNRS UMR 5208, Institut Camille Jordan, 43 blvd. du 11 novembre 1918, F-69622 Villeurbanne cedex, France (masnou@math.univ-lyon1.fr, http://math.univ-lyon1.fr/~masnou).

where $\omega^{in} = \bigcup \omega_n^{in}$ and $\omega^{out} = \bigcup \omega_n^{out}$, and J is either the perimeter or the Willmore energy, depending on the applications and the desired smoothness of the boundary ∂E^*, see [11, 32] where a similar viewpoint is used.

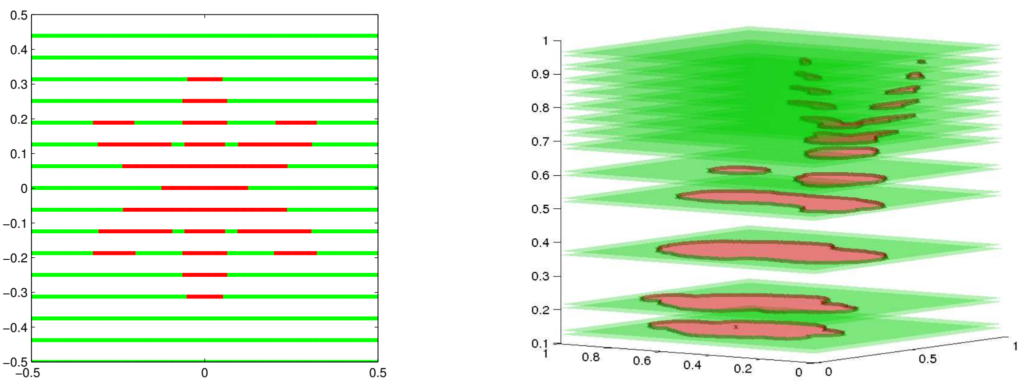

Figure 1: Examples of 2D (left) and 3D (right) reconstruction problems. Inner constraints are shown in red color and outer constraints in green color.

Beyond the theoretical issues raised by this model, which will be discussed later, we also tackle its numerical approximation. We use a phase-field formulation [5, 7, 6, 18, 37] which allows the approximation in the sense of Γ-convergence of the geometric energy J by a sequence of diffuse energies J_ε which are easier to handle numerically. The notion of Γ-convergence was introduced by De Giorgi as a suitable notion of convergence in a variational setting for functionals defined on metric spaces. If (X, d) is a metric space and (G_n) is a sequence of functionals mapping X onto $\overline{\mathbb{R}}$, one says that (G_n) Γ-converges in X to $G : X \to \overline{\mathbb{R}}$ as $n \to \infty$ if both following conditions hold [10]:

$$[\Gamma - \liminf]\ \forall u \in X,\ \forall (u_n) \subset X,\ u_n \to u \Longrightarrow G(u) \leqslant \liminf_{n\to\infty} G_n(u_n);$$

$$[\Gamma - \limsup]\ \forall u \in X,\ \exists (u_n) \subset X,\ u_n \to u,\ \limsup_{n\to\infty} G_n(u_n) \leqslant G(u).$$

A nice connection between Γ-convergence and minimizers is the following: if (G_n) Γ-converges to G, and $(x_n) \in X$ is such that x_n minimizes G_n, then every cluster point of (x_n) is a minimizer of G.

In the case J=perimeter=P, a phase field approximation is the celebrated Van der Waals-Cahn-Hilliard energy

$$P_\varepsilon(u) = \int_{\mathbb{R}^d} \left(\frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \right) \mathrm{d}x.$$

where $\varepsilon > 0$ is a small parameter, and W is the double well potential defined by $W(s) = \frac{1}{2} s^2 (1-s)^2$, $s \in \mathbb{R}$.

Modica and Mortola [36, 37] have shown the Γ-convergence with respect to the L^1-topology of $(P_\varepsilon)_{\varepsilon>0}$ (the convergence is intended for a sequence (ε_n) converging to 0 as $n \to \infty$) to the L^1 extension of λP defined as

$$u \longmapsto \begin{cases} \lambda P(\{u = 1\}) & \text{if } u \text{ is the characteristic function of a measurable set} \\ +\infty & \text{otherwise,} \end{cases}$$

with $\lambda = \int_0^1 \sqrt{2W(s)} ds$. A key element in Modica-Mortola's proof, which will be used in this paper, is the property that the indicator function of any set E (of finite perimeter) can be approximated by a sequence (u_ε) defined by $u_\varepsilon = q\left(\frac{1}{\varepsilon} d(x, E)\right)$ and such that $P_\varepsilon(u_\varepsilon) \to \lambda P(E)$. Here, $d(\cdot, E)$ is the signed distance function to E in $\mathbb{R}^d$ (negative in E, positive outside), and q is the so-called *profile function* associated to W and defined by $q(s) = \frac{1}{2}(1 - \tanh(s/2))$, $s \in \mathbb{R}$.

The second regularization energy that can be used in our framework is the Willmore energy, defined for any set $E \subset \mathbb{R}^d$ with smooth boundary by

$$\mathcal{W}(E) = \frac{1}{2} \int_{\partial E} |H|^2 \mathrm{d}\mathcal{H}^{d-1},$$

with H the mean curvature on the boundary ∂E. The (L^1-extension of the) Willmore energy can be approximated by various phase field models, see [21, 5, 6, 38, 39, 41, 43] for more details. We shall consider here the most classical one [21, 6, 41] defined as:

$$\mathcal{W}_\varepsilon(u) = \frac{1}{2\varepsilon} \int_{\mathbb{R}^d} \left(\varepsilon \Delta u - \frac{1}{\varepsilon} W'(u) \right)^2 \mathrm{d}x.$$

Using phase field methods for the reconstruction of 3D surfaces from cross-sections has been recently proposed in [20], where the inclusion-exclusion constraints are imposed with a penalization technique. Our idea is slightly different, and consists in incorporating the constraints $\omega^{in} \subset E$ and $E \cap \omega^{out} = \emptyset$ in the phase field approximation as the following **linear obstacle constraints** on u_ε:

$$u_\varepsilon^{in} \leqslant u_\varepsilon \leqslant u_\varepsilon^{out},$$

where $u_\varepsilon^{in} = q\left(d(x, \omega^{in})/\varepsilon\right)$ and $u_\varepsilon^{out} = 1 - q\left(d(x, \omega^{out})/\varepsilon\right)$. Having linear obstacle constraints makes the theoretical analysis of the model much easier, and opens the way to simple and very accurate numerical schemes. The direct phase field formulation of the original optimization problem (1) reads now as

$$u_\varepsilon^* = \operatorname*{argmin}_{u_\varepsilon^{in} \leqslant u_\varepsilon \leqslant u_\varepsilon^{out}} J_\varepsilon(u_\varepsilon), \tag{2}$$

where J_ε is a Γ-converging approximation of the L^1-extension of J.

We will show, however, that such direct phase field formulation is actually not appropriate, in the sense that the reconstructed sets do not satisfy strictly the constraints. We will therefore propose a variant model, based on a suitable fattening of the constraints, with the following nice properties which were the main motivations for this paper:

1. The minimizers of our model, using either the perimeter or the Willmore energy, can be approximated with a simple and accurate numerical method. Numerical results (see Section 3) confirm that fattening the constraints yields a better approximation of the relaxation of problem (1), and that using the Willmore energy gives smoother and more natural reconstructed surfaces.

2. A rigorous convergence analysis of the model can be provided (at least in the case of perimeter), which is new in this context to the best of our knowledge.

The characterization of limit energies when the Willmore functional is used is an open problem. This is due to the high non locality of the L^1-relaxation of the unconstrained Willmore energy, see [5, 6]. In particular, we expect that the contribution to the energy of volume-less ghost parts cannot be represented by integration.

The paper is organized as follows: the next section is devoted to setting the problem in the particular case of perimeter, for which the necessity of fattening the constraints can be easily justified. The same fattening method can be used as well for the Willmore-based formulation. In Section 2, and in the very case of the perimeter, we prove the convergence of our phase-field formulation to a limit energy which coincides (up to a multiplicative constant) with the relaxation in L^1 of the perimeter under constraints. In Section 3 we address the numerical approximation of solutions to Problem (2) when J_ε is a phase-field approximation of either the perimeter or the Willmore energy, and when the constraints are either previously fattened or not. We introduce two numerical schemes, and we provide a series of numerical simulations illustrating the performances of our approach in various situations, with either the perimeter or the Willmore energy as regularization criterion.

1 Problem setting: the perimeter case

1.1 Constrained perimeter, relaxation, and phase field approximation

1.1.1 Geometric constrained problem and relaxation

Recall that, for any measurable set $E \subset \mathbb{R}^d$ and $t \in [0,1]$, the set of points with density t with respect to E is

$$E^t = \left\{ x \in \mathbb{R}^d, \lim_{r \to 0} \frac{|E \cap B(x,r)|}{|B(x,r)|} = t \right\},$$

where $B(x,r)$ is an open ball and $|\cdot|$ the Lebesgue measure. If E has locally finite perimeter then the limit in the definition of E^t exists $\mathcal{H}^{d-1}$-almost everywhere, and Federer's Theorem states that $\mathbb{R}^d = E^0 \cup E^{1/2} \cup E^1$ up to a $\mathcal{H}^{d-1}$-negligible subset, see [2] for a full account on sets of finite perimeter. The set E^1 is the measure-theoretic interior of E, E^0 is the exterior and $E^{1/2}$ is the boundary. If E has locally finite perimeter then, for all Borel sets $A \subset \mathbb{R}^d$, the perimeter of E in A satisfies $P(E,A) = \mathcal{H}^{d-1}(E^{1/2} \cap A)$. As usual, we denote $P(E) = P(E, \mathbb{R}^d)$ the total perimeter of E.

Problem (1) is not well defined for sets of finite perimeter because the constraints are imposed on Lebesgue-negligible sets. A more convenient reformulation reads as follows:

$$E^* = \operatorname*{argmin}_{\substack{\omega^{in} \subset E^1 \\ \omega^{out} \subset E^0}} P(E), \tag{3}$$

The natural energy associated with this new constrained problem is

$$P_{\omega^{in},\omega^{out}}(E) = \begin{cases} P(E) & \text{if } \omega^{in} \subset E^1 \text{ and } \omega^{out} \subset E^0, \\ +\infty & \text{otherwise.} \end{cases}$$

It is not difficult to see that $P_{\omega^{in},\omega^{out}}$ is not lower semicontinuous with respect to the L^1-topology, thus problem (3) is ill-posed. To be convinced, consider in $\mathbb{R}^2$ the sequence $E_h = [-1,1] \times [-h,h]$ with $\omega^{in} = [-1,1] \times \{0\}$ and $\omega^{out} = \emptyset$. Then (for the L^1 convergence of characteristic functions) $E_h \to \emptyset$ as $h \to 0$, but $P_{\omega^{in},\omega^{out}}(E_h) \to 4$ whereas $P_{\omega^{in},\omega^{out}}(\emptyset) = +\infty$. To define a well-posed problem, we proceed as usual and consider the relaxation $\overline{P}_{\omega^{in},\omega^{out}}$ of $P_{\omega^{in},\omega^{out}}$ with respect to the L^1-topology, defined as

$$\overline{P}_{\omega^{in},\omega^{out}}(E) = \inf_{E_h \to E} \left\{ \liminf_{h \to 0} \left\{ P_{\omega^{in},\omega^{out}}(E_h) \right\} \right\}.$$

In the remaining of the paper, we address the new following minimization problem

$$E^* = \operatorname*{argmin}_{E} \overline{P}_{\omega^{in},\omega^{out}}(E). \tag{4}$$

We shall see in Theorem 2.2 that the relaxed energy $\overline{P}_{\omega^{in},\omega^{out}}$ can be identified with $F_{2,\omega^{in},\omega^{out}}$, where

$$F_{2,\omega^{in},\omega^{out}}(E) = P(E) + 2\mathcal{H}^{d-1}(E^0 \cap \omega^{in}) + 2\mathcal{H}^{d-1}(E^1 \cap \omega^{out}).$$

In particular, in our previous example with $E_h = [-1,1] \times [-h,h]$, we have $F_{2,\omega^{in},\omega^{out}}(\emptyset) = 4$.

1.1.2 Phase field approximation

We define the phase field constrained Van der Waals-Cahn-Hilliard energy on $L^1(\mathbb{R}^d)$ as

$$P_{\varepsilon,u_1,u_2}(u) = \begin{cases} \displaystyle\int_{\mathbb{R}^d} \left(\frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \right) \mathrm{d}x & \text{if } u \in H^1(\mathbb{R}^d) \text{ and } u_1 \leqslant u \leqslant u_2, \\ +\infty & \text{otherwise,} \end{cases}$$

and a natural minimization problem to address is

$$u_\varepsilon^* = \operatorname*{argmin}_{u} \{ P_{\varepsilon, u_\varepsilon^{in}, u_\varepsilon^{out}}(u) \}, \tag{5}$$

where $u_\varepsilon^{in} = q\left(d(x,\omega^{in})/\varepsilon\right)$ and $u_\varepsilon^{out} = 1 - q\left(d(x,\omega^{out})/\varepsilon\right)$.

However, we will see in subsection 1.3 that $P_{\varepsilon,u_\varepsilon^{in},u_\varepsilon^{out}}$ does not Γ-converge with respect to the L^1-topology to $\lambda F_{2,\omega^{in},\omega^{out}}$. More precisely, we conjecture that $P_{\varepsilon,u_\varepsilon^{in},u_\varepsilon^{out}}$ Γ-converges to $\lambda F_{1,\omega^{in},\omega^{out}}$ where we denote for every $u \in L^1$:

$$F_{1,\omega^{in},\omega^{out}}(u) = \begin{cases} F_{1,\omega^{in},\omega^{out}}(E) & \text{if } u = \mathbb{1}_E \\ +\infty & \text{otherwise.} \end{cases}$$

with

$$F_{1,\omega^{in},\omega^{out}}(E) = P(E) + \mathcal{H}^{d-1}(E^0 \cap \omega^{in}) + \mathcal{H}^{d-1}(E^1 \cap \omega^{out}).$$

We will see in this paper that a technique to obtain the good relaxation, i.e. $\lambda F_{2,\omega^{in},\omega^{out}}$, consists in fattening the inclusion-exclusion constraints. We will denote $\Omega_{\varepsilon^\alpha}^{in}$ and $\Omega_{\varepsilon^\alpha}^{out}$ the fattened constraint sets (see Section 1.2.1), which depend on a thickness parameter $\alpha \in]0,1[$ so that the associated fattened diffuse fields have width of order ε^α, i.e. larger than the initial phase field approximation whose width is of order ε. Therefore, we now consider the new phase field constraints $u_{\varepsilon,\alpha}^{in} \leqslant u \leqslant u_{\varepsilon,\alpha}^{out}$ with

$$u_{\varepsilon,\alpha}^{in} = q\left(d(x,\Omega_{\varepsilon^\alpha}^{in})/\varepsilon\right) \text{ and } u_{\varepsilon,\alpha}^{out} = 1 - q\left(d(x,\Omega_{\varepsilon^\alpha}^{out})/\varepsilon\right).$$

The main theoretical result of this paper is the Γ-convergence (with respect to the L^1-topology) of $P_{\varepsilon,u_{\varepsilon,\alpha}^{in},u_{\varepsilon,\alpha}^{out}}$ to $\lambda F_{2,\omega^{in},\omega^{out}}$.

Remark 1.1. As already mentioned, we will prove in Theorem 2.2 that $\overline{P}_{\omega^{in},\omega^{out}}$, i.e. the relaxation of $P_{\omega^{in},\omega^{out}}$ with respect to the L^1 convergence of characteristic functions, coincides with $F_{2,\omega^{in},\omega^{out}}$. We actually believe that $F_{1,\omega^{in},\omega^{out}}$ does also correspond to a relaxation of $P_{\omega^{in},\omega^{out}}$ when phase fields are used for approximation instead of binary functions. Recalling that, when a set E satisfies the constraints, $P_{\omega^{in},\omega^{out}}(E)$ coincides with the total variation $|D\mathbb{1}_E|(\mathbb{R}^d)$, one can define the following relaxation of the constrained perimeter:

$$\tilde{P}_{\omega^{in},\omega^{out}}(E) = \inf\left\{ \liminf_{\varepsilon\to 0} \int_{\mathbb{R}^d} |\nabla u_\varepsilon| dx, \quad u_\varepsilon \in C_c^\infty,\ u_\varepsilon \to \mathbb{1}_E \text{ in } L^1,\ u_\varepsilon \geqslant \frac{1}{2} \text{ on } \omega^{in},\ u_\varepsilon \leqslant \frac{1}{2} \text{ on } \omega^{out} \right\}.$$

i.e., the total variation of $\mathbb{1}_E$ is approximated by the total variations of smooth functions which are constrained on ω^{in} and ω^{out}. An equivalent perspective is the following: $P_{\omega^{in},\omega^{out}}(E)$ coincides with the mass of the varifold $V_E = |D\mathbb{1}_E| \otimes \delta_{(D\mathbb{1}_E)^\perp}$ which can be approximated (with respect to the weak-$\star$ convergence of measures) by diffuse varifolds $V_\varepsilon = |\nabla u_\varepsilon| dx \otimes \delta_{(\nabla u_\varepsilon)^\perp}$ with $u_\varepsilon \in C_c^\infty$, $u_\varepsilon \geqslant \frac{1}{2}$ on ω^{in}, and $u_\varepsilon \leqslant \frac{1}{2}$ on ω^{out}. In view of the example discussed in Section 1.3, it is natural to conjecture that $\tilde{P}_{\omega^{in},\omega^{out}}(E) = F_{1,\omega^{in},\omega^{out}}$.

