Gradient Young measures, varifolds, and a generalized Willmore functional

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Abstract

Being Ω an open and bounded Lipschitz domain of $\mathbb{R}^n$, we consider the *generalized Willmore functional* defined on $\mathrm{L}^1(\Omega)$ as

$$F(u) = \begin{cases} \int_{\mathbb{R}^2} |\nabla u| (\alpha + \beta |\operatorname{div} \frac{\nabla u}{|\nabla u|}|^p)\, dx & \text{if } u \in \mathrm{C}^2(\Omega), \\ +\infty & \text{else,} \end{cases}$$

where $p > 1$, $\alpha > 0$, $\beta \geq 0$. We propose a new framework, that combines varifolds and Young measures, to study the relaxation of F in $\mathrm{BV}(\Omega)$ with respect to the strong topology of L^1.

1 Introduction

Let Ω be an open bounded Lipschitz domain of $\mathbb{R}^n$. We address in this paper the problem of identifying the relaxation (with respect to the strong topology of $\mathrm{L}^1(\Omega)$) of the functional

$$F(\cdot, \Omega): \ u \in \mathrm{BV}(\Omega) \mapsto \begin{cases} \int_\Omega |\nabla u| \left(\alpha + \beta \left| \operatorname{div} \frac{\nabla u}{|\nabla u|} \right|^p \right) dx & \text{if } u \in \mathrm{C}^2(\Omega) \\ +\infty & \text{otherwise} \end{cases}$$

with $p > 1$, $\alpha > 0$, $\beta \geq 0$ and the convention that the integrand is 0 wherever $|\nabla u| = 0$. Here, $\mathrm{BV}(\Omega)$ denotes the space of functions of bounded variation in Ω, see [3]. Without loss of generality and to simplify the notations, we shall assume in the sequel that $\alpha = \beta = 1$.

This functional appears, under various forms, in the context of optimal design of shapes or digital surfaces in 3D [5], modeling and approximation of elastic membranes, or folding in multi-layered materials [10], image or surface processing [22, 23, 12, 5]. In particular, it has been introduced in [22, 23] as a variational model in the context of digital image inpainting, i.e. the problem of recovering an image that is known only out of a given domain. It is also related to a model of amodal completion in a neurogeometric description of the visual cortex [13].

The functional F has a strong geometric meaning. Indeed, by the coarea formula [15, 3],

$$F(u, \Omega) = \int_{\mathbb{R}} \left[\int_{\partial\{u>t\}\cap\Omega} \left(1 + \left| \mathbf{H}_{\partial\{u>t\}\cap\Omega} \right|^p \right) d\mathscr{H}^{n-1} \right] dt \qquad \forall u \in \mathrm{C}^2(\Omega) \tag{1}$$

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where $\mathbf{H}_{\partial\{u>t\}\cap\Omega}(x) = -\operatorname{div}\frac{\nabla u}{|\nabla u|}\frac{\nabla u}{|\nabla u|}(x)$ is the mean curvature vector at a point $x \in \partial\{u>t\} \cap \Omega$. We call F a *generalized Willmore functional* for it naturally relates to the celebrated Willmore energy of an immersed compact oriented surface $f : \Sigma \to \mathbb{R}^N$ without boundary, defined as

$$\mathscr{W}(f) = \int_\Sigma |H|^2 dA$$

with dA the induced area metric on Σ.

Minimizing F (for instance under fat boundary constraints) raises immediate difficulties for a simple reason: the functional is not lower semicontinuous with respect to the strong convergence in L^1, as can be seen immediately from the following classical example.

Example 1.1 Being E and Ω the planar sets drawn on Figure 1, left, let $u = \mathbb{1}_E$. Obviously, $u \in \mathrm{BV}(\Omega)$ can be approximated in L^1 by a sequence $\{u_h\} \subset \mathrm{C}^2_\mathrm{c}(\Omega)$ such that every level line of every function u_h looks like the curve represented in Figure 1, right. It is easy to check that $\liminf_{h\to\infty} F(u_h) < \infty$ but, since $u \notin \mathrm{C}^2(\Omega)$, we have $F(u,\Omega) = \infty$ so F is not lower semicontinuous.