1.1.3 Comparing minimizers of $F_{1,\omega^{in},\omega^{out}}$ and $F_{2,\omega^{in},\omega^{out}}$

To understand the main difference between $F_{1,\omega^{in},\omega^{out}}$ and $F_{2,\omega^{in},\omega^{out}}$, we focus on the local configurations illustrated in Figure 2, and we compare their energies.

In the left example of Figure 2, $P(E) = AB + BC$, $\mathcal{H}^{d-1}(E^0 \cap \omega^{in}) = 0$, and $\mathcal{H}^{d-1}(E^1 \cap \omega^{out}) = BD$. In the right configuration, $P(E) = AD + DC$ and $\mathcal{H}^{d-1}(E^0 \cap \omega^{in}) = \mathcal{H}^{d-1}(E^1 \cap \omega^{out}) = 0$. Thus

$$F_{1,\omega^{in},\omega^{out}}(\text{left configuration}) = AB + BC + BD < AD + DC = F_{1,\omega^{in},\omega^{out}}(\text{right configuration}).$$

For $F_{2,\omega^{in},\omega^{out}}$, using the triangular inequality, we have

$$F_{2,\omega^{in},\omega^{out}}(\text{left}) = AB + 2BD + BC > AD + DC = F_{2,\omega^{in},\omega^{out}}(\text{right}).$$

This simple example indicates that, in general, minimizers E of $F_{1,\omega^{in},\omega^{out}}$ need not be suitable for our reconstruction problem for they will not satisfy the constraints $\omega^{in} \subset E^1$ and $E^0 \cap \omega^{out} = \emptyset$. For this very example, $F_{2,\omega^{in},\omega^{out}}$ behaves better. More generally, numerical experiments suggest that

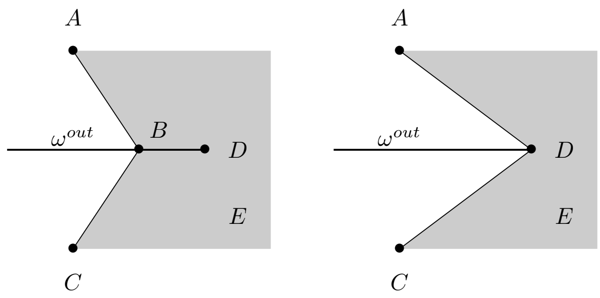

Figure 2: Two local configurations where E is in gray and ω^{out} is the horizontal half-line starting at D.

minimizers of $F_{2,\omega^{in},\omega^{out}}$ do satisfy the constraints on reasonable examples where minimizers of $F_{1,\omega^{in},\omega^{out}}$ do not. Although no theoretical characterization of the minimizers of $F_{2,\omega^{in},\omega^{out}}$ which satisfy the constraints is known, here is another simple comparison example. Let $\omega^{in} = [0,1] \times (\{0\} \cup \{h\})$ and $\omega^{out} = \emptyset$. Then for every h, a minimizer of $F_{1,\omega^{in},\omega^{out}}$ is the empty set. In contrast, for $h \leqslant 1$, the set $[0,1] \times [0,h]$ minimizes $F_{2,\omega^{in},\omega^{out}}$ and do satisfy the constraints. But for $h > 1$, the empty set is the unique minimizer. Summarizing, using $F_{2,\omega^{in},\omega^{out}}$ does not guarantee that the constraints are always satisfied by the minimizers, but this energy behaves better than $F_{1,\omega^{in},\omega^{out}}$.

Reformulating for the phase field approximations, and in view of the discussion in the previous paragraph, the constraints $u_\varepsilon^{in} \leqslant u \leqslant u_\varepsilon^{out}$ are not enough coercive to give a good approximation of our geometric problem. It will be necessary to consider the fattened constraints $u_{\varepsilon,\alpha}^{in} \leqslant u \leqslant u_{\varepsilon,\alpha}^{out}$, see figure 6 for a numerical illustration.

1.2 Definitions and notations

1.2.1 Interior-exterior constraints and thickness

Let $d \in \mathbb{N}^*$ and Q be an open bounded subset of $\mathbb{R}^d$. We consider a finite collection of restrictions to Q of hyperplanes, denoted as $\{\Pi_n\}_n$, and we assume that $\Pi_n \cap \Pi_{n'} = \emptyset$ for any $n \neq n'$. Our constraints are given as two subsets $\omega^{in}, \omega^{out} \subset Q$ such that, for any n, both $\omega^{in} \cap \Pi_n$ and $\omega^{out} \cap \Pi_n$ are finite unions $\bigcup_k \omega_{n,k}^{in}$ and $\bigcup_\ell \omega_{n,\ell}^{out}$ of disjoint, connected, open, bounded and Lipschitz sets in Π_n as a subset of $\mathbb{R}^{d-1}$.

Let d_n denote the signed distance function to subsets of Π_n (as a subset of $\mathbb{R}^{d-1}$), i.e. for any $A \subset \Pi_n$ and $y \in \Pi_n$, $d_n(y,A) = dist(y,A) - dist(y, \Pi_n \setminus A)$, with $dist$ the Euclidean distance in $\mathbb{R}^d$. In a suitable orthonormal system of coordinates in $\mathbb{R}^d$ such that

$$\Pi_n = \{(y,0) \mid y \in \mathbb{R}^{d-1}\}$$

we define for every k, ℓ the fattened constraints with thickness parameter $h > 0$ (see figure 3, where in, out, and k are dropped for simplicity, and $h = \varepsilon^\alpha$):

$$\begin{aligned} \Omega_{n,k,h}^{in} &= \Big\{ (y,z) \in (\mathbb{R}^{d-1} \times \mathbb{R}) \cap Q, \quad y \in \omega_{n,k}^{in},\ |z| < h |d_n(y, \omega_{n,k}^{in})| \Big\} \\ \Omega_{n,\ell,h}^{out} &= \Big\{ (y,z) \in (\mathbb{R}^{d-1} \times \mathbb{R}) \cap Q, \quad y \in \omega_{n,\ell}^{out},\ |z| < h |d_n(y, \omega_{n,\ell}^{out})| \Big\}. \end{aligned} \tag{6}$$

Being the $\Pi_n's$ pairwise disjoint, there exists $h_0 > 0$ small enough such that any two elements of the united collection $\{\Omega_{n,k,h}^{in}, \Omega_{n,\ell,h}^{out}\}_{n,k,\ell}$ are disjoint. Lastly, we define

$$\Omega_h^{in} = \bigcup_{n,k} \Omega_{n,k,h}^{in} \qquad \text{and} \qquad \Omega_h^{out} = \bigcup_{n,\ell} \Omega_{n,\ell,h}^{out}.$$

Remark 1.2. We opt for a conic fattening rather than a rectangular fattening in the normal direction in order not to prescribe the tangential direction of the reconstructed domain around the slices. Indeed, if slices are tied (i.e. if $\overline{\omega_n^{in}} \cup \overline{\omega_n^{out}} = \Pi_n$) then a rectangular fattening forces the reconstructed domain

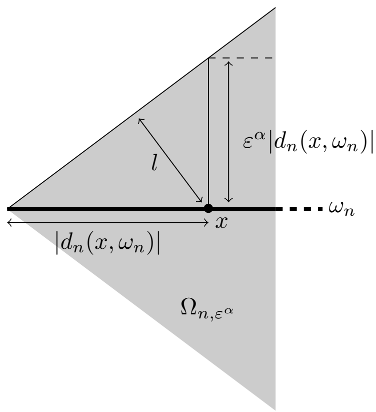

Figure 3: Constraint fattening with thickness parameter $h = \varepsilon^\alpha$: being ω_n the horizontal half-line, the fattened set $\Omega_{n,\varepsilon^\alpha} = \{(y,z) \in (\mathbb{R}^{d-1} \times \mathbb{R}) \cap Q, \quad y \in \omega^{in}_{n,k},\ |z| < \varepsilon^\alpha |d_n(y,\omega_n)|\}$ is (partially) represented in gray.

to be orthogonal to slice hyperplanes. This is not really a problem if we use the perimeter because the domain may have singularities (typically angles) to balance these tangential constraints. However, for the Willmore energy, the control of mean curvature prevents angles, thus a rectangular fattening forbids non-orthogonal reconstructions. This issue is avoided with a conic fattening.

Remark 1.3. A truncated distance function can be used to avoid the overlap of fattened constraints sets. This does not change the theoretical analysis, but it can be useful for numerical purposes.

1.2.2 Phase field approximation with constraints

Let $q : \mathbb{R} \to [0,1]$ be the optimal profile associated with W, defined as the solution to the Cauchy problem

$$\begin{cases} q' &= -\sqrt{2W(q)} \\ q(0) &= 1/2. \end{cases}$$

For the aforementioned potential $W(s) = \frac{1}{2}s^2(1-s)^2$, it can be proven that $q(t) = \frac{1}{2}(1-\tanh(t/2))$. Given $\varepsilon > 0$ and $\alpha \in]0,1[$, we define as before

$$u^{in}_{\varepsilon,\alpha}(x) = q\left(\frac{1}{\varepsilon} d(x, \Omega^{in}_{\varepsilon^\alpha})\right) \qquad \text{and} \qquad u^{out}_{\varepsilon,\alpha}(x) = 1 - q\left(\frac{1}{\varepsilon} d(x, \Omega^{out}_{\varepsilon^\alpha})\right).$$

where, again, d denotes the signed distance function to subsets of Q. For all subset $E \subset Q$ and the associated phase field function $u_\varepsilon(x) = q(d(x,E)/\varepsilon)$, we have the equivalence

$$u^{in}_{\varepsilon,\alpha} \leqslant u \leqslant u^{out}_{\varepsilon,\alpha} \quad \Longleftrightarrow \quad \Omega^{in}_{\varepsilon^\alpha} \subset E \subset \mathbb{R}^d \setminus \Omega^{out}_{\varepsilon^\alpha}.$$

Remark 1.4. Using standard density arguments, the following convergence results can be proven

$$u^{in}_{\varepsilon,\alpha}(x) \overset{\varepsilon\to 0}{\longrightarrow} \begin{cases} 1 & \text{if } x \in \omega^{in} \\ 1/2 & \text{if } x \in \bigcup\limits_{n,k} \partial_n \omega^{in}_{n,k} \\ 0 & \text{otherwise} \end{cases}, \quad \text{and} \quad u^{out}_{\varepsilon,\alpha}(x) \overset{\varepsilon\to 0}{\longrightarrow} \begin{cases} 0 & \text{if } x \in \omega^{out} \\ 1/2 & \text{if } x \in \bigcup\limits_{n,\ell} \partial_n \omega^{out}_{n,\ell} \\ 1 & \text{otherwise} \end{cases}$$

where ∂_n is the boundary within the subspace Π_n. On the contrary, without the fattening constraints, there holds

$$u^{in}_{\varepsilon}(x) \overset{\varepsilon\to 0}{\longrightarrow} \begin{cases} 1/2 & \text{if } x \in \omega^{in} \\ 1/2 & \text{if } x \in \bigcup\limits_{n,k} \partial_n \omega^{in}_{n,k} \\ 0 & \text{otherwise} \end{cases}, \quad \text{and} \quad u^{out}_{\varepsilon}(x) \overset{\varepsilon\to 0}{\longrightarrow} \begin{cases} 1/2 & \text{if } x \in \omega^{out} \\ 1/2 & \text{if } x \in \bigcup\limits_{n,\ell} \partial_n \omega^{out}_{n,\ell} \\ 1 & \text{otherwise} \end{cases}$$

These convergence results can be of interest for the numerical approximation when a sharp representation of the set contraints is difficult to compute (for instance in 3D) and can be replaced instead by smooth approximations with phase field functions.

1.3 Necessity of constraints fattening

In this subsection, we give a simple example for which the limit of $P_{\varepsilon,u_\varepsilon^{in},u_\varepsilon^{out}}$ appears to be $\lambda F_{1,\omega^{in},\omega^{out}}$ and not $\lambda F_{2,\omega^{in},\omega^{out}}$. In particular, one expects that the limit of $P_{\varepsilon,u_\varepsilon^{in},u_\varepsilon^{out}}$, if it exists, is at most equal to $\lambda F_{1,\omega^{in},\omega^{out}}$. We already gave the drawbacks of using $F_{1,\omega^{in},\omega^{out}}$, and this example motivates the necessity of fattening the constraints so as to rather approximate $F_{2,\omega^{in},\omega^{out}}$.

Let $E = B \subset \mathbb{R}^2$ be a ball, ω^{in} be a segment outside the ball and $\omega^{out} = \emptyset$ (see Figure 4).

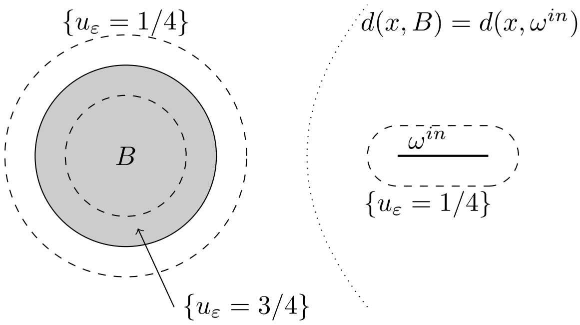

Figure 4: An example for which the limit of $P_{\varepsilon,u_\varepsilon^{in},u_\varepsilon^{out}}$ is $\lambda F_{1,\omega^{in},\omega^{out}}$

We consider the phase field profiles associated with B and ω^{in}:

$$u_\varepsilon^B(x) = q\left(\frac{1}{\varepsilon}d(x,B)\right) \quad \text{and} \quad u_\varepsilon^{in}(x) = q\left(\frac{1}{\varepsilon}d(x,\omega^{in})\right).$$

As ω^{in} has no interior, $d(x,\omega^{in}) \geqslant 0$ for all x in $\mathbb{R}^2$, therefore the level lines $\{u_\varepsilon^{in} = t\}$ are empty for $1/2 < t < 1$. Then, we define $u_\varepsilon = \max(u_\varepsilon^B, u_\varepsilon^{in})$, which is equal to u_ε^B near B, and u_ε^{in} near ω^{in}. In particular, u_ε is smooth except on the set $\{u_\varepsilon^{in} = u_\varepsilon^B\}$ which is given by $\{x \mid d(x,B) = d(x,\omega^{in})\}$. In our case, this set is negligible and coincides with some parabolic line between B and ω^{in} (see Figure 4). In addition, u_ε is a phase field profile i.e.

$$\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon) = |\nabla u_\varepsilon|\sqrt{2W(u_\varepsilon)}$$

except on the parabolic line, and it satisfies $u_\varepsilon^{in} \leqslant u_\varepsilon$ and $u_\varepsilon \leqslant u_\varepsilon^{out} = 1$. By the coarea formula,

$$P_{\varepsilon,u_\varepsilon^{in},u_\varepsilon^{out}}(u_\varepsilon) = \int_{\mathbb{R}^2}\left(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\right)\mathrm{d}x = \int_0^1 \sqrt{2W(t)}P(\{u_\varepsilon \geqslant t\})\mathrm{d}t.$$

For $t \in]0,1[$, there exists $\varepsilon > 0$ small enough such that $\{u_\varepsilon \geqslant t\}$ remains far from the parabolic line $\{d(x,B) = d(x,\omega^{in})\}$ and so

$$P(\{u_\varepsilon \geqslant t\}) = \begin{cases} \mathcal{H}^1(\{u_\varepsilon^B = t\}) + \mathcal{H}^1(\{u_\varepsilon^{in} = t\}) & \text{if } 0 < t < 1/2 \\ \mathcal{H}^1(\{u_\varepsilon^B = t\}) & \text{if } 1/2 < t < 1. \end{cases}$$

As $\{u_\varepsilon^B = t\}$ is a ball and $\{u_\varepsilon^{in} = t\}$ is a stadium, we see that

$$\mathcal{H}^1(\{u_\varepsilon^B = t\}) \to P(B) \quad \text{and} \quad \mathcal{H}^1(\{u_\varepsilon^{in} = t\}) \to 2\mathcal{H}^1(\omega^{in})$$

when $\varepsilon \to 0$. By the Dominated Convergence Theorem,

$$P_{\varepsilon,u_\varepsilon^{in},u_\varepsilon^{out}}(u_\varepsilon) \overset{\varepsilon\to 0}{\longrightarrow} \int_0^{1/2}\sqrt{2W(t)}\Big(P(B) + 2\mathcal{H}^1(\omega^{in})\Big)\mathrm{d}t + \int_{1/2}^1 \sqrt{2W(t)}P(B)\mathrm{d}t.$$

Using the symmetry relation $W(1-t) = W(t)$, we get that

$$\lim_{\varepsilon\to 0} P_{\varepsilon, u_\varepsilon^{in}, u_\varepsilon^{out}}(u_\varepsilon) = \lambda\Big(P(B) + \mathcal{H}^1(\omega^{in})\Big) = \lambda F_{1,\omega^{in},\emptyset}(B).$$

The key point in this example is that the thin constraint ω^{in} generates t-level lines only for $t \in]0, 1/2[$, but not for $t \in]1/2, 1[$.