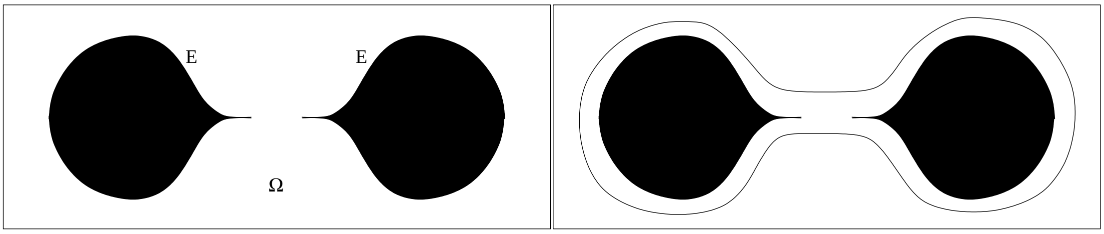

Figure 1: Left: $u = \mathbb{1}_E$ with $\overline{F}(u,\Omega) < \infty$. Right: Approximation by regular functions
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The usual technique in calculus of variations to overcome this difficulty consists in relaxing F, i.e., introducing the functional

$$\overline{F}(u,\Omega) = \inf\left\{ \liminf_{h\to\infty} F(u_h,\Omega) : u_h \xrightarrow{\mathrm{L}^1(\Omega)} u \right\}.$$

As a relaxation, this functional has the interesting property to be lower semicontinuous in L^1 [25]. Together with the compactness of BV, it guarantees that the infimum of F coincides with a minimum of $\overline{F}$, which somewhat solves the minimization problem. It remains however that not much can be said neither about the minimizers of $\overline{F}$ nor, more generally, about $\overline{F}(u)$ for a general function u of bounded variation.

Partial results have been obtained in [4, 21] in the case where u is smooth. Combining the techniques used in these papers with the more recent [26], it can be proved that, in any space dimension n and for any $p \geq 1$, $F(u) = \overline{F}(u)$ when u is C^2. What about more general functions?

Examining again the previous example, it is clear that $\overline{F}(u) < +\infty$ since (u_h) has uniformly bounded energy and converges to u. Besides, it is equivalent to study $\overline{F}$ for the function $u = \mathbb{1}_E$ or to study the relaxation at E of the following functional that acts on measurable sets (in our example $n = 2$):

$$A \subset \mathbb{R}^n \;\mapsto\; W(A) = \begin{cases} \int_{\partial A} (1 + |\mathbf{H}_{\partial A}|^p)\, d\mathscr{H}^{\mathrm{n}-1} & \text{if } \partial A \text{ is smooth} \\ +\infty & \text{otherwise} \end{cases}$$

The relaxed functional associated with W is

$$\overline{W}(A) = \inf\{\liminf_{h\to\infty} W(A_h),\ (A_h) \text{ smooth},\ |A_h \Delta A| \to 0\}$$

The properties of bounded sets $A \subset \mathbb{R}^2$ such that $\overline{W}(A) < +\infty$ and the explicit representation of $\overline{W}(A)$ have been carefully studied in [7, 8, 9]. Such sets have finite perimeter (by definition of the energy) and, by explicit representation, we mean that $\overline{W}(A)$ can be written in terms of the $\mathrm{W}^{2,p}$ norms of a collection of curves that cover the essential boundary $\partial^* A$ of A. This can again be easily understood from Example 1.1 and Figures 1, 2: a "good" way to approximate E in measure consists in choosing a set E_h whose boundary Γ^h is represented in Figure 2. These sets have uniformly bounded energy and, as shown in [7], $\overline{W}(E)$ coincides with $W(\Gamma) = \int_\Gamma (1 + |\kappa_\Gamma|^p) d\mathscr{H}^1$ (with κ_Γ the curvature along Γ) where Γ is the limit curve represented in Figure 2, right, with its multiplicity.

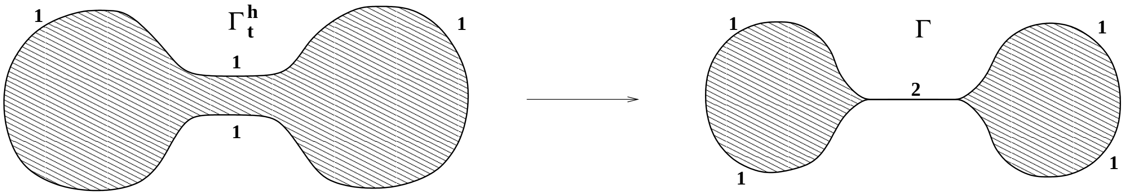

Figure 2: Accumulation at the limit of the boundaries of sets that approximate in measure the set E of Figure 1.

Having in mind the expression (1) of F through the coarea formula, it is natural to expect that, at least in dimension 2, the relaxed energy $\overline{F}(u)$ of a function u of bounded variation can be written in terms of the energies $W(\Gamma_t)$ of systems of curves (Γ_t) that cover the essential boundaries $\partial^*\{u > t\}$. This is exactly what happens, as we proved in the companion paper [24], among other results, by generalizing techniques that were proposed in [23] for a more restrictive boundary value problem involving the same energy. The precise statements will be recalled in Section 2.

The techniques developed in [7, 8, 9, 23, 24] depend strongly on parameterizations of curves and can hardly be generalized to higher space dimensions. Indeed, in dimension > 2, parameterizations of hypersurfaces are much harder to handle in our context, if not impossible, especially since the energy of interest controls the mean curvature vector only.

This was the main motivation for the new framework that we introduce in this paper. It involves two specific tools: the Young measures, that play a fundamental role in many problems of the calculus of variations, and the varifolds, that appear to be very useful to handle generalized surfaces and a weak notion of mean curvature.

Varifolds We shall introduce the varifolds with more details in Section 4. The basic idea is that each rectifiable k-subset $M \subset \mathbb{R}^n$ can be endowed with a multiplicity function θ_M and associated with the measure $\theta_M \mathscr{H}^{\mathrm{k}} \llcorner M$. For instance, the curve Γ in Figure 2 can be associated with the measure $\theta_\Gamma \mathscr{H}^1 \llcorner \Gamma$ where θ_Γ denotes the multiplicity function represented in the figure. This measure is actually the *mass* of the varifold naturally associated with Γ, which is denoted by $\mathbf{v}(\Gamma, \theta_\Gamma)$ and defined as the measure $\theta_\Gamma \mathscr{H}^1 \llcorner \Gamma \otimes \delta_{T_\Gamma(x)}$ on $G_1(\Omega) = \Omega \times G(2,1)$ where $G(2,1)$ is the Grassmaniann of 1-subspaces of $\mathbb{R}^2$ and $T_\Gamma(x)$ the tangent space to Γ at $x \in \Gamma$. Similarly, a k-rectifiable set $M \subset \mathbb{R}^n$ and a multiplicity function θ_M on M define the varifold $V_M = \mathbf{v}(M, \theta_M) = \theta_M \mathscr{H}^{\mathrm{k}} \llcorner \Gamma \otimes \delta_{T_M(x)}$ on the product space

$G_k(\Omega) = \Omega \times G(n,k)$. Varifolds have nice properties, among which the possibility to use a weak notion of mean curvature, the continuity of the mass, a compactness property, the lower semicontinuity of some useful second-order energies, etc. They are actually much more adapted than parameterizations for handling sequences of k-surfaces in $\mathbb{R}^n$ when there is a control on the mean curvature.

Denoting $\mu_V(\cdot) = V(\cdot \times G(n,k))$ the mass of a k-varifold V in $\mathbb{R}^n$, we show in the following table how notions that are naturally defined for smooth k-sets can be easily translated in terms of k-varifolds. Here, X denotes a smooth vector field with compact support.