1.4 Approximation with fattened constraints

In view of the above discussion, and extending also to the Willmore regularity criterion, we propose as a relaxation of the initial problem (1) the following model:

$$u_\varepsilon^* = \underset{u_{\varepsilon,\alpha}^{in} \leqslant u \leqslant u_{\varepsilon,\alpha}^{out}}{\operatorname{argmin}} \; J_\varepsilon(u),$$

where J_ε is a phase-field approximation of either the perimeter or the Willmore energy.

2 Convergence and relaxation results for the perimeter-based formulation

The main theoretical results of this paper are given in the following theorems, whose proofs are detailed in the subsequent sections. As mentioned in the introduction, no similar results are known for the Willmore-based formulation.

Theorem 2.1 (Fat constraints case)**.** *Let Q be a bounded open subset of $\mathbb{R}^d$, and let $\omega^{in}, \omega^{out}$ and $u_{\varepsilon,\alpha}^{in}, u_{\varepsilon,\alpha}^{out}$ be defined as in Sections 1.2.1 and 1.2.2. Then, for all $\alpha \in]0, 1[$,*

$$P_{\varepsilon, u_{\varepsilon,\alpha}^{in}, u_{\varepsilon,\alpha}^{out}} \quad \Gamma - \textit{converges to} \quad \lambda\, F_{2,\omega^{in},\omega^{out}},$$

with respect to the $L^1(Q)$-topology as ε goes to 0.

Theorem 2.2 (Identification of the relaxation)**.** *With the notations above, for all $E \subset\subset Q$, we have*

$$\overline{P}_{\omega^{in},\omega^{out}}(E) = F_{2,\omega^{in},\omega^{out}}(E),$$

Remark 2.3. This result implies that for all $E_\varepsilon \to E \subset\subset Q$, we have

$$\liminf_{\varepsilon\to 0} P_{\omega^{in},\omega^{out}}(E_\varepsilon) \geqslant F_{2,\omega^{in},\omega^{out}}(E),$$

and for all $E \subset\subset Q$, there exists a sequence $E_\eta \to E$ such that

$$\limsup_{\eta\to 0} P_{\omega^{in},\omega^{out}}(E_\eta) \leqslant F_{2,\omega^{in},\omega^{out}}(E).$$

2.1 Proof of Theorem 2.1: $\Gamma - \limsup$ inequality

Let E be a set with finite perimeter in Q. We now give a construction of a sequence $(u_\varepsilon)_{\varepsilon>0}$ which converges in $L^1(Q)$ to the characteristic function of E and satisfies

$$\limsup_{\varepsilon\to 0} P_{\varepsilon, u_{\varepsilon,\alpha}^{in}, u_{\varepsilon,\alpha}^{out}}(u_\varepsilon) \leqslant \lambda F_{2,\omega^{in},\omega^{out}}(E).$$

Let $\eta > 0$ and consider the set $E_\eta = (E \cup \Omega_\eta^{in}) \setminus \Omega_\eta^{out}$. Notice that for ε sufficiently small and satisfying $\varepsilon^\alpha < \eta$, the set E_η satisfies the fat constraint, i.e

$$\Omega_{\varepsilon^\alpha}^{in} \subset E_\eta \subset Q \setminus \Omega_{\varepsilon^\alpha}^{out}.$$

The canonical phase field function associated with E_η is

$$u_\varepsilon^\eta(x) = q\left(\frac{d(x, E_\eta)}{\varepsilon}\right),$$

which satisfies phase field constraints, i.e. $u^{in}_{\varepsilon,\alpha} \leqslant u^{\eta}_{\varepsilon} \leqslant u^{out}_{\varepsilon,\alpha}$ as soon as $\varepsilon^{\alpha} < \eta$. Then, using the original result of Modica and Mortola [37], we get that

$$P_{\varepsilon,u^{in}_{\varepsilon,\alpha},u^{out}_{\varepsilon,\alpha}}(u^{\eta}_{\varepsilon}) \xrightarrow{\varepsilon\to 0} \lambda P(E_{\eta}, Q).$$

Let us now consider the sequence $(v_{\varepsilon})_{\varepsilon>0}$ defined by $v_{\varepsilon} = u^{2\varepsilon^{\alpha}}_{\varepsilon}$ which satisfies $v_{\varepsilon} \xrightarrow{\varepsilon\to 0} \mathbb{1}_E$ almost everywhere on Q. Indeed,

- if $x \in Q \setminus (\overline{E} \cup \overline{\omega^{in}})$ then $d(x, E \cup \omega^{in}) > 0$ and there exists ε_0 small enough such that $d(x, E_{2\varepsilon_0}) > 0$. Then, for all $\varepsilon < \varepsilon_0$, $d(x, E_{2\varepsilon}) \geqslant d(x, E_{2\varepsilon_0}) > 0$, which leads to
$$v_{\varepsilon}(x) \xrightarrow{\varepsilon\to 0} \lim_{s\to+\infty} q(s) = 0.$$
- if $x \in \overset{\circ}{E} \setminus \omega^{out}$ then $d(x, E \cup \omega^{out}) < 0$, and there exists ε_0 small enough such that $d(x, E_{2\varepsilon_0}) < 0$ and, for all $\varepsilon < \varepsilon_0$, $d(x, E_{2\varepsilon}) \leqslant d(x, E_{2\varepsilon_0}) < 0$, which implies that
$$v_{\varepsilon}(x) \xrightarrow{\varepsilon\to 0} \lim_{s\to-\infty} q(s) = 1.$$

Without loss of generality, we can assume thanks to [34, Proposition 12.19] that $\mathcal{L}^d(\partial E) = 0$. Since, in addition, $|\omega^{in}| = |\omega^{out}| = 0$, we have $v_{\varepsilon} \xrightarrow{\varepsilon\to 0} \mathbb{1}_E$ almost everywhere on Q.
Furthermore, the sequence (v_{ε}) satisfies the constraints $u^{in}_{\varepsilon} \leqslant v_{\varepsilon} \leqslant u^{out}_{\varepsilon}$ as $2\varepsilon^{\alpha} > \varepsilon^{\alpha}$, which implies together with Lemma 2.4 below that

$$\begin{aligned}\limsup_{\varepsilon\to 0} P_{\varepsilon,u^{in}_{\varepsilon,\alpha},u^{out}_{\varepsilon,\alpha}}(v_{\varepsilon}) &\leqslant \limsup_{\eta\to 0}\limsup_{\varepsilon\to 0} P_{\varepsilon,u^{in}_{\varepsilon,\alpha},u^{out}_{\varepsilon,\alpha}}(u^{\eta}_{\varepsilon}) \\ &\leqslant \lambda \limsup_{\eta\to 0} P(E_{\eta}, Q) \leqslant \lambda F_{2,\omega^{in},\omega^{out}}(E),\end{aligned}$$

and the proof is complete.

Lemma 2.4. *The sequence $(E_{\eta})_{\eta}$ with $E_{\eta} = (E \cup \Omega^{in}_{\eta}) \setminus \Omega^{out}_{\eta}$ satisfies*

$$\limsup_{\eta\to 0} P(E_{\eta}, Q) \leqslant P(E, Q) + 2\mathcal{H}^{d-1}(E^0 \cap \omega^{in}) + 2\mathcal{H}^{d-1}(E^1 \cap \omega^{out})$$

Proof. See Section A.1 in the Appendix. □

2.2 Proof of Theorem 2.1: $\Gamma - \liminf$ inequality

Let $(u_{\varepsilon})_{\varepsilon>0}$ be a sequence such that

$$u_{\varepsilon} \xrightarrow{\varepsilon\to 0} u \quad \text{in } L^1(Q)$$

and

$$\liminf_{\varepsilon\to 0} P_{\varepsilon,u^{in}_{\varepsilon,\alpha},u^{out}_{\varepsilon,\alpha}}(u_{\varepsilon}) \quad < \quad +\infty.$$

Using the same arguments as in the original proof of Modica and Mortola [36], we can prove that there exists a Borel set E such that $u = \mathbb{1}_E$ and

$$\liminf_{\varepsilon\to 0} P_{\varepsilon,u^{in}_{\varepsilon,\alpha},u^{out}_{\varepsilon,\alpha}}(u_{\varepsilon}) \quad \geqslant \quad \int_{-\infty}^{+\infty} \sqrt{2W(s)} \liminf_{\varepsilon\to 0} P(\{u_{\varepsilon} \geqslant s\}, Q) \mathrm{d}s.$$

Using Lemma 2.5 below, it follows that

$$\begin{aligned}\liminf_{\varepsilon\to 0} P_{\varepsilon,u^{in}_{\varepsilon,\alpha},u^{out}_{\varepsilon,\alpha}}(u_{\varepsilon}) &\geqslant \int_{-\infty}^{+\infty} \sqrt{2W(s)} \liminf_{\varepsilon\to 0} P(\{u_{\varepsilon} \geqslant s\}, Q) \mathrm{d}s \\ &\geqslant \int_{0}^{1} \sqrt{2W(s)} \liminf_{\varepsilon\to 0} P(\{u_{\varepsilon} \geqslant s\}, Q) \mathrm{d}s \\ &\geqslant \int_{0}^{1} \sqrt{2W(s)} \Big(P(E, Q) + 2\mathcal{H}^{d-1}(E^0 \cap \omega^{in}) + 2\mathcal{H}^{d-1}(E^1 \cap \omega^{out})\Big) \mathrm{d}s \\ &= \lambda F_{2,\omega^{in},\omega^{out}}(\mathbb{1}_E),\end{aligned}$$

which completes the proof.

Lemma 2.5. *Assume that* $\alpha \in]0,1[$ *and* u_ε *satisfies* $u^{in}_{\varepsilon,\alpha} \leqslant u_\varepsilon \leqslant u^{out}_{\varepsilon,\alpha}$. *Then, for almost every* s *in* $[0,1]$, $\liminf_{\varepsilon\to 0} P(\{u_\varepsilon \geqslant s\}, Q) \geqslant P(E,Q) + 2\mathcal{H}^{d-1}(E^0 \cap \omega^{in}) + 2\mathcal{H}^{d-1}(E^1 \cap \omega^{out})$.

Proof. See section A.2. □

2.3 Proof of Theorem 2.2

Take a sequence $(E_\varepsilon)_{\varepsilon>0}$ converging to E for the $L^1(Q)$ topology and satisfying

$$P_{\omega^{in},\omega^{out}}(E_\varepsilon) < +\infty.$$

Notice that for all $\varepsilon > 0$, we have $\omega^{in} \subset E^1_\varepsilon$ and $\omega^{out} \subset E^0_\varepsilon$. Let $\alpha \in]0,1[$ and consider the sequence $(u_\varepsilon)_\varepsilon$ defined by $u_\varepsilon = \mathbb{1}_{E_\varepsilon}$ for all $\varepsilon > 0$. It is not difficult to see that u_ε satisfies the constraints $u^{in}_{\varepsilon,\alpha} \leqslant u_\varepsilon \leqslant u^{out}_{\varepsilon,\alpha}$ as soon as ε is sufficiently small. In particular, we can apply Lemma 2.5 to deduce that

$$\liminf_{\varepsilon\to 0} P_{\omega^{in},\omega^{out}}(E_\varepsilon) = \liminf_{\varepsilon\to 0} P(\{u_\varepsilon \geqslant 1/2\}, Q) \geqslant F_{2,\omega^{in},\omega^{out}}(E).$$

Let E be a set with finite perimeter in Q. The sequence $E_\eta = (E \cup \Omega^{in}_\eta) \setminus \Omega^{out}_\eta \to E$ for the $L^1(Q)$ topology. Furthermore, for all $\eta > 0$, E_η satisfies the constraints $\omega^{in} \subset E^1_\eta$ and $\omega^{out} \subset E^0_\eta$, and Lemma 2.4 implies that

$$\limsup_{\eta\to 0} P_{\omega^{in},\omega^{out}}(E_\eta) \leqslant F_{2,\omega^{in},\omega^{out}}(E),$$

which completes the proof of Theorem 2.2.

3 Numerical approximations for both perimeter- and Willmore-based formulations

This section is devoted to the design of a numerical algorithm for the approximation of local minimizers to the following phase field optimization problem:

$$u^*_\varepsilon = \underset{u_1 \leqslant u \leqslant u_2}{\text{argmin}}\ J_\varepsilon(u), \tag{2}$$

where J_ε is a phase-field approximation of either the perimeter or the Willmore energy. More precisely, we consider three different problems in space dimension $d = 2$ or $d = 3$ (with $Q = [0,1]^d$ the computation box, see the beginning of Section 1.2.2 for the other notations):

(P1) Perimeter with sharp constraints:

$$J_\varepsilon(u) = P_\varepsilon(u) = \int_Q \left(\varepsilon|\nabla u|^2 + \frac{1}{\varepsilon}W(u)\right) \mathrm{d}x \quad \text{and} \quad u_1 = u^{in}_{\varepsilon,+\infty},\ u_2 = u^{out}_{\varepsilon,+\infty}.$$

(P2) Perimeter with fat constraints:

$$J_\varepsilon(u) = P_\varepsilon(u) = \int_Q \left(\varepsilon|\nabla u|^2 + \frac{1}{\varepsilon}W(u)\right) \mathrm{d}x, \quad \text{and} \quad u_1 = u^{in}_{\varepsilon,1/2},\ u_2 = u^{out}_{\varepsilon,1/2}.$$

(P3) Willmore energy with sharp constraints:

$$J_\varepsilon(u) = \mathcal{W}_\varepsilon(u) = \frac{1}{2\varepsilon}\int_Q \left(\varepsilon\Delta u - \frac{1}{\varepsilon}W'(u)\right)^2 \mathrm{d}x \quad \text{and} \quad u_1 = u^{in}_{\varepsilon,+\infty},\ u_2 = u^{out}_{\varepsilon,+\infty}.$$

3.1 Approximation of phase field constraints

We assume that inner constraints ω^{in} and outer constraints ω^{out} belong to the box Q (see Section 1.2.1 for notations). In order to simplify the numerical construction of $\Omega^{in}_{\varepsilon,\alpha}$ and $\Omega^{out}_{\varepsilon,\alpha}$, we extend the constraint only in the direction orthogonal to the slice plane Π_n with a thickness of size ε^{α} (see Figure 5). A Fast Marching method [42] can be used to compute the signed distance functions $d(\cdot,\Omega^{in}_{\varepsilon,\alpha})$ and $d(\cdot,\Omega^{out}_{\varepsilon,\alpha})$ on Q, and the phase field functions $u^{in}_{\varepsilon,\alpha}$ and $u^{out}_{\varepsilon,\alpha}$ are simply estimated using the expressions

$$u^{in}_{\varepsilon,\alpha} = q\left(\frac{d(x,\Omega^{in}_{\varepsilon\alpha})}{\varepsilon}\right) \quad \text{and} \quad u^{out}_{\varepsilon,\alpha} = 1 - q\left(\frac{d(x,\Omega^{out}_{\varepsilon\alpha})}{\varepsilon}\right) \quad \text{with } q(s) = \frac{1}{2}(1-\tanh(s/2)).$$

Notice that from a numerical point of view, it is sometimes easier to consider the following approximating phase field constraints (see Section 1.2.1):

$$u^{in}_{\varepsilon,\alpha} = \frac{1}{2}\mathbb{1}_{\Omega^{in}_{\varepsilon\alpha}} \quad \text{and} \quad u^{out}_{\varepsilon,\alpha} = 1 - \frac{1}{2}\mathbb{1}_{\Omega^{out}_{\varepsilon\alpha}}.$$

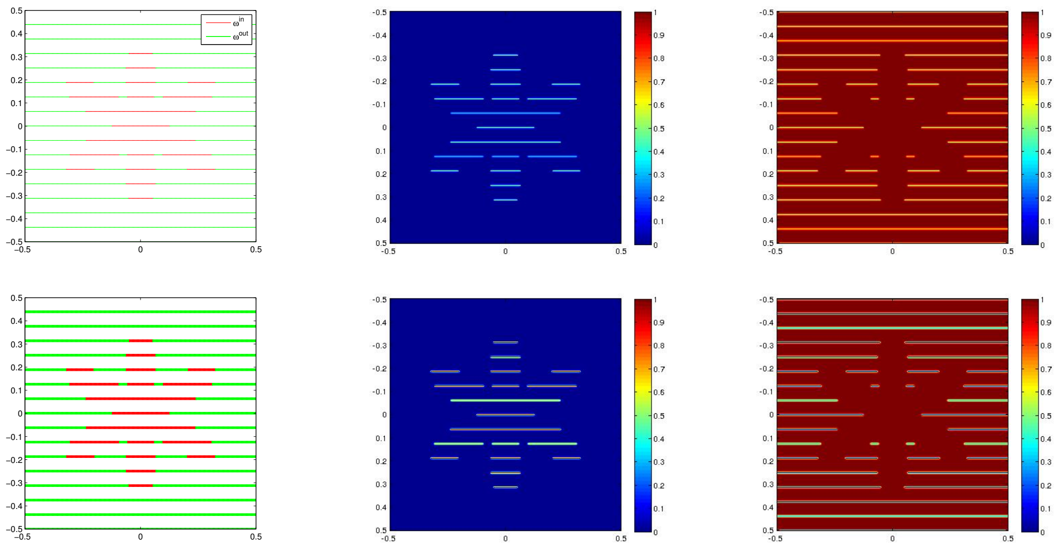

Figure 5: Examples of inner/outer constraints and the associated phase fields. On first row with $\alpha = \infty$, on second row with $\alpha = 1/2$. Left image: $\Omega^{in}_{\varepsilon\alpha}$ (in red) and $\Omega^{out}_{\varepsilon\alpha}$ (in green), middle image: $u^{in}_{\varepsilon,\alpha}$, right image: $u^{out}_{\varepsilon,\alpha}$.

3.2 A projected gradient descent approach

We propose a classical iterative splitting approach to approximate (local) solutions of the problem:

$$u^*_{\varepsilon} = \underset{u_1 \leqslant u \leqslant u_2}{\operatorname{argmin}}\ J_{\varepsilon}(u).$$

The approach consists in alternatively applying one step of a L^2 gradient descent for the energy J_{ε}, followed by a projection onto the set of inclusion-exclusion constraints. Therefore, we introduce an approximating sequence $(u^n)_n$ defined recursively with an artificial time step δ_t as:

1. u^0 is the phase field function $u^0 = q(d(x,E_0)/\varepsilon)$, where E_0 is an initial set built so as to satisfy the constraints, i.e. $\Omega^{in}_{\varepsilon\alpha} \subset E_0 \subset \mathbb{R}^d \setminus \Omega^{out}_{\varepsilon\alpha}$.

2. $u^{n+1/2}$ is an approximation of $v(\delta_t)$, with v the solution to the Cauchy problem

$$\begin{cases} v_t = -\nabla J_{\varepsilon}(v) & \text{on } Q \\ v(x,0) = u^n(x); \end{cases}$$

3. u^{n+1} is the L^2 orthogonal projection of $u^{n+1/2}$ onto the constraints, i.e.

$$u^{n+1} = \underset{u}{\text{argmin}} \left\{ \frac{1}{2\delta_t} \int_Q |u - u^{n+1/2}|^2 dx + \chi_{u_1,u_2}(u) \right\} = \min(\max(u^{n+1/2}, u_1), u_2).$$

Here, χ_{u_1,u_2} is the constraints indicator function defined by

$$\chi_{u_1,u_2}(u) = \begin{cases} 0 & \text{if } u_1 \leqslant u \leqslant u_2 \\ +\infty & \text{otherwise.} \end{cases}$$

Remark 3.1. Our formulation being non convex, the initial choice for E_0 and the initial value of ε have a strong influence on the final solution. The non convexity arises in particular from the locality of the energies integrands. In practice, choosing initially a large value for ε reduces the influence of E_0 for it tends to convexify the problem, in the sense that the locality of the energy is reduced and more topological changes are allowed. Therefore, a possible strategy is to set a large initial value for ε and to reduce it gradually during the evolution, in order to improve both accuracy and sharpness of the phase field. This multiscale approach for ε gives more freedom for the initial choice of E_0, and it is coherent with the Γ-convergence of the approximating energies. In practice, however, for all the numerical experiments presented in this paper, using a fixed value for ϵ seems to be enough to obtain consistent results.

3.3 Numerical schemes for the L^2 gradient flow

Various numerical methods have been proposed to approximate the solution to the Cauchy problem:

$$\begin{cases} \partial_t v = -\nabla J_\varepsilon(v) & \text{on } Q \\ v(x,0) = u^n(x). \end{cases}$$

For the particular case of Allen-Cahn equation, i.e. J = perimeter, there are efficient finite difference methods [4, 19, 30], or finite element methods [27, 28, 23]. We opt as in [16, 24] for a Fourier method with periodic boundary conditions on Q, which yields very accurate solutions and is computationally light. The method is based on the following Euler implicit discretization in time:

$$u^{n+1/2} - u^n = -\delta_t \nabla J_\varepsilon(u^{n+1/2}), \tag{7}$$

which ensures that the energy J_ε decreases. Indeed, the solution $u^{n+1/2}$ also satisfies

$$u^{n+1/2} = \underset{v}{\text{argmin}} \left\{ \frac{1}{2\delta_t} \int_Q (v - u^n)^2 \text{d}x + J_\varepsilon(v) \right\},$$

therefore

$$J_\varepsilon(u^{n+1/2}) \leqslant \frac{1}{2\delta_t} \int_Q (u^{n+1/2} - u^n)^2 \text{d}x + J_\varepsilon(u^{n+1/2}) \leqslant J_\varepsilon(u^n).$$

Schemes to solve (7) using either the perimeter or the Willmore energy are described in the two following sections.

3.3.1 Perimeter case

For the perimeter case, the L^2 gradient flow of P_ε reads as the rescaled Allen Cahn equation

$$\partial_t v = \varepsilon \Delta v - \frac{1}{\varepsilon} W'(v).$$

Equation (7) gives that $u^{n+1/2} - u^n = \delta_t \left(\varepsilon \Delta u^{n+1/2} - \frac{1}{\varepsilon} W'(u^{n+1/2}) \right)$, thus

$$(I_d - \delta_t \Delta) u^{n+1/2} = u^n + \frac{\delta_t}{\varepsilon} W'(u^{n+1/2}).$$

It follows that $u^{n+1/2}$ is a fixed point of

$$\phi_P(v) = (I_d - \delta_t \varepsilon \Delta)^{-1} \left[\left(u^n + \frac{\delta_t}{\varepsilon} W'(v) \right) \right].$$

Moreover, as $\phi_P'(v)(w) = (I_d - \delta_t \varepsilon \Delta)^{-1} \frac{\delta_t}{\varepsilon} W''(v) w$, it is not difficult to see that

$$\|\phi_P'(v)\| \leqslant \frac{\delta_t}{\varepsilon} c_2 \quad \text{ where } c_2 = \sup_{s \in [0,1]} \{|W''(s)|\}.$$

Typically, when $W(s) = \frac{1}{2} s^2 (1-s)^2$, we have $c_2 = 1$. We then deduce that the fixed point iteration converges as soon as $\delta_t < \varepsilon$. In practice, the operator $(I_d - \delta_t \varepsilon \Delta)^{-1}$ can be computed in Fourier domain using the Fast Fourier Transform and by remarking that its associated symbol simply reads as

$$\sigma_P(\xi) = \frac{1}{1 + 4\pi^2 \varepsilon \delta_t |\xi|^2}.$$

3.3.2 Case of Willmore energy

In the case of the Willmore energy, notice that the L^2-gradient flow of $\mathcal{W}_\varepsilon$ reads as

$$\begin{cases} \partial_t v = \Delta \mu - \frac{1}{\varepsilon^2} W''(v) \mu, \\ \mu = \frac{1}{\varepsilon} W'(v) - \varepsilon \Delta v, \end{cases}$$

and Equation (7) becomes

$$\begin{cases} u^{n+1/2} = u^n + \delta_t \left[\Delta \mu^{n+1/2} - \frac{1}{\varepsilon^2} W''(u^{n+1/2}) \mu^{n+1/2} \right] \\ \mu^{n+1/2} = \frac{1}{\varepsilon} W'(u^{n+1/2}) - \varepsilon \Delta u^{n+1/2}. \end{cases}$$

Obviously, $(u^{n+1/2}, \mu^{n+1/2})$ is a fixed point of

$$\phi_{\mathcal{W}} \begin{pmatrix} u \\ \mu \end{pmatrix} = \begin{pmatrix} I_d & -\delta_t \Delta \\ +\varepsilon \Delta & I_d \end{pmatrix}^{-1} \begin{pmatrix} u^n - \frac{\delta_t}{\varepsilon} W''(u) \mu \\ \frac{1}{\varepsilon^2} W'(u) \end{pmatrix},$$

where the operator $\begin{pmatrix} I_d & -\delta_t \Delta \\ +\varepsilon \Delta & I_d \end{pmatrix}^{-1} = \left(I_d + \delta_t \varepsilon \Delta^2 \right)^{-1} \begin{pmatrix} I_d & \delta_t \Delta \\ -\varepsilon \Delta & I_d \end{pmatrix}$ can be easily computed in Fourier domain. Note that the fixed point iterative scheme is locally stable if

$$\delta_t \leqslant C \min \left\{ \varepsilon^3, \frac{\varepsilon}{N^2} \right\}$$

where N is the number of Fourier modes in each direction and C is a constant which depends only on the double well potential W [12].

As above, the operator $\begin{pmatrix} I_d & -\delta_t \Delta \\ +\varepsilon \Delta & I_d \end{pmatrix}^{-1}$ can be computed in Fourier domain using Fast Fourier Transform and observing that its associated symbol simply reads as:

$$\sigma_{\mathcal{W}}(\xi) = \frac{1}{1 + 16\pi^4 \delta_t \varepsilon |\xi|^4} \begin{pmatrix} 1 & -4\pi^2 \delta_t |\xi|^2 \\ 4\pi^2 \varepsilon |\xi|^2 & 1 \end{pmatrix}.$$

3.4 A first experiment in dimension 2

In this first example, we compare various results obtained by solving numerically problems (P1), (P2), and (P3) with the following numerical parameters: $N = 2^9$, and $\varepsilon = 1.5/N$. Concerning time-step, we set $\delta_t = \varepsilon$ for problems (P1), (P2) which involve the perimeter, and $\delta_t = 1/N^3$ for problem (P3) where the Willmore energy is used as regularization criterion.

In the first picture (top left) of figure 6, inner constraints are shown in red and outer constraints in green. The second picture (top right) shows the reconstruction obtained with the sharply constrained perimeter where the boundary of the optimal set E^* is plotted in black. We can observe that E^* is not smooth and does not satisfy the constraints. For the perimeter with fat constraints (problem (P2)) the numerical solution is plotted on the third picture (bottom left). As expected, E^* satisfies the constraints but the use of constraints fattening can be seen on the numerical solution.

Lastly, the last example on the bottom right figure shows a result obtained with the Willmore energy (problem (P3)). The reconstructed curve is visually smooth, and it satisfies the constraints despite they are not fattened.

Another numerical experiment using the Willmore energy is plotted on Figure 7, for which we use non complementary inner-outer constraints, i.e. $\omega_n^{in} \cap \omega_n^{out} \neq \Pi_n$. As explained in the introduction, our approach does not require that the two constraints be mutually complementary. Again, the reconstructed surface looks smooth and natural, and it satisfies the constraints (right picture in Figure 7).

To conclude, we can deduce from this first set of experiments that the Willmore approach allows the reconstruction of very natural sets, and we will focus on problem (P3) for 3D cases in the sequel.

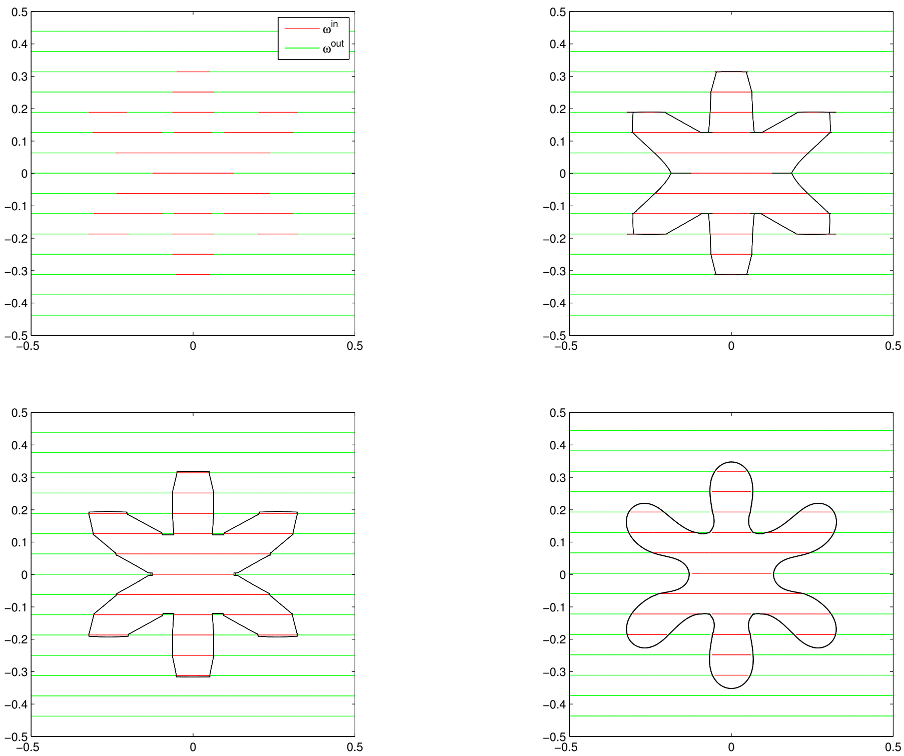

Figure 6: Reconstruction example with three different methods: (top left) input data: interior/exterior constraints - (top right): reconstruction using perimeter with sharp constraints - (bottom left): reconstruction using perimeter with fat constraints - (bottom right): reconstruction using the constrained Willmore energy.

3.5 3D numerical experiments using Willmore energy

We present here a few numerical tests in space dimension 3 using the reconstruction with the constrained Willmore energy from a given set of 2D slices. In each experiment we used $N = 2^7$, $\varepsilon = 1.5/N$ and $\delta_t = 1/N^3$.

In the experiment of Figure 8, we use as initial data two sets of 2D slices of the Standford Bunny: as can be observed, the reconstruction is effective even when the number of slices is low, although in this latter case rabbit's feet cannot be reconstructed since the associated information is fully missing in the slices input data.

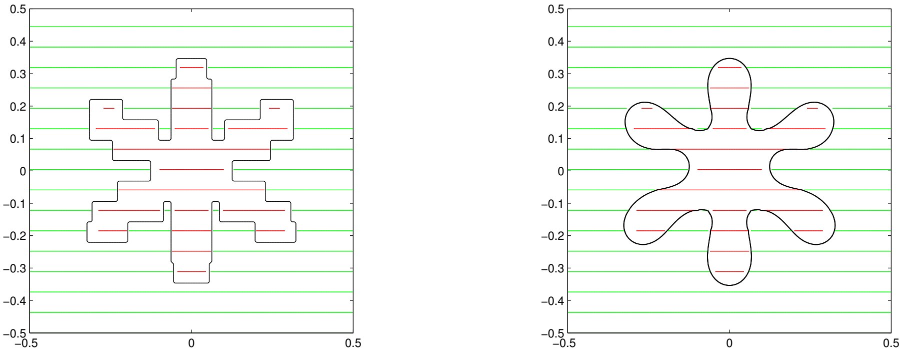

Figure 7: Reconstruction example where interior-exterior constraints are not complementary (i.e., $\omega_n^{in} \cup \omega_n^{out} \neq \Pi_n$). Left: input data constraints and initial reconstructed set E_0 - Right: final reconstructed surface using the constrained Willmore energy.

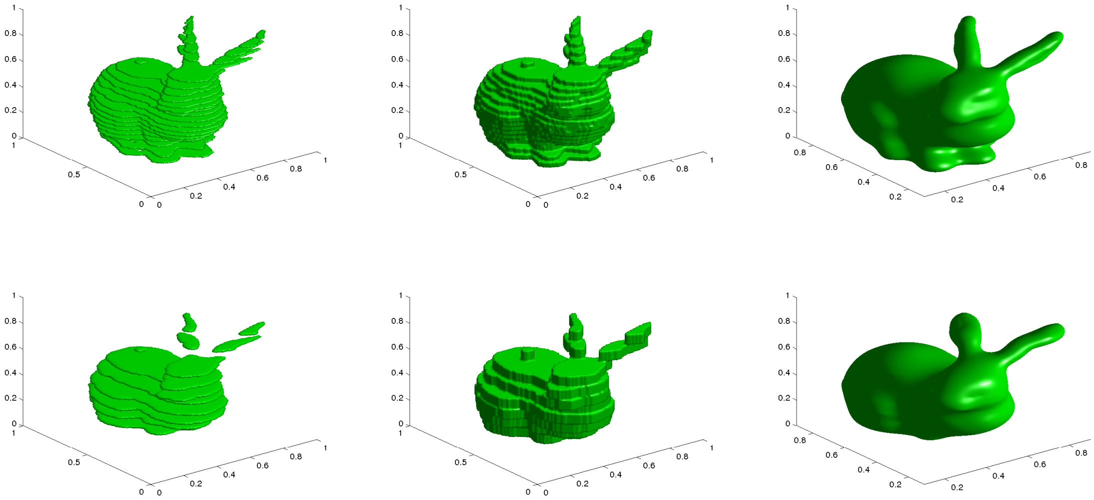

Figure 8: Two experiments with Stanford bunny using either 24 slices (first line) or 12 slices (second line). On each row, the given interior constraints are shown on the left column, the initial reconstruction E_0 is shown in middle, and the final reconstructed surface is provided on the right column. Observe on the second line that bunny's feet cannot be properly reconstructed since they are missing in the initial cross-sections.

In the second example (see Figure 9), we test how the method behaves with respect to topology changes between slices, each slice being either a circle of the union of two circles. Similar configurations are very common in medical imaging. Our experiment illustrates the efficiency of the reconstruction using Willmore energy, which yields a surface close to branching cylinders when the density of slices is large enough. In contrast, even with many slices, using perimeter as regularization energy will yield undesirable surfaces made of glued catenoids. Remark also that the Willmore energy ensures a smooth closure of the surface at top and bottom, whereas minimizing with the perimeter and fat constraints yields a surface with almost two disks as closing caps.