	M =closed, smooth k-set	$V = k$-varifold
Mass	$\mathscr{H}^{\mathrm{k}}(M)$	$\mu_V = V(\Omega \times G(n,k))$
First variation	$\int_M \mathrm{div}_M X d\mathscr{H}^{\mathrm{k}}$	$\delta V(X) = \int_{G_k(\Omega)} \mathrm{div}_S X dV(x,S)$
Mean curvature vector	H_M s.t $\int_M \mathrm{div}_M X d\mathscr{H}^{\mathrm{k}} = -\int_M H_M \cdot X d\mathscr{H}^{\mathrm{k}}$	$H_V = -\dfrac{\delta V}{\mu_V}$

In particular, this table shows the divergence theorem that relates (the integral of) the tangential divergence of a smooth vector field with (the integral of) the mean curvature vector. Recalling that, for a smooth function u, $-(\mathrm{div}\, \frac{\nabla u}{|\nabla u|}) \frac{\nabla u}{|\nabla u|}(x)$ is the mean curvature vector at $x \in \partial\{u > t\}$, and

$$\mathrm{div}_{\{y,\, u(x)=u(x)\}} X = \mathrm{div}_{\nabla u^\perp} X$$

we calculate

$$\begin{aligned} \int_\Omega |\nabla u|\, \mathrm{div}_{\nabla u^\perp} X\, dx &= \int_{-\infty}^{+\infty} \int_{\Omega \cap \{u=t\}} \mathrm{div}_{\nabla u^\perp} X\, dx\, dt \\ &= -\int_{-\infty}^{+\infty} \int_{\Omega \cap \{u=t\}} H_{\{u=t\}} \cdot X\, dx\, dt \\ &= -\int_\Omega |\nabla u| H_{\{y,\, u(y)=u(x)\}} \cdot X\, dx \end{aligned}$$

This looks exactly like the formula provided by the divergence theorem, except that the Hausdorff measure has been replaced by the measure $|\nabla u| dx$. This observation is the core of our approach in a less regular context: roughly speaking, given a function u of bounded variation, we will define a varifold associated with the mass represented by the total variation. Then the first variation of the mass can be computed (like above), whose Radon-Nikodym derivative with respect to the mass finally yields the mean curvature.

The first delicate issue is to extend properly to BV the quantity $\int_\Omega |\nabla u|\, \mathrm{div}_{\nabla u^\perp} X\, dx$ that belongs to the general family of mappings $u \mapsto \int_\Omega f(x, \nabla u) dx$. Studying such mappings in BV is the purpose of [20], where suitable tools are defined based on the theory of Young measures.

Young measures They were introduced by L.C. Young [30, 31, 32] to describe limits of minimizing sequences for integrals of the type

$$\int f(x,u)\, dx \quad \text{or} \quad \int f(x,u,\nabla u)\, dx$$

Young measures are particularly useful when classical minimizers do not exist. They can handle complex situations with concentration, oscillation, or diffusion phenomena. They find many applications in calculus of variations, optimal control theory, optimal design, variational modeling of nonlocal interactions, etc. [29, 11, 27].

The typical situation where they arise is the following: Ω being bounded, take a sequence (v_h) that converges weakly to v in $\mathrm{L}^\infty(\Omega,\mathbb{R}^n)$, and look at $f(x,v_h(x))$ with f continuous and nonlinear. Then a classical theorem states that there exists a family of probability measures $(\nu_x)_{x\in\Omega}$, called Young measure generated by the sequence (v_h), such that

$$\int_{\mathbb{R}^n} z\, d\nu_x(z) = v(x) \qquad \mathscr{L}^n - \text{a.e. } x$$

and, up to a subsequence,

$$\int_\Omega f(x,v_h)\, dx = \int_\Omega \int_{\mathbb{R}^n} f(x,z)\delta_{v_h}(z)\, dx \to \int_\Omega \int_{\mathbb{R}^n} f(x,z)\nu_x(z)\, dx.$$

In other words, the impossibility to use the continuity of f is overcome by introducing a measure that, in some sense, carries the information out of f.