The last example shows an application to the reconstruction of real MRI data (Figure 10), which was the initial main motivation of this work. Although no information is provided about the topology of the surface, using the constrained Willmore energy yields a very realistic solution.

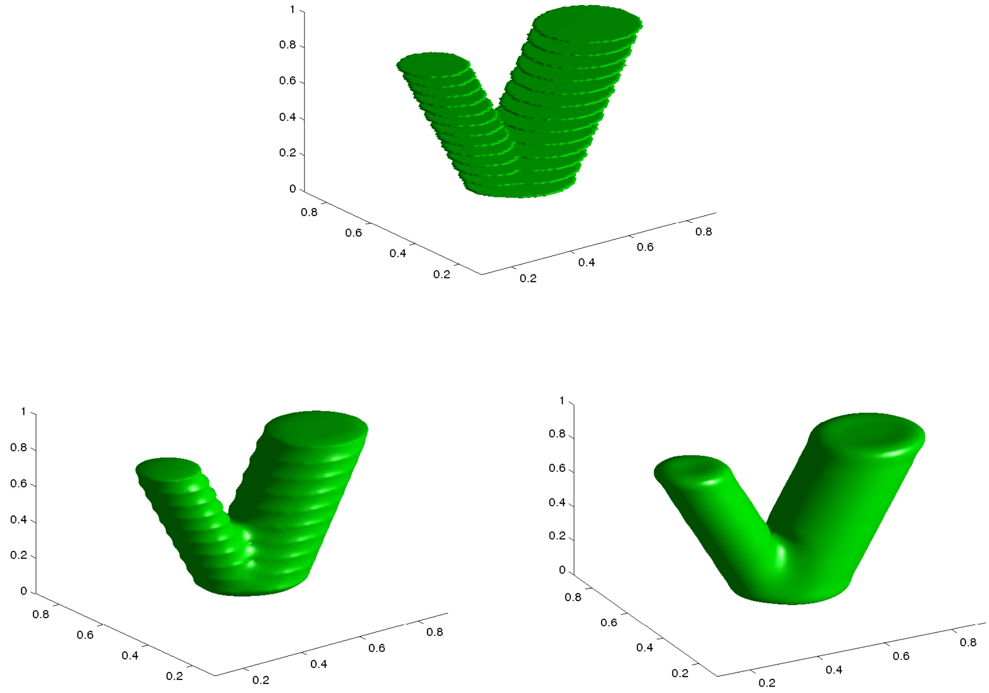

Figure 9: Reconstruction of branching cylinders. Top: initial data (inner constraints). Bottom left: reconstruction using perimeter and fat constraints. Bottom right: reconstruction using Willmore energy and fat constraints.

4 Conclusion

We propose in this paper a phase field model for the consistent reconstruction of d-dimensional surfaces from $(d-1)$-dimensional slices, using approximations of either the Willmore energy or the perimeter as a regularization criterion. A crucial property of the phase field approximation in this context is that constraints admit a linear obstacle formulation $u^{\text{in}} \leqslant u \leqslant u^{\text{out}}$, which is very useful both for the convergence analysis and the design of our numerical scheme. Due to the dimensionality of the constraints, we show that no consistency can be expected unless the constraints are fattened in the phase field model. In the particular case of perimeter, we prove that the associated phase field approximation with fat constraints Γ-converges to some L^1-relaxation of perimeter with sharp constraints. Our approach is flexible enough so that similar convergence results can be expected for more general, not necessarily parallel and not necessarily $(d-1)$-dimensional, input slices, e.g. non planar slices, slices living on a manifold, volumetric point clouds, etc.

In the case of the Willmore energy as regularization criterion, a fine characterization of the limit energy remains an open problem due to the non locality of the L^1-relaxation of the (unconstrained) Willmore energy.

We propose efficient schemes for the numerical approximation of critical points of both constrained phase-field models, either Willmore-based or perimeter-based. Our schemes are not restricted to planar or parallel constraint slices, as illustrated in Figure 11 where non planar slices or volumetric point clouds are used as constraints. We also emphasize that constraints need not be complementary, i.e. some freedom can be allowed between inner and outer constraints (see figure 7). This is potentially very useful for considering noisy or statistical data for which uncertainties arise naturally.

The extension of our approach to multiphase and/or anisotropic situations is a work in progress. Preliminary reconstruction results of multiphase surfaces using Willmore energy is shown in Figure 12, which

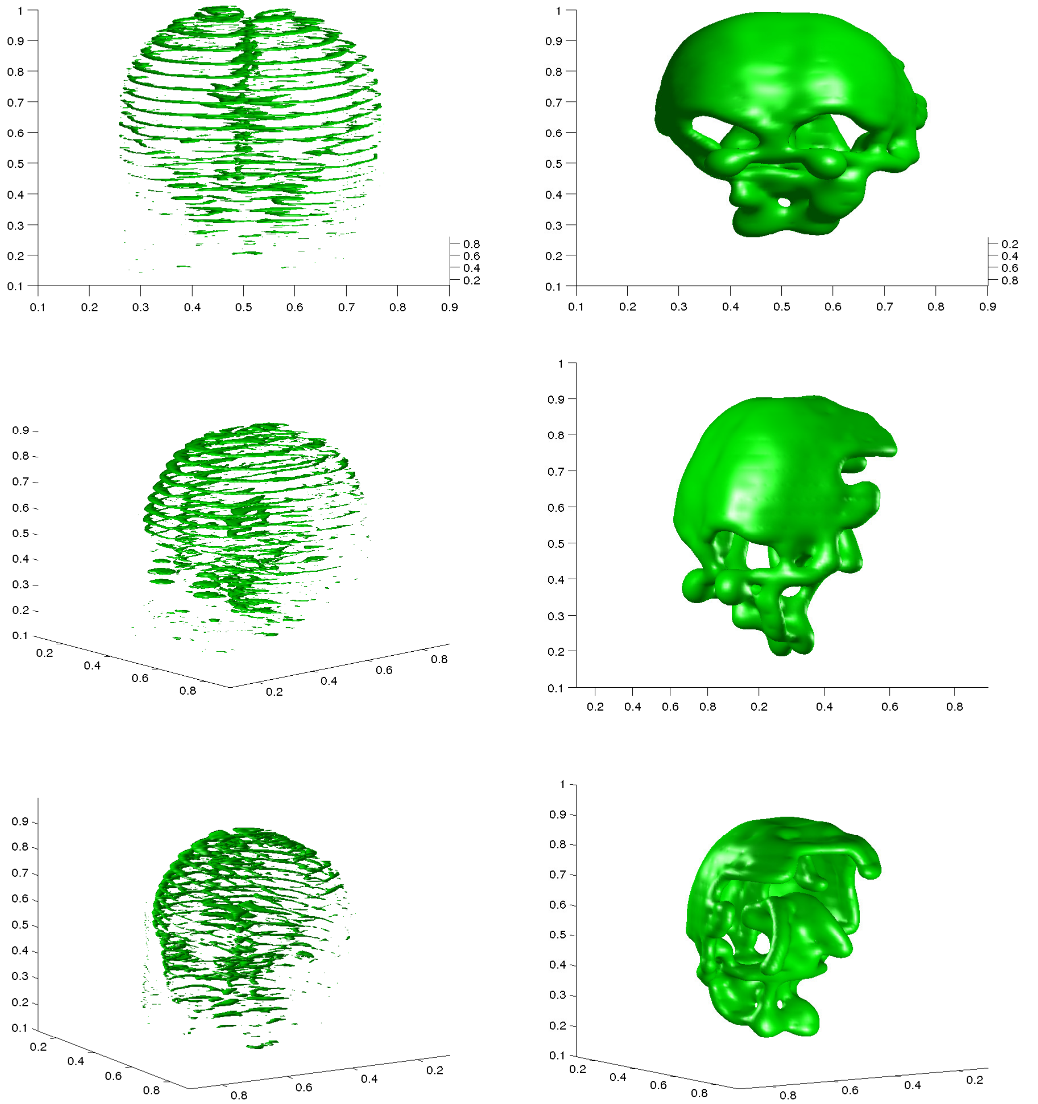

Figure 10: Three different views of a reconstruction from real MRI data - Left: input data (interior constraints) - Right : reconstructed surface using the constrained Willmore energy.

opens interesting perspectives for the reconstruction of segmented data. More generally, we strongly believe that our method is very promising for numerous applications to surface reconstruction or interpolation.

Acknowledgements

The authors acknowledge the support of the French National Research Agency (ANR) under grant ANR-12-BS01-0014-01 (project GEOMETRYA). This work was also supported by the LABEX MILYON (ANR-10-LABX-0070) of Université de Lyon, within the program "Investissements d'Avenir" (ANR-11-IDEX-0007) operated by the French National Research Agency (ANR).

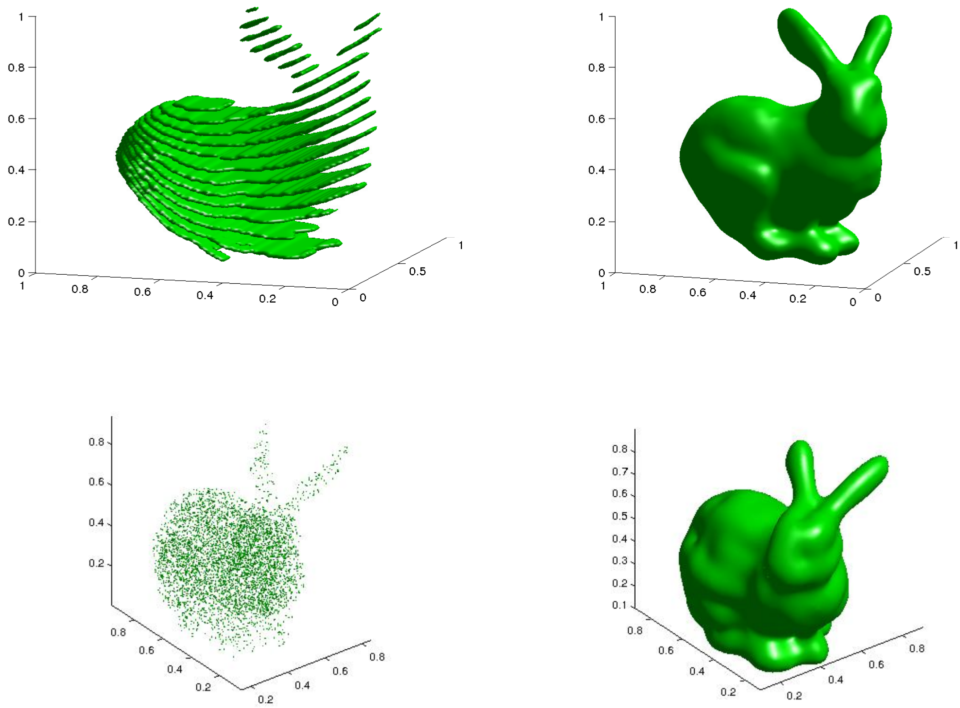

Figure 11: Possible adaptations of the method - Left : interior constraints - Right : reconstructed surface - First line with non parallel bowed surfaces - Second line : with point clouds filling the interior of the domain.

A Technical proofs

A.1 Proof of lemma 2.4

Let $\eta > 0$ and consider the sequence $(E_\eta)_{\eta>0}$ defined by $E_\eta = (E \cup \Omega_\eta^{in}) \setminus \Omega_\eta^{out}$. We will now prove that

$$\limsup_{\eta \to 0} P(E_\eta, Q) \leqslant P(E, Q) + 2\mathcal{H}^{d-1}(E^0 \cap \omega^{in}) + 2\mathcal{H}^{d-1}(E^1 \cap \omega^{out}) \tag{8}$$

The idea consists in introducing a partition of $Q = B_0 \cup (\cup_{n,k} B_{n,k}^{in}) \cup (\cup_{n,\ell} B_{n,\ell}^{out})$ where the boxes $B_{n,k}^{in}$ and $B_{n,\ell}^{out}$ are associated with $\omega_{n,k}^{in}$ and $\omega_{n,\ell}^{out}$, respectively. We recall that $\omega^{in} \cap \Pi_n$ and $\omega^{out} \cap \Pi_n$ both are a finite union of disjoint, connected, open, bounded and Lipschitz sets in Π_n denoted as $\omega_{n,k}^{in}$ and $\omega_{n,\ell}^{out}$, respectively.

More precisely, for each (n, k, ℓ), we can define respectively (see definition 6) the boxes $B_{n,k}^{in} = \Omega_{n,k,h}^{in}$ and $B_{n,\ell}^{out} = \Omega_{n,\ell,h}^{out}$ where $h > 0$ is assumed to be well chosen (see Lemma A.1 below). We define also the free part box $B_0 = Q \setminus \Big((\cup_{n,k} B_{n,k}^{in}) \cup (\cup_{n,k} B_{n,k}^{out}) \Big)$. For convenience, we introduce the boxes $\{\tilde{B}_m\}$ as a simple reindexation of boxes $B_{n,k}^{in}$ and $B_{n,\ell}^{out}$, i.e.

$$Q = B_0 \cup \left(\bigcup_{n,k} B_{n,k}^{in} \right) \cup \left(\bigcup_{n,\ell} B_{n,\ell}^{out} \right) = \bigcup_m \tilde{B}_m.$$

We use the following lemma to set $h > 0$ (proof is given in Section A.3 of Appendix):

Lemma A.1. *There exists $h > 0$ such that the family of sets $\{\tilde{B}_m\}_m$ is pairwise disjoint and, for all m, $\mathcal{H}^{d-1}(E^{1/2} \cap \tilde{B}_m) = \emptyset$.*

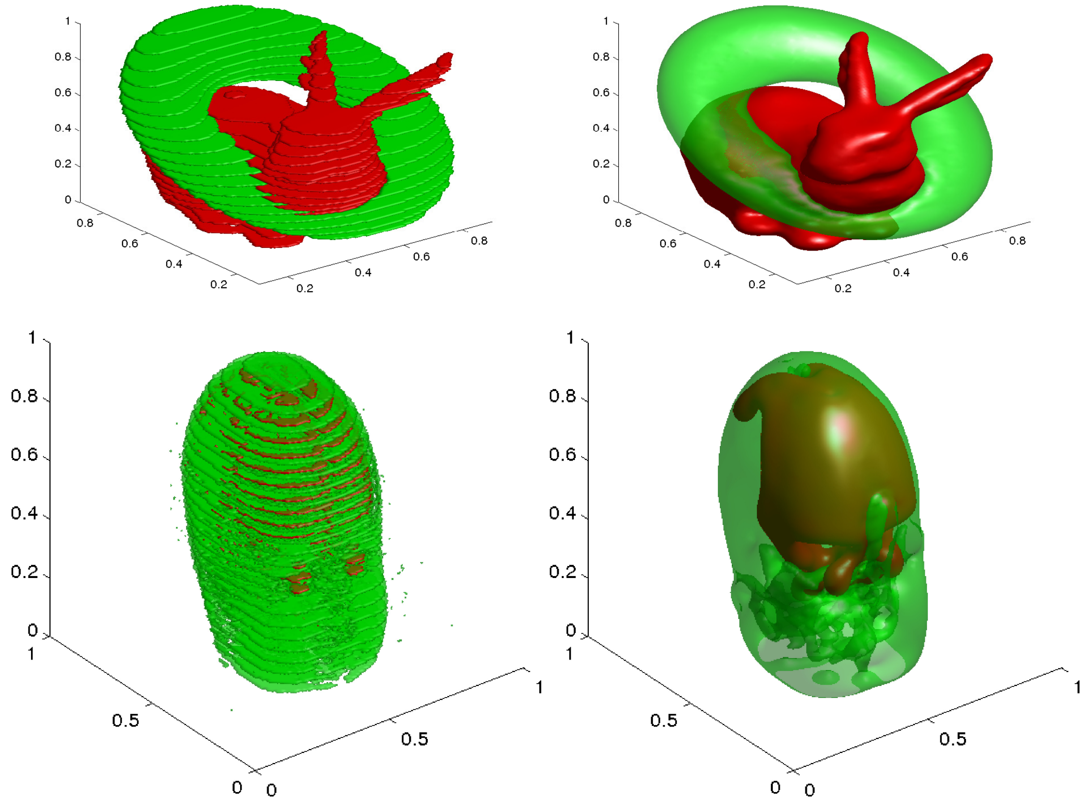

Figure 12: Possible adaptations of the method in a multiphase context - Reconstruction of two surfaces E^1 (red surface) and E^2 (green surface) with two sets of inclusion-exclusion constraints ; First line with $E^1 \cap E^2 = \emptyset$ - Second line with $E^1 \subset E^2$; Left - inclusion constraints - Right : reconstructed surfaces.

Therefore h is independent of ε and (n,k,ℓ), and it satisfies that each intersection of two boxes is empty and for all (n,k,ℓ) and $\mathcal{H}^{d-1}(E^{1/2} \cap \partial B^{in}_{n,k}) = \mathcal{H}^{d-1}(E^{1/2} \cap \partial B^{out}_{n,\ell}) = 0$. Then, as $\{\tilde{B}_m\}_m$ is a pairwise disjoint family of sets, we have $P(E_\eta, Q) = \sum_{m \geqslant 0} P(E_\eta, \tilde{B}_m)$.

Inequality (8) will be established locally by considering the following **three cases**:

1) if $\tilde{B}_m = B_0$, prove that $\limsup_{\eta \to 0} P(E_\eta, \tilde{B}_m) \leqslant P(E, \tilde{B}_m)$.

2) if $\tilde{B}_m = B^{in}_{n,k}$, prove that $\limsup_{\eta \to 0} P(E_h, \tilde{B}_m) \leqslant P(E, \tilde{B}_m) + 2\mathcal{H}^{d-1}(\omega^{in}_{n,k} \cap E^0)$.

3) if $\tilde{B}_m = B^{out}_{n,\ell}$, prove that $\limsup_{\eta \to 0} P(E_h, \tilde{B}_m) \leqslant P(E, \tilde{B}_m) + 2\mathcal{H}^{d-1}(\omega^{out}_{n,\ell} \cap E^1)$.

We then gather these three cases to have a global inequality.

1) On the free part: $\tilde{B}_m = B_0$.
In this case, the result follows immediately as $P(E_\eta, B_0) = P(E, B_0)$.

2) Near an inside constraint: $\tilde{B}_m = B^{in}_{n,k}$.
First, notice that $P(E_\eta, \tilde{B}_m) = P(E \cup \Omega^{in}_{n,k,\eta}, \tilde{B}_m)$. We use the classical following inequality for perimeter (see [2, Proposition 3.38]):

Proposition A.2.

$$P(A \cup B, \Omega) + P(A \cap B, \Omega) \leqslant P(A, \Omega) + P(B, \Omega).$$

Taking $A = E$, $B = \Omega^{in}_{n,k,\eta}$ and $\Omega = \tilde{B}_m$, we obtain

$$P(E \cup \Omega^{in}_{n,k,\eta}, \tilde{B}_m) \leqslant P(E, \tilde{B}_m) + P(\Omega^{in}_{n,k,\eta}, \tilde{B}_m) - P(E \cap \Omega^{in}_{n,k,\eta}, \tilde{B}_m)$$

We can also apply the following lemma (see Section A.3 for a proof) in the case $A = E \cap \Omega^{in}_{n,k,\eta}$ and $\Pi = \Pi_n$.

Lemma A.3. *Let A be a set with finite perimeter and Π be a hyperplane of $\mathbb{R}^d$. Then*

$$P(A) \geqslant 2\mathcal{H}^{d-1}(\Pi \cap (A^1 \cup A^{1/2})).$$

As $E \cap \Omega^{in}_{n,k,\eta}$ is included in $\tilde{B}_m$ (except possibly along $\partial_n \omega^{in}_{n,k}$ which has zero $\mathcal{H}^{d-1}$-measure), we have $P(E \cap \Omega^{in}_{n,k,\eta}, \tilde{B}_m) = P(E \cap \Omega^{in}_{n,k,\eta})$. Notice also that on Π_n, we have $(E \cap \Omega^{in}_{n,k,\eta})^1 = E^1 \cap \Omega^{in}_{n,k,\eta}$ up to a $\mathcal{H}^{d-1}$- negligible set because $\Omega^{in}_{n,k,\eta}$ is open. Indeed, using the notation

$$\theta^r_x(A) = \frac{|A \cap B(x,r)|}{|B(x,r)|},$$

we have:

- if $x \in \Pi_n \cap \Omega^{in}_{n,k,\eta}$ then $\theta^r_x(E \cap \Omega^{in}_{n,k,\eta}) = \theta^r_x(E)$;
- if $x \in \Pi_n \cap (\Omega \setminus \overline{\Omega^{in}_{n,k,\eta}})$ then $\theta^r_x(E \cap \Omega^{in}_{n,k,\eta}) \leqslant \theta^r_x(\Omega^{in}_{n,k,\eta}) = 0$ for r small enough;
- otherwise x belongs to $\partial\Omega^{in}_{n,k,\eta} \cap \Pi_n = \partial_n \omega^{in}_{n,k}$ which has zero $\mathcal{H}^{d-1}$-measure.

Similarly we have $(E \cap \Omega^{in}_{n,k,\eta})^{1/2} = E^{1/2} \cap \Omega^{in}_{n,k,\eta}$. Consequently, we obtain

$$\begin{aligned}\mathcal{H}^{d-1}\left(\Pi_n \cap \left((E \cap \Omega^{in}_{n,k,\eta})^1 \cup (E \cap \Omega^{in}_{n,k,\eta})^{1/2}\right)\right) &= \mathcal{H}^{d-1}\left(\Pi_n \cap \Omega^{in}_{n,k,\eta} \cap (E^1 \cup E^{1/2})\right) \\ &= \mathcal{H}^{d-1}(\omega^{in}_{n,k} \cap (E^1 \cup E^{1/2})).\end{aligned}$$

which shows that $P(E \cap \Omega^{in}_{n,k,\eta}, \tilde{B}_m) \geqslant 2\mathcal{H}^{d-1}(\omega^{in}_{n,k} \cap (E^1 \cup E^{1/2}))$, and then

$$P(E \cup \Omega^{in}_{n,k,\eta}, \tilde{B}_m) \leqslant P(E, \tilde{B}_m) + P(\Omega^{in}_{n,k,\eta}, \tilde{B}_m) - 2\mathcal{H}^{d-1}(\omega^{in}_{n,k} \cap (E^1 \cup E^{1/2})).$$

Finally, taking the lim sup when $\eta \to 0$ and using the following lemma (see Section A.3 for a proof) leads to

$$\begin{aligned}\limsup_{\eta \to 0} P(E \cup \Omega^{in}_{n,k,\eta}, \tilde{B}_m) &\leqslant P(E, \tilde{B}_m) + 2\mathcal{H}^{d-1}(\omega^{in}_{n,k}) - 2\mathcal{H}^{d-1}(\omega^{in}_{n,k} \cap (E^1 \cup E^{1/2})) \\ &\leqslant P(E, \tilde{B}_m) + 2\mathcal{H}^{d-1}(\omega^{in}_{n,k} \cap E^0).\end{aligned},$$

as $E^1 \cup E^{1/2} \cup E^0$ is a $\mathcal{H}^{d-1}$-almost partition of $\mathbb{R}^d$.

Lemma A.4. *The fat constraints $\Omega^{in}_{n,k,\eta}$ and $\Omega^{out}_{n,\ell,\eta}$ satisfy $P(\Omega^{in}_{n,k,\eta}, B^{in}_{n,k}) \xrightarrow{\eta \to 0} 2\mathcal{H}^{d-1}(\omega^{in}_{n,k})$ and $P(\Omega^{out}_{n,\ell,\eta}, B^{out}_{n,\ell}) \xrightarrow{\eta \to 0} 2\mathcal{H}^{d-1}(\omega^{out}_{n,\ell})$.*

3) Near an outside constraint: $\tilde{B}_m = B^{out}_{n,\ell}$.
There holds $P(E_\eta, \tilde{B}_m) = P(E \setminus \Omega^{out}_{n,\ell,\eta}, \tilde{B}_m)$, and taking the complementary set in Q leads to

$$\begin{aligned}P(E \setminus \Omega^{out}_{n,\ell,\eta}, \tilde{B}_m) &= P(E \cap (Q \setminus \Omega^{out}_{n,\ell,\eta}), \tilde{B}_m) = P((Q \setminus E) \cup \Omega^{out}_{n,\ell,\eta}, \tilde{B}_m) \\ &\leqslant P((Q \setminus E), \tilde{B}_m) + P(\Omega^{out}_{n,\ell,\eta}, \tilde{B}_m) - P((Q \setminus E) \cap \Omega^{out}_{n,\ell,\eta}, \tilde{B}_m).\end{aligned}$$

We use exactly the same argument as in the previous case with $Q \setminus E$ instead of E and we obtain

$$\begin{aligned}\limsup_{\eta \to 0} P(E \setminus \Omega^{out}_{n,\ell,\eta}, \tilde{B}_m) &\leqslant P(Q \setminus E, \tilde{B}_m) + 2\mathcal{H}^{d-1}(\omega^{out}_{n,\ell} \cap (Q \setminus E)^0) \\ &\leqslant P(E, \tilde{B}_m) + 2\mathcal{H}^{d-1}(\omega^{out}_{n,\ell} \cap E^1).\end{aligned}$$

From local to global

Then we have

$$\begin{aligned}
\limsup_{\eta\to 0} P(E_\eta, Q) &= \limsup_{\eta\to 0} \sum_{m\geqslant 0} P(E_\eta, \tilde{B}_m) \\
&\leqslant \limsup_{\eta\to 0} P(E_\eta, B_0) + \sum_{n,k} \limsup_{\eta\to 0} P(E_\eta, B^{in}_{n,k,\eta}) + \sum_{n,\ell} \limsup_{\eta\to 0} P(E_\eta, B^{out}_{n,\ell,\eta}) \\
&\leqslant P(E_\eta, B_0) + \sum_{n,k} \left(P(E, B^{in}_{n,k}) + 2\mathcal{H}^{d-1}(\omega^{in}_{n,k} \cap E^0)\right) \\
&\qquad + \sum_{n,\ell} \left(P(E, B^{out}_{n,\ell}) + 2\mathcal{H}^{d-1}(\omega^{out}_{n,\ell} \cap E^1)\right) \\
&\leqslant P(E, \Omega) + 2\mathcal{H}^{d-1}(E^0 \cap \omega^{in}) + 2\mathcal{H}^{d-1}(E^1 \cap \omega^{out}).
\end{aligned}$$

A.2 Proof of Lemma 2.5

As in the proof of the Γ-limsup, we consider a partition of the computation box $Q = B_0 \cup \left(\bigcup_{n,k} B^{in}_{n,k}\right) \cup \left(\bigcup_{n,\ell} B^{out}_{n,\ell}\right) = \bigcup_m \tilde{B}_m$ and we will first prove the lemma locally in each box $\tilde{B}_m$ and then globally.

A.2.1 Near an inside constraint $\tilde{B}_m = B^{in}_{n,k}$

Let us look at one constraint $\omega^{in}_{n,k}$ and its box $\tilde{B}_m = B^{in}_{n,k}$. As $u_\varepsilon \to \mathbb{1}_E$ in L^1 we have, up to a subsequence, for almost every $s \in [0,1]$

$$\{u_\varepsilon \geqslant s\} \xrightarrow{\varepsilon\to 0} E \quad \text{in measure in } Q.$$

Due to $P_{\varepsilon, u^{in}_{\varepsilon,\alpha}, u^{out}_{\varepsilon,\alpha}}(u_\varepsilon) < +\infty$, we have $u^{in}_\varepsilon \leqslant u_\varepsilon$ and then $\{u^{in}_\varepsilon \geqslant s\} \subset \{u_\varepsilon \geqslant s\}$. Notice that $u^{in}_\varepsilon(x) \geqslant s \quad \Leftrightarrow \quad d(x, \Omega^{in}_{n,k,\varepsilon^\alpha}) \leqslant \varepsilon q^{-1}(s)$.

- If $s \leqslant q(0) = 1/2$ then $\varepsilon q^{-1}(s) \geqslant 0$ and so, for every $\varepsilon^\alpha < h$,

  $$\Omega^{in}_{n,k,\varepsilon^\alpha} \subset \{x \in B_{n,k} \mid d(x, \Omega^{in}_{n,k,\varepsilon^\alpha}) \leqslant \varepsilon q^{-1}(s)\}.$$

  That means $\Omega^{in}_{n,k,\varepsilon^\alpha}$ is an open subset enclosing $\omega^{in}_{n,k}$ and separating $B^{in}_{n,k}$ into two parts (see Figure 13 [left])

  $$B^+_{n,k} = \left\{(y,z) \in (\mathbb{R}^{d-1} \times \mathbb{R}) \cap Q \;\middle|\; y \in \omega^{in}_{n,k},\ z > 0,\ |z| < \varepsilon^\alpha |d_n(y, \omega^{in}_{n,k})|\right\}$$

  $$\text{and } B^-_{n,k} = \left\{(y,z) \in (\mathbb{R}^{d-1} \times \mathbb{R}) \cap Q, \;\middle|\; y \in \omega^{in}_{n,k},\ z < 0,\ |z| < \varepsilon^\alpha |d_n(y, \omega^{in}_{n,k})|\right\}.$$

  We denote $S_{n,k,\varepsilon} = \Omega^{in}_{n,k,\varepsilon^\alpha}$.

- If $s > q(0) = 1/2$ then $\varepsilon q^{-1}(s) < 0$ and so $\{x \in \tilde{B}_m \mid d(x, \Omega^{in}_{n,k,\varepsilon^\alpha}) \leqslant \varepsilon q^{-1}(s)\}$ does not enclose necessarily $\omega^{in}_{n,k}$ anymore (see Figure 13 [right]). Thus we have to reduce $B_{n,k}$ in order to be in the previous case (see Figure 14). Let $\delta > 0$ and consider $B^{in}_{n,k,\delta} = \Omega^{in}_{n,k,\varepsilon_0^\alpha,\delta}$ and $S_{n,k,\varepsilon,\delta}$ defined by

  $$\Omega_{n,k,\varepsilon^\alpha,\delta} = \left\{(y,z) \in (\mathbb{R}^{d-1} \times \mathbb{R}) \cap Q,\ y \in \omega^{in}_{n,k},\ |d_n(y, \omega^{in}_{n,k})| > \delta,\ |z| < \varepsilon^\alpha |d_n(y, \omega^{in}_{n,k})|\right\},$$

  $$\text{and } S_{n,k,\varepsilon,\delta} = \left\{(y,z) \in (\mathbb{R}^{d-1} \times \mathbb{R}) \cap Q,\ y \in \omega^{in}_{n,k},\ |z| < \ell_\varepsilon,\ |d_n(y, \omega^{in}_{n,k})| > \delta\right\}$$

  where $\ell_\varepsilon = \delta\varepsilon^\alpha - \varepsilon q^{-1}(s)\sqrt{1 + \varepsilon^{2\alpha}}$ (see Figure 14). For ε small enough, $S_{n,k,\varepsilon,\delta}$ is an open subset enclosing $\omega^{in}_{n,k} \cap \Omega_{n,k,\varepsilon^\alpha,\delta}$ and separating $B^{in}_{n,k,\delta}$ into two parts $B^+_{n,k,\delta}$ and $B^-_{n,k,\delta}$.

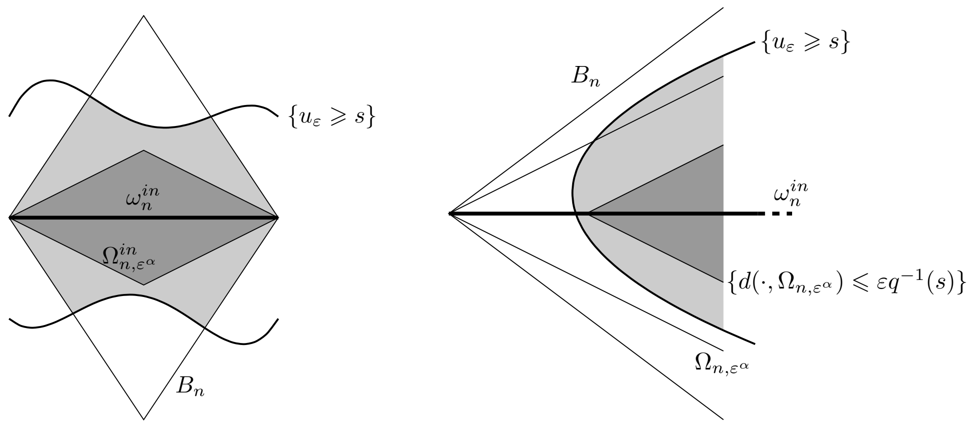

Figure 13: Level sets.

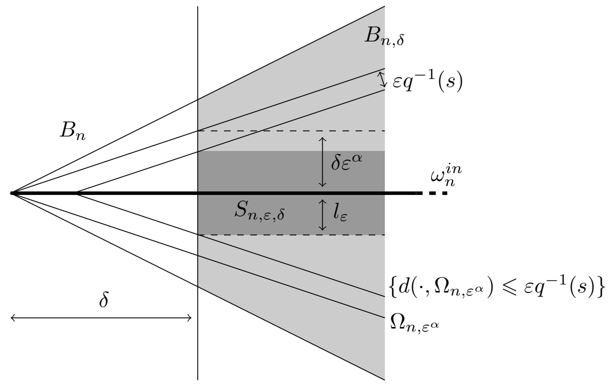

Figure 14: The small open $S_{n,k,\varepsilon,\delta}$ for the second case.