A frequent situation in the calculus of variations concerns the case when v_h are gradients, i.e. $v_h = \nabla u_h$ and $v = \nabla u$ for some $u_h, u \in \mathrm{W}^{1,p}(\Omega)$. As above, every sequence of gradients that weakly converges in L^p generates a Young measure, called gradient Young measure.

Then it is natural to ask which families of probability measures are generated by sequences of gradients or, in other words, can one characterize the set of gradient Young measures?

In [18, 19] the authors study the gradient Young measures generated by a sequence of gradients converging weakly in $\mathrm{L}^p(\Omega,\mathbb{R}^m)$ $(p>1)$ and their characterization essentially depends on the condition

$$\int_\Omega \int_{\mathbb{R}^n} |z|^p\, d\nu_x(z) dx < \infty.$$

Their results are generalizable to $p=\infty$ and to the vectorial case (i.e. for $\mathbb{R}^d$-valued functions, with $d>1$) , see [27] for precise statements.

In the applications, if $p>1$, the weak convergence follows from a uniform bound on the $\mathrm{W}^{1,p}$ norm of the gradients, but in the case $p=1$ the space $\mathrm{W}^{1,1}$ is not reflexive so, to infer weak relative compactness, the sequence $\{\nabla u_h\}$ should be equi-integrable, which is hard to establish in the applications. As an alternative, one can consider the weak-* topology of $\mathrm{BV}(\Omega)$ and this leads to an extension of the concept of Young measures.

In [14, 1, 20] the authors introduce a new formulation for Young measures which extends the classical theory to the framework of functions of bounded variation. A generalized gradient Young measure ν is defined as a triplet of measures $\nu = (\nu_x, \nu_x^\infty, \lambda_\nu)$ where $(\nu_x)_{x\in\mathbb{R}^n}$ is a family of probability measures on $\mathbb{R}^n$, λ_ν is a positive bounded Radon measure on $\overline{\Omega}$ and $(\nu_x^\infty)_{x\in\mathbb{R}^n}$ is a family of probability measures on $\mathbb{S}^{n-1}$, the unit sphere of $\mathbb{R}^n$. The Young measure representation is extended to

$$\langle\langle \nu, f\rangle\rangle := \int_\Omega \int_{\mathbb{R}^n} f(x,z)\, d\nu_x(z)\, dx + \int_{\overline{\Omega}} \int_{\mathbb{S}^{n-1}} f^\infty(x,z)\, d\nu_x^\infty(z)\, d\lambda_\nu(x)$$

where f^∞ is the recession function defined as

$$f^\infty(x,z) := \lim_{\substack{x'\to x\\ z'\to z\\ t\to\infty}} \frac{f(x',tz')}{t}.$$

In [20] a characterization theorem for generalized Young measures is proved. As in the case of classical Young measures, the condition for having a generalized gradient Young measure (i.e. generated by a sequence of gradients of functions that converge weakly-* in BV) is

$$\int_\Omega \int_{\mathbb{R}^n} |z|\, d\nu_x(z)dx + \lambda_\nu(\overline{\Omega}) < \infty.$$

We refer to [20] for general results in the vectorial context of $\mathrm{BV}(\mathbb{R}^n, \mathbb{R}^m)$. In Section 5 we recall the main results in the real-valued case $m = 1$.

The examples given in [1, 20], and those from Section 5, show that Young measures are totally suitable for describing the concentration and oscillations effects generated by the weak convergence of gradients. In fact, limit Young measures contain analytic and geometric information; they depend on the converging sequence (and not only on its weak limit!) so they carry some information about the weak limit of the sequence of gradients and the intrinsic features of the sequence.