In both cases, the proof is the same and we denote B and S to treat both cases together ($B_{n,k}^{in}$ and $S_{n,k,\varepsilon}$ for the first and $B_{n,k,\delta}^{in}$ and $S_{n,k,\varepsilon,\delta}$ for the second). As S is an open subset included in $\{u_\varepsilon \geqslant s\}$ and containing $\omega_{n,k}^{in} \cap B$ the interface between B^+ and B^-, we have the following relation on the perimeter:

$$\begin{aligned} P(\{u_\varepsilon \geqslant s\}, B) &= P(\{u_\varepsilon \geqslant s\}, B^+) + P(\{u_\varepsilon \geqslant s\}, B^-) \\ &= P(\{u_\varepsilon \geqslant s\} \cup B^-, B^+) + P(\{u_\varepsilon \geqslant s\} \cup B^+, B^-) \\ &= P(\{u_\varepsilon \geqslant s\} \cup B^-, B) + P(\{u_\varepsilon \geqslant s\} \cup B^+, B). \end{aligned} \tag{9}$$

Moreover, recall that for almost every s, $\{u_\varepsilon \geqslant s\} \overset{\varepsilon\to 0}{\longrightarrow} E$ in measure in Q. Then, it holds that $\{u_\varepsilon \geqslant s\} \cup B^+ \overset{\varepsilon\to 0}{\longrightarrow} E \cup B^+$ in measure in B, and $\{u_\varepsilon \geqslant s\} \cup B^- \overset{\varepsilon\to 0}{\longrightarrow} E \cup B^-$ in measure in B. So, taking the $\liminf$ in (9), we have

$$\liminf_{\varepsilon\to 0} P(\{u_\varepsilon \geqslant s\}, B) \geqslant P(E \cup B^-, B) + P(E \cup B^+, B). \tag{10}$$

In the case $B = B_{n,k}^{in}$, it means that $\liminf_{\varepsilon\to 0} P(\{u_\varepsilon \geqslant s\}, B_{n,k}^{in}) \geqslant P(E \cup B_{n,k}^-, B_{n,k}^{in}) + P(E \cup B_{n,k}^+, B_{n,k}^{in})$. In the case of $B = B_{n,k,\delta}^{in}$, let us consider (δ_p) a decreasing sequence converging to 0, then $B_{n,k,\delta_p}^{in} \subset B_{n,k,\delta_{p+1}}^{in} \subset B_{n,k}^{in}$, $P(E \cup B_{n,k,\delta_p}^-, B_{n,k,\delta_p}) = P(E \cup B_{n,k}^-, B_{n,k,\delta_p})$ and $P(E \cup B_{n,k,\delta_p}^+, B_{n,k,\delta_p}) = P(E \cup B_{n,k}^+, B_{n,k,\delta_p})$. Using the increasing limit property of measures $B_{n,k}^{in} = \bigcup_{p\in\mathbb{N}} B_{n,k,\delta_p}$ for $A \mapsto P(\{u_\varepsilon \geqslant s\}, A)$, $A \mapsto P(E \cup B_{n,k}^+, A)$ and $A \mapsto P(E \cup B_{n,k}^-, A)$, we can take the limit $p \to +\infty$, and thus $P(\{u_\varepsilon \geqslant s\}, B_{n,k,\delta_p}) \overset{p\to+\infty}{\longrightarrow} P(\{u_\varepsilon \geqslant s\}, B_{n,k}^{in})$, $P(E \cup B_{n,k}^-, B_{n,k,\delta_p}) \overset{p\to+\infty}{\longrightarrow} P(E \cup B_{n,k}^-, B_{n,k}^{in})$, and

$P(E\cup B^+_{n,k}, B_{n,k,\delta_p}) \overset{p\to+\infty}{\longrightarrow} P(E\cup B^+_{n,k}, B^{in}_{n,k})$. So, taking the lim inf on $\delta_p \to 0$ in (10), we have

$$\liminf_{\delta\to 0}\liminf_{\varepsilon\to 0} P(\{u_\varepsilon \geqslant s\}, B_{n,k,\delta}) \geqslant P(E\cup B^-_{n,k}, B_n) + P(E\cup B^+_{n,k}, B^{in}_{n,k}).$$

Noticing that the quantities are increasing as δ_p and ε decrease to 0, we have:

$$\begin{aligned}
\liminf_{\delta\to 0}\liminf_{\varepsilon\to 0} P(\{u_\varepsilon \geqslant s\}, B_{n,k,\delta}) &= \sup_{p\in\mathbb{N}}\sup_{\varepsilon>0}\inf_{\varepsilon'<\varepsilon} P(\{u_{\varepsilon'} \geqslant s\}, B_{n,k,\delta_p})\\
&= \sup_{\varepsilon>0}\sup_{p\in\mathbb{N}}\inf_{\varepsilon'<\varepsilon} P(\{u_{\varepsilon'} \geqslant s\}, B_{n,k,\delta_p})\\
&\leqslant \sup_{\varepsilon>0}\inf_{\varepsilon'<\varepsilon}\sup_{p\in\mathbb{N}} P(\{u_{\varepsilon'} \geqslant s\}, B_{n,k,\delta_p})\\
&= \sup_{\varepsilon>0}\inf_{\varepsilon'<\varepsilon} P(\{u_{\varepsilon'} \geqslant s\}, B^{in}_{n,k})\\
&= \liminf_{\varepsilon\to 0} P(\{u_\varepsilon \geqslant s\}, B^{in}_{n,k}).
\end{aligned}$$

Thus, (10) is valid for almost every s in $[0,1]$ with $B = B^{in}_{n,k}$. We use the following lemma (see Section A.3 for a proof).

Lemma A.5. *Let E be a set with locally finite perimeter. If $B_{n,k}$ is a box enclosing $\omega^{in}_{n,k}$, then $P(E\cup B^-_{n,k}, B_{n,k}) + P(E\cup B^+_{n,k}, B_{n,k}) = P(E, B_{n,k}) + 2\mathcal{H}^{d-1}(E^0\cap\omega^{in}\cap B_{n,k})$.*

We finally obtain $\liminf_{\varepsilon\to 0} P(\{u_\varepsilon \geqslant s\}, B^{in}_{n,k}) \geqslant P(E, B^{in}_{n,k}) + 2\mathcal{H}^{d-1}(E^0\cap\omega^{in}\cap B^{in}_{n,k})$.

A.2.2 Near an outside constraint $\tilde{B}_m = B^{out}_{n,\ell}$

For a constraint $\omega^{out}_{n,\ell}$ and its associated box $B^{out}_{n,\ell}$, we apply the "inside constraint" result to the complementary set in Q. As $u_\varepsilon \to \mathbb{1}_E$ in $L^1(Q)$ we have, up to a subsequence, for almost every $s\in[0,1]$

$$Q\setminus\{u_\varepsilon \geqslant s\} \overset{\varepsilon\to 0}{\longrightarrow} Q\setminus E \quad \text{in measure in } Q.$$

The constraint $u_\varepsilon \leqslant u^{out}_\varepsilon$ leads us to $Q\setminus\{u^{out}_\varepsilon \geqslant s\} \subset Q\setminus\{u_\varepsilon \geqslant s\}$. The same proof using these complementary sets shows that

$$\liminf_{\varepsilon\to 0} P(Q\setminus\{u_\varepsilon \geqslant s\}, B^{out}_{n,\ell}) \;\geqslant\; P(Q\setminus E, B^{out}_{n,\ell}) + 2\mathcal{H}^{d-1}((Q\setminus E)^0\cap\omega^{out}\cap B^{out}_{n,\ell}).$$

Notice also that as for any Borel set A, $Q\setminus A$ coincides with $B^{out}_{n,\ell}\setminus A$ in $B^{out}_{n,\ell}$, we have $P(A, B_n) = P(Q\setminus A, B_n)$. Moreover, as $(Q\setminus E)^0\cap B^{out}_{n,\ell} = E^1\cap B^{out}_{n,\ell}$ implies that

$$\liminf_{\varepsilon\to 0} P(\{u_\varepsilon \geqslant s\}, B^{out}_{n,\ell}) \;\geqslant\; P(E, B^{out}_{n,\ell}) + 2\mathcal{H}^{d-1}(E^1\cap\omega^{out}_{n,\ell}\cap B^{out}_{n,\ell}).$$

A.2.3 On the free part $\tilde{B}_m = B_0$

As $u_\varepsilon \to \mathbb{1}_E$ in $L^1(Q)$, then, for almost every s, $\{u_\varepsilon \geqslant s\} \overset{\varepsilon\to 0}{\longrightarrow} E$ in measure in B_0, and $\liminf_{\varepsilon\to 0} P(\{u_\varepsilon \geqslant s\}, B_0) \geqslant P(E, B_0)$.

A.2.4 From local to global.

As $(\tilde{B}_m)$ are pairwise disjoint, we can gather the results of the previous subsections in one inequality:

$$\liminf_{\varepsilon\to 0} P(\{u_\varepsilon \geqslant s\}, \tilde{B}_m) \;\geqslant\; P(E, \tilde{B}_m) + 2\mathcal{H}^{d-1}(E^0\cap\omega^{in}\cap\tilde{B}_m) + 2\mathcal{H}^{d-1}(E^1\cap\omega^{out}\cap\tilde{B}_m).$$

Remembering we have chosen our boxes $\tilde{B}_m$ in order to have $\mathcal{H}^{d-1}(E^{1/2}\cap\partial\tilde{B}_m) = 0$ and knowing that $\left\{\{\omega^{in}\cap\tilde{B}_m\}_m, \{\omega^{out}\cap\tilde{B}_m\}_m\right\}$ is a pairwise disjoint family of set covering ω^{in} and ω^{out}, we can write

$$\begin{aligned}
\liminf_{\varepsilon\to 0} P(\{u_\varepsilon \geqslant s\}, Q) &\geqslant \liminf_{\varepsilon\to 0} P(\{u_\varepsilon \geqslant s\}, \bigcup_{m\geqslant 0}\tilde{B}_m) = \liminf_{\varepsilon\to 0}\sum_{m\geqslant 0} P(\{u_\varepsilon \geqslant s\}, \tilde{B}_m)\\
&\geqslant \sum_m \liminf_{\varepsilon\to 0} P(\{u_\varepsilon \geqslant s\}, \tilde{B}_m)\\
&\geqslant \sum_m [P(E,\tilde{B}_m) + 2\mathcal{H}^{d-1}(E^0\cap\omega^{in}\cap\tilde{B}_m) + 2\mathcal{H}^{d-1}(E^1\cap\omega^{out}\cap\tilde{B}_m)]\\
&= P(E, Q) + 2\mathcal{H}^{d-1}(E^0\cap\omega^{in}) + 2\mathcal{H}^{d-1}(E^1\cap\omega^{out}),
\end{aligned}$$

which concludes the proof of lemma 2.5.

A.3 Sublemmas

Lemma (Lemma A.1)**.** *There exists $h > 0$ such that the family of sets $\{\tilde{B}_m\}_m$ is pairwise disjoint and, for all m, $\mathcal{H}^{d-1}(E^{1/2} \cap \tilde{B}_m) = \emptyset$.*

Proof. Firstly, consider $\tilde{B}_m = B^{in}_{n,k}$. Remember $h_0 > 0$ was defined in (6) to ensure that the family $\{\Omega^{in}_{n,k,h}, \Omega^{out}_{n,\ell,h}\}_{n,k,\ell}$ is pairwise disjoint. Using the affine system coordinates of Π_n, let $\varphi_{n,k} : \Omega^{in}_{n,k,h_0} \to \mathbb{R}$ be defined by $\varphi_{n,k}(y,z) = \dfrac{|z|}{|d_n(y, \omega^{in}_{n,k})|}$. The function $\varphi_{n,k}$ is bounded by h_0 and its level set $\{\varphi_{n,k} < \varepsilon^\alpha\}$ is exactly the open set $\Omega^{in}_{n,k,\varepsilon^\alpha}$. However it is only locally Lipschitz continuous and we want to apply a result requiring a Lipschitz continuous function. Let $\delta > 0$ and $\Omega^{in}_{n,k,h_0,\delta} = \Big\{(y,z) \;\Big|\; y \in \omega^{in}_{n,k},\; |d_n(y,\omega^{in}_{n,k})| > \delta,\; |z| < h_0 |d_n(y,\omega^{in}_{n,k})|\Big\}$. The restriction of $\varphi_{n,k}$ to $\Omega^{in}_{n,k,h_0,\delta}$ is $1/\delta^2$-Lipschitz continuous and then, using [2, Lemma 2.95 p.102] and $|E^{1/2}| = 0$, we have, for almost every ε such that $0 < \varepsilon^\alpha < h_0$,

$$\mathcal{H}^{d-1}\Big(E^{1/2} \cap \Omega^{in}_{n,k,h_0,\delta} \cap \varphi^{-1}_{n,k}(\{\varepsilon^\alpha\})\Big) = 0. \tag{11}$$

Let (δ_p) be a decreasing sequence converging to 0 and $N_{n,k,p}$ the negligible set of $]0,h_0[$ where (11) is not true. Then, except for the negligible set $N_{n,k} = \bigcup_{p\in\mathbb{N}} N_{n,k,p}$, we have

$$\mathcal{H}^{d-1}\Big(E^{1/2} \cap \Omega^{in}_{n,k,h_0,\delta_p} \cap \varphi^{-1}_{n,k}(\{\varepsilon^\alpha\})\Big) = 0$$

and $\Omega^{in}_{n,k,h_0,\delta_p} \subset \Omega^{in}_{n,k,h,\delta_{p+1}}$. Thus, by increasing limit $\Omega^{in}_{n,k,h_0} = \bigcup_{p\in\mathbb{N}} \Omega^{in}_{n,k,h_0,\delta_p}$, we recover, for all $\varepsilon^\alpha \in]0,h_0[\setminus N_{n,k}$, $\mathcal{H}^{d-1}\Big(E^{1/2} \cap \Omega^{in}_{n,k,h_0} \cap \varphi^{-1}_{n,k}(\{\varepsilon^\alpha\})\Big) = 0$. We have the same result with $\Omega^{out}_{n,\ell,h_0}$ and a similar function $\varphi_{n,\ell}$ with a negligible set denoted by $N_{n,\ell}$.

Secondly, since there are finitely many indexes n,k,ℓ, we can define a negligible set $N = \Big(\bigcup_{n,k} N_{n,k}\Big) \cup \Big(\bigcup_{n,\ell} N_{n,\ell}\Big)$ and choose $h \in]0,h_0[\setminus N$ such that the equality above holds for all n,k,ℓ with $\varepsilon^\alpha = h$. We have then $B^{in}_{n,k} = \Omega^{in}_{n,k,h_0} \cap \varphi^{-1}_{n,k}(\{h\}) = \Omega^{in}_{n,k,h}$ and $B^{out}_{n,\ell} = \Omega^{out}_{n,\ell,h_0} \cap \varphi^{-1}_{n,\ell}(\{h\}) = \Omega^{out}_{n,\ell,h}$ which satisfies $\mathcal{H}^{d-1}(E^{1/2} \cap \partial\tilde{B}_m) = 0$ as expected, for all m, independently of ε. □

Lemma (Lemma A.3)**.** *Let A be a bounded set with finite perimeter and Π be a hyperplane of $\mathbb{R}^d$. Then $P(A) \geqslant 2\mathcal{H}^{d-1}(\Pi \cap (A^1 \cup A^{1/2}))$.*

Proof. We denote by Π^+ and Π^- the open half-spaces whose boundary is Π. If $|A\cap\Pi^+| > 0$ and $|A\cap\Pi^-| > 0$ then, using [34, Proposition 19.22] we have $P(A,\Pi^+) = P(A\cap\Pi^+,\Pi^+) > \mathcal{H}^{d-1}\Big(\partial^*(A\cap\Pi^+)\cap\Pi\Big)$ and $P(A,\Pi^-) = P(A\cap\Pi^-,\Pi^-) > \mathcal{H}^{d-1}\Big(\partial^*(A\cap\Pi^-)\cap\Pi\Big)$.

We recall that for a set A with locally finite perimeter, we denote $\partial^* A$ the set of points for which the generalised normal ν exists. Using [34, Theorem 16.3] we obtain $\partial^*(A\cap\Pi^+) = \Big((\Pi^+)^1 \cap \partial^* A\Big) \cup \Big(A^1 \cap \partial^*(\Pi^+)\Big) \cup \{\nu_A = \nu_{\Pi^+}\}$ and $\partial^*(A\cap\Pi^-) = \Big((\Pi^-)^1 \cap \partial^* A\Big) \cup \Big(A^1 \cap \partial^*(\Pi^-)\Big) \cup \{\nu_A = \nu_{\Pi^-}\}$, where $\{\nu_A = \nu_{\Pi^+}\} = \{x \in \partial^* A \cap \partial^*(\Pi^+) \mid \nu_A(x) = \nu_{\Pi^+}(x)\}$.