Young varifolds We have now the material to introduce the Young varifolds, i.e. a suitable class of varifolds generated by Young measures which allows us to formalize our problem in the varifolds framework. [1]

For every $f \in \mathrm{C_c}(G_{n-1}(\Omega))$ let

$$g: \ (x,z) \in \Omega \times \mathbb{R}^n \mapsto g(x,z) = |z| f(x, z^\perp)$$

where $z^\perp$ is the element of $G_{n-1}(\Omega)$ perpendicular to z. It is easy to check that for every $k \in \mathbb{R}$ we have $(kz)^\perp = z^\perp$ (as elements of $G(n,n-1)$) so we get

$$g^\infty(x,z) = f(x,z^\perp).$$

A varifold V is a Young varifold if there exists a Young measure ν such that

$$\int_{G_{n-1}(\Omega)} f(x,S)\, dV_\nu(x,S) = \int_\Omega \int_{\mathbb{R}^n} |z| f(x,z^\perp)\, d\nu_x\, dx + \int_\Omega \int_{\mathbb{S}^{n-1}} f(x,z^\perp)\, d\nu_x^\infty\, d\lambda_\nu$$

for every $f \in \mathrm{C_c}(G_{n-1}(\Omega))$. $V = V_\nu$ is called the Young varifold associated to ν.

The definition of a Young varifold is particularly explicit for smooth functions. If $u \in \mathrm{C}^2(\Omega)$ we consider the Young measure $\nu = (\nu_x, \nu_x^\infty, \lambda_\nu)$ defined by

$$\nu_x = \delta_{\nabla u_x}\,, \quad \nu_x^\infty = 0\,, \quad \lambda_\nu = 0$$

and it follows that

$$\int_{G_{n-1}} f(x,S)\, dV_\nu(x,S) = \int_\Omega |\nabla u| f(x, (\nabla u)^\perp(x))\, dx.$$

The mass of the varifold is defined by

$$\mu_{V_\nu}(E) = V_\nu(G_{n-1}(E)) = \int_E \int_{\mathbb{R}^n} |z|\, d\nu_x(z)\, dx + \lambda_\nu(\overline{E}) \quad \forall E \subseteq \Omega.$$

and the first variation is

$$\delta V_\nu(X) = \int_{G_{n-1}(\Omega)} \mathrm{div}_S X(x)\ dV_\nu(x,S) = \int_\Omega \int_{\mathbb{R}^n} |z| \,\mathrm{div}_{z^\perp} X\ d\nu_x(z)\, dx + \int_\Omega \int_{\mathbb{S}^{n-1}} \mathrm{div}_{z^\perp} X\ d\nu_x^\infty(z)\, d\lambda_\nu$$

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[1] M. Novaga kindly brought to our attention, while the current paper was in the final correction phase, the reference [6] where a generalization of Almgren's theory of varifolds in a lorentzian setting is proposed. In a different context and for different purposes, it shares with our work the idea of disintegrating and indexing the measures that we borrowed from [20] while it is done using ad-hoc varifolds in [6].

We can now, as we did for sets, show how the usual notions for smooth functions can be extended to the framework of Young varifolds.

	$u \in C^2$	Young varifold V_ν
Mass	$\lvert Du\rvert$	μ_{V_ν}
First variation	$\int_{\mathbb{R}^2} \lvert\nabla u\rvert \mathrm{div}_{\nabla u^\perp} X dx$	$\delta V_\nu(X) = \int_{G_{n-1}(\Omega)} \mathrm{div}_S X dV_\nu(x,S)$
Mean curvature vector	$-(\mathrm{div}\frac{\nabla u}{\lvert\nabla u\rvert})\frac{\nabla u}{\lvert\nabla u\rvert}$	$\mathbf{H}_V = -\frac{\delta V_\nu}{\mu_{V_\nu}}$