Moreover, as we consider open half-spaces, we have $(\Pi^+)^1 = \Pi^+$, $(\Pi^-)^1 = \Pi^-$, $\partial^*(\Pi^+) = \partial^*(\Pi^-) = \Pi$ and $\nu_{\Pi^+} = -\nu_{\Pi^-}$. Furthermore,

$$\mathcal{H}^{d-1}\Big(\partial^*(A\cap\Pi^+)\cap\Pi\Big) = \mathcal{H}^{d-1}(A^1\cap\Pi) + \mathcal{H}^{d-1}(\{\nu_A = \nu_{\Pi^+}\})$$

$$\text{and} \quad \mathcal{H}^{d-1}\Big(\partial^*(A\cap\Pi^-)\cap\Pi\Big) = \mathcal{H}^{d-1}(A^1\cap\Pi) + \mathcal{H}^{d-1}(\{\nu_A = -\nu_{\Pi^+}\}).$$

Denoting e a normal vector to Π, we have $\{\nu_A = \nu_{\Pi^+}\} \cup \{\nu_A = -\nu_{\Pi^+}\} = \Pi \cap \{\nu_A = \pm e\}$ and, using [34, Proposition 10.5] as in the previous lemma, we obtain

$$\mathcal{H}^{d-1}\Big(\partial^* A \cap H\Big) = \mathcal{H}^{d-1}(\{\nu_A = \nu_{\Pi^+}\}) + \mathcal{H}^{d-1}(\{\nu_A = -\nu_{\Pi^+}\}).$$

Therefore

$$\begin{array}{rcl} P(A) & = & P(A,\Pi^+) + P(A,\Pi^-) + \mathcal{H}^{d-1}(\partial^* A \cap \Pi) \\ & = & P(A \cap \Pi^+, \Pi^+) + P(A \cap \Pi^-, \Pi^-) + \mathcal{H}^{d-1}(\partial^* A \cap \Pi) \\ & \geqslant & \mathcal{H}^{d-1}\Big(\partial^*(A\cap\Pi^+)\cap\Pi\Big) + \mathcal{H}^{d-1}\Big(\partial^*(A\cap\Pi^-)\cap\Pi\Big) + \mathcal{H}^{d-1}(\partial^* A \cap \Pi) \\ & = & 2\mathcal{H}^{d-1}(A^1\cap\Pi) + \mathcal{H}^{d-1}(\{\nu_A = \nu_{\Pi^+}\}) + \mathcal{H}^{d-1}(\{\nu_A = -\nu_{\Pi^+}\}) + \mathcal{H}^{d-1}(\partial^* A \cap \Pi) \\ & = & 2\mathcal{H}^{d-1}(A^1\cap\Pi) + 2\mathcal{H}^{d-1}(\partial^* A \cap \Pi) \\ & = & 2\mathcal{H}^{d-1}\Big((A^1 \cup \partial^* A)\cap\Pi\Big). \end{array}$$

To conclude, we use the fact that $\partial^* A = A^{1/2}$ up to a $\mathcal{H}^{d-1}$-negligible set.

Notice that if $|A \cap \Pi^+| = 0$ then $A^1 \cap \Pi^+ = \emptyset$ and $\{\nu_A = \nu_{\Pi^+}\} = \emptyset$. The proof remains valid in that case. The same holds if $|A \cap \Pi^-| = 0$. If both $|A\cap\Pi^+| = 0$ and $|A \cap \Pi^-| = 0$ then $|A| = 0$ and the statement of the lemma is clearly true. □

Lemma (Lemma A.4)**.** *The fat constraints $\Omega^{in}_{n,k,\eta}$ and $\Omega^{out}_{n,\ell,\eta}$ satisfy $P(\Omega^{in}_{n,k,\eta}, B^{in}_{n,k}) \xrightarrow{\eta\to 0} 2\mathcal{H}^{d-1}(\omega^{in}_{n,k})$ and $P(\Omega^{out}_{n,\ell,\eta}, B^{out}_{n,\ell}) \xrightarrow{\eta\to 0} 2\mathcal{H}^{d-1}(\omega^{out}_{n,\ell})$.*

Proof. The key point is that the boundary of $\Omega^{in}_{n,k,\eta}$ is Lipschitz continuous and we can write it as two graphs over $\omega^{in}_{n,k}$. Indeed, we recall that $\Omega^{in}_{n,k,\eta} = \{(y,z) \mid y \in \omega^{in}_{n,k},\ |z| < \eta |d_n(y, \omega^{out}_{n,k})|\}$. Introducing, for $y \in \Pi_n$, $f^{\pm}_\eta(y) = \pm \eta d_n(y, \omega^{in}_{n,k})$, we have

$$\partial\Omega^{in}_{n,k,\eta} = \mathrm{graph}(f^+_\eta, \omega^{in}_{n,k}) \cup \mathrm{graph}(f^-_\eta, \omega^{in}_{n,k}) \cup \partial_n \omega^{in}_{n,k},$$

where sets are disjoint. The functions f^+_η and f^-_η are differentiable $\mathcal{H}^{d-1}$-almost everywhere on Π_n and as $\mathcal{H}^{d-1}(\partial_n\omega^{in}_{n,k}) = 0$, we have

$$\mathcal{H}^{d-1}(\partial\Omega^{in}_{n,k,\eta}) = \int_{\omega^{in}_{n,k}} \sqrt{1 + |\nabla f^+_\eta|^2}\mathrm{d}\mathcal{H}^{d-1} + \int_{\omega^{in}_{n,k}} \sqrt{1 + |\nabla f^-_\eta|^2}\mathrm{d}\mathcal{H}^{d-1}.$$

Moreover, $|\nabla d_n(y, \omega^{in}_{n,k})| \leqslant 1$ so $\sqrt{1 + |\nabla f^\pm_\eta|^2} \xrightarrow{\eta\to 0} 1$. The set $\omega^{in}_{n,k}$ is bounded so, by Lebesgue dominated convergence theorem, we have $\mathcal{H}^{d-1}(\partial\Omega^{in}_{n,k,\eta}) \xrightarrow{\eta\to 0} 2\mathcal{H}^{d-1}(\omega^{in}_{n,k})$. Recall that $P(\Omega^{in}_{n,k,\eta}, B_{n,k}) = \mathcal{H}^{d-1}(\partial\Omega^{in}_{n,k,\eta} \cap B_{n,k}) = \mathcal{H}^{d-1}(\partial\Omega^{in}_{n,k,\eta})$ because $\Omega^{in}_{n,k,\eta} \subset \overline{B_{n,k}}$ and $\partial\Omega^{in}_{n,k,\eta} \cap \partial B_{n,k} = \partial_n \omega^{in}_{n,k}$ has zero $\mathcal{H}^{d-1}$-measure. We have exactly the same result for $\Omega^{out}_{n,\ell}$ and then, the lemma is proved. □

Lemma (Lemma A.5)**.** *Let E be a set with locally finite perimeter. If $B_{n,k}$ is a box enclosing $\omega^{in}_{n,k}$, then $P(E \cup B^-_{n,k}, B_{n,k}) + P(E \cup B^+_{n,k}, B_{n,k}) = P(E, B_{n,k}) + 2\mathcal{H}^{d-1}(E^0 \cap \omega^{in} \cap B_{n,k})$.*

Proof. Using [34, Theorem 16.3], we have

$$P(E \cup B^-_{n,k}, B_{n,k}) = P(E, (B^-_{n,k})^0 \cap B_{n,k}) + P(B^-_{n,k}, E^0 \cap B_{n,k}) + \mathcal{H}^{d-1}\Big(\{\nu_E = \nu_{B^-_{n,k}}\} \cap B_{n,k}\Big)$$

where ν_E stands for the generalized exterior normal to E and

$$\{\nu_E = \nu_{B^-_{n,k}}\} = \{x \in \partial^* E \cap \partial^* B^-_{n,k} \mid \nu_E(x) = \nu_{B^-_{n,k}}(x)\}$$

with $\partial^* E$ the set of points where ν_E exists. Notice, for set with finite perimeter, $\partial^* E = E^{1/2}$ modulo a $\mathcal{H}^{d-1}$-negligible set. As we work with open boxes $B_{n,k}$, we have $(B^-_{n,k})^0 \cap B_{n,k} = B^+_{n,k}$ and $\partial^* B^-_{n,k} \cap B_{n,k} = \omega^{in}_{n,k}$. We define e the unit vector orthogonal to $\omega^{in}_{n,k}$ pointing in $B^+_{n,k}$. Then, we can write

$$P(E \cup B^-_{n,k}, B_{n,k}) = P(E, B^+_{n,k}) + \mathcal{H}^{d-1}(\omega^{in}_{n,k} \cap E^0) + \mathcal{H}^{d-1}\Big(\{x \in \partial^* E \cap \omega^{in}_{n,k} \mid \nu_E(x) = e\}\Big). \qquad (12)$$

Similarly, we have

$$P(E\cup B^+_{n,k},B_{n,k}) = P(E,B^-_{n,k}) + \mathcal{H}^{d-1}(\omega^{in}_{n,k}\cap E^0) + \mathcal{H}^{d-1}\Big(\{x\in\partial^*E\cap\omega^{in}_{n,k} \mid \nu_E(x)=-e\}\Big). \quad (13)$$

Moreover, $\omega^{in}_{n,k}$ and ∂^*E are $(d-1)$-rectifiable then, using [34, Proposition 10.5], we have, for $\mathcal{H}^{d-1}$-every $x\in\partial^*E\cap\omega^{in}_{n,k}$, $\nu_E(x)=\pm e$. Therefore,

$$\mathcal{H}^{d-1}(\partial^*E\cap\omega^{in}_{n,k}) = \mathcal{H}^{d-1}\Big(\{x\in\partial^*E\cap\omega^{in}_{n,k} \mid \nu_E(x)=e\}\Big) + \mathcal{H}^{d-1}\Big(\{x\in\partial^*E\cap\omega^{in}_{n,k} \mid \nu_E(x)=-e\}\Big).$$

Finally, summing (12) and (13), we obtain

$$P(E\cup B^-_{n,k},B_{n,k}) + P(E\cup B^+_{n,k},B_{n,k}) = P(E) + 2\mathcal{H}^{d-1}(\omega^{in}_{n,k}\cap E^0)$$

since $P(E,B_{n,k}) = P(E,B^+_{n,k}) + P(E,B^-_{n,k}) + \mathcal{H}^{d-1}(\partial^*E\cap\omega^{in}_{n,k})$. □

References

[1] S. Akkouche and E. Galin, *Fast surface reconstruction from contours using implicit surfaces*, in Implicit Surfaces, vol. 98, 1998, pp. 139–144.

[2] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems.*, Oxford University Press, 2006.

[3] M. Amrani, B. Crespin, and B. Shariat, *Skeletal implicit surface reconstruction from sections for flexible body simulation*, in Information Visualisation, 2001. Proceedings. Fifth International Conference on, 2001, pp. 723–728.

[4] P. Bates, S. Brown, and J. L. Han, *Numerical analysis for a nonlocal Allen-Cahn equation*, International Journal of Numerical Analysis and Modeling, 6 (2009), pp. 33–49.

[5] G. Bellettini, G. Dal Maso, and M. Paolini., *Semi-continuity and relaxation properties of a curvature depending functional in 2d.*, Annali della Scuola Normale Superior di Pisa, Classe de scienze, 20 (1993), pp. 247–297.

[6] G. Bellettini and L. Mugnai, *On the approximation of the elastica functional in radial symmetry*, Calculus of Variations and Partial Differential Equations, 24 (2005), pp. 1–20.

[7] G. Bellettini and M. Paolini., *Anisotropic motion by mean curvature in the context of Finsler geometry*, Hokkaido Mathematical Journal, 25 (1996), pp. 537–566.

[8] M. Bello, J. Carson, W. Chiu, G. Eichele, I. Kakadiaris, T. Ju, C. Thaller, and J. Warren, *Building 3d surface networks from 2d curve networks with application to anatomical modeling*, The Visual Computer, 21 (2005), pp. 764–773.

[9] J.-D. Boissonnat, *Shape reconstruction from planar cross sections*, Computer Vision, Graphics, and Image Processing. An International Journal, 44 (1988), pp. 1–29.

[10] A. Braides, *Γ-convergence for beginners.*, Oxford University Press, 2002.

[11] E. Bretin, J.-O. Lachaud, and E. Oudet, *Regularization of discrete contour by Willmore energy.*, Journal of Mathematical Imaging and Vision, 40 (2011), pp. 214–229.

[12] É. Bretin, S. Masnou, and É. Oudet, *Phase-field approximations of the Willmore functional and flow*, Numerische Mathematik, 131 (2014), pp. 115–171.

[13] M.-P. Cani, O. Palombi, and A. Pihuit, *Reconstruction implicite de surfaces 3d à partir de régions 2d dans des plans parallèles*, in AFIG 2009 - 22e Journées de l'Association Française d'Informatique Graphique, Novembre 2009.

[14] N. Carr, M. Holloway, T. Ju, and M. Zou, *Topology-constrained surface reconstruction from cross-sections*, ACM Transactions on Graphics, 34 (2015), pp. 128:1–128:10.

[15] D. Chen, B. Morse, P. Rheingans, K. R.Subramanian, and T. Yoo, *Interpolating implicit surfaces from scattered surface data using compactly supported radial basis functions*, in ACM SIGGRAPH 2005 Courses, SIGGRAPH '05, 2005.

[16] L. Q. Chen and J. Shen, *Applications of semi-implicit Fourier-spectral method to phase field equations*, Computer Physics Communications, 108 (1998), pp. 147–158.

[17] M. Chen and M. Jones, *A new approach to the construction of surfaces from contour data*, Computer Graphics Forum, 13 (1994), pp. 75–84.

[18] X. Chen, *Generation and propagation of interfaces for reaction-diffusion equations*, Journal of Differential Equations, 96 (1992), pp. 116–141.

[19] X. Chen, C. Elliott, A. Gardiner, and J. J. Zhao, *Convergence of numerical solutions to the Allen-Cahn equation*, Applicable Analysis. An International Journal, 69 (1998), pp. 47–56.

[20] Y. Choi, Y. Li, J. Kim, and J. Shin, *Three-dimensional volume reconstruction from slice data using phase-field models*, Computer Vision and Image Understanding, 137 (2015), pp. 115 – 124.

[21] E. De Giorgi, *Some remarks on Γ-convergence and least squares method*, in Composite media and homogenization theory (Trieste, 1990), vol. 5 of Progr. Nonlinear Differential Equations Appl., Birkhäuser Boston, Boston, MA, 1991, pp. 135–142.

[22] T. DeRose, T. Duchamp, H. Hoppe, J. McDonald, and W. Stuetzle, *Surface reconstruction from unorganized points*, SIGGRAPH Computer Graphics, 26 (1992), pp. 71–78.

[23] Q. Du and J. Zhang, *Numerical studies of discrete approximations to the Allen-Cahn equation in the sharp interface limit*, SIAM Journal on Scientific Computing, 31 (2009), pp. 3042–3063.

[24] Q. Du and W.-X. Zhu, *Stability analysis and application of the exponential time differencing schemes*, Journal of Computational Mathematics, 22 (2004), pp. 200–209.

[25] A. B. Ekoule, F. Peyrin, and C. Odet, *A triangulation algorithm from arbitrary shaped multiple planar contours*, ACM Transactions on Graphics, 10 (1991), pp. 182–199.

[26] R. Fedkiw, S. Osher, and H.-K. Zhao, *Fast surface reconstruction using the level set method*, in Variational and Level Set Methods in Computer Vision, 2001. Proceedings. IEEE Workshop on, 2001, pp. 194–201.

[27] X. Feng and A. Prohl, *Analysis of a fully discrete finite element method for the phase field model and approximation of its sharp interface limits*, Mathematics of Computation, (2004), pp. 541–567.

[28] X. Feng and H.-J. Wu, *A posteriori error estimates and an adaptive finite element method for the Allen-Cahn equation and the mean curvature flow*, Journal of Scientific Computing, 24 (2005), pp. 121–146.

[29] H. Fuchs, Z. M. Kedem, and S. P. Uselton, *Optimal surface reconstruction from planar contours*, Communications of the Association for Computing Machinery, 20 (1977), pp. 693–702.

[30] D. Jeong, J. Kim, H. G. Lee, and Y. Li, *An unconditionally stable hybrid numerical method for solving the Allen–Cahn equation*, Computers Mathematics with Applications, 60 (2010), pp. 1591–1606.

[31] E. Keppel, *Approximating complex surfaces by triangulation of contour lines*, IBM Journal of Research and Development, 19 (1975), pp. 2–11.

[32] S.-U. Kim and C.-O. Lee, *Accurate surface reconstruction in 3d using two-dimensional parallel cross sections*, Journal of Mathematical Imaging and Vision, 53 (2015), pp. 182–195.

[33] L. Liu, C. Bajaj, J. O. Deasy, D. A. Low, and T. Ju, *Surface Reconstruction From Non-parallel Curve Networks*, Computer Graphics Forum, (2008).

[34] F. Maggi, *Sets of Finite Perimeter and Geometric Variational Problems.*, Cambridge University Press, 2012.

[35] D. Meyers, S. Skinner, and K. Sloan, *Surfaces from contours*, ACM Transactions on Graphics, 11 (1992), pp. 228–258.

[36] L. Modica and S. Mortola, *Il limite nella* Γ*-convergenza di una famiglia di funzionali ellittici*, Unione Matematica Italiana. Bollettino. A. Serie V, 14 (1977), pp. 526–529.

[37] L. Modica and S. Mortola, *Un esempio di* Γ*-convergenza.*, Unione Matematica Italiana. Bollettino. B. Serie V, 14 (1977), p. 285–299.

[38] R. Moser, *A higher order asymptotic problem related to phase transitions*, SIAM Journal of Mathematical Analysis, 37 (2005), pp. 712–736.

[39] Y. Nagase and Y. Tonegawa, *A singular perturbation problem with integral curvature bound*, Hiroshima Mathematical Journal, 37 (2007), pp. 455–489.

[40] J. O'Brien and G. Turk, *Shape transformation using variational implicit functions*, in Proceedings of the 26th Annual Conference on Computer Graphics and Interactive Techniques, SIGGRAPH '99, 1999, pp. 335–342.

[41] M. Röger and R. Schätzle, *On a modified conjecture of De Giorgi*, Mathematische Zeitschrift, 254 (2006), pp. 675–714.

[42] J. A. Sethian, *Level set methods and fast marching methods : evolving interfaces in computational geometry, fluid mechanics, computer vision, and materials science*, Cambridge monographs on applied and computational mathematics, Cambridge University Press, 1999.

[43] Y. Tonegawa, *Phase field model with a variable chemical potential*, Proceedings of the Royal Society of Edinburgh: Section A Mathematics, 132 (2002), pp. 993–1019.