Finally, we define the generalized Willmore energy associated with a Young varifold as

$$W(V) = \int_\Omega (1 + |\mathbf{H}_V|^p)\ d\mu_V.$$

The paper is devoted to defining the Young varifolds, studying some of their properties and investigating the relationship between Young varifolds and the relaxation problems for F and W. We will provide several examples of converging sequences of functions for which there is *continuity* of the energy of the associated Young varifolds. We will prove in Theorem 8.1 that, in any dimension,

$$\overline{F}(u,\Omega) \geq \mathrm{Min}\,\{W(V) : V \in \mathbb{V}(u)\}$$

where $\mathbb{V}(u)$ is the set of Young varifolds V such that $\|\delta V\| << \mu_V$ and V is associated with the gradient Young measures generated by sequences of regular functions that converges weakly-* in BV. We conjecture that the equality actually holds but we were not able to prove it so far. It is actually very delicate to manipulate limit Young varifolds and adequate tools are still missing (it is the purpose of ongoing research). The situation is much better in dimension 2 where, using the results of [24], we prove (Theorems 8.4 and8.10) that for every u a special function of bounded variation with compact support, there exists an associated Young varifold whose energy coincides with $\overline{F}(u)$.

The plan of the paper is as follows: the first sections are dedicated to a careful introduction of all notions that we have roughly described so far. More precisely, in Section 2 we recall the main definitions and results obtained in [24]. Section 3 and 4 are devoted, respectively, to the general theory of functionals depending on measures and to the varifold theory. In Section 5 we recall the general theory of Young measures, following [20]. In Section 6 and 7 we define the Young varifolds and the Willmore functional for Young varifolds and we give also several examples showing that Young varifolds allow to get information about geometric phenomena, like oscillations and concentration, for minimizing sequences. Finally in Section 8 we study the relationship between the Willmore functional for Young varifolds and $\overline{F}$.

General notations

$\mathbb{R}^n$ is equipped with the Euclidean norm and we will denote by either $\mathscr{L}^n$ or $|\cdot|$ the Lebesgue measure on $\mathbb{R}^n$. $\mathscr{H}^k$ is the k-dimensional Hausdorff measure. The restriction of a measure μ to a set A is denoted by $\mu \llcorner A$ and spt μ is the support of μ.

For two open sets $E, F \subset \mathbb{R}^n$ the notation $E \subset\subset F$ means that $\overline{E} \subset F$ and $\overline{E}$ is compact.

If X is a locally compact separable metric space we denote by $\mathscr{M}(X, \mathbb{R}^n)$ the space of $\mathbb{R}^n$-valued bounded Radon measures and by $\mathscr{M}^+(X)$, $\mathscr{M}^1(X)$ the spaces of positive Radon measures and probability measures, respectively. Moreover, given $\mu \in \mathscr{M}(X,\mathbb{R}^n)$ and $\nu \in \mathscr{M}^+(X)$ we denote by $\frac{\mu}{\nu}$ the derivative of μ with respect to ν and the Radon-Nikodym decomposition of μ with respect to ν is $\mu = \mu^a + \mu^s = \frac{\mu}{\nu}\nu + \mu^s$.

$\mathrm{C_c}, \mathrm{C}^k, \mathrm{L}^p, \mathrm{W}^{k,p}, \mathrm{BV}, \mathrm{SBV}$ are the usual function spaces. For a detailed study of the spaces BV and SBV of functions with bounded variation, the reader may refer to [3]. If $\Omega \subset \mathbb{R}^n$, we say that $\partial\Omega \in \mathrm{C}^k$ (resp. $\mathrm{W}^{k,p}$) if we can represent locally its boundary as a graph of a C^k (resp. $\mathrm{W}^{k,p}$) function. In particular, Ω is called a Lipschitz domain if $\partial\Omega \in \mathrm{C}^{0,1}$.

2 Relaxation by a coarea-type formula in dimension two

We recall in this section the main results proved in [24] which will be used in the following. Let us start with the notion of system of curves of class $W^{2,p}$:

Definition 2.1 *By a system of curves of class* $\mathrm{W}^{2,p}$ *we mean a finite family* $\Gamma = \{\gamma_1, ..., \gamma_N\}$ *of closed curves of class* $\mathrm{W}^{2,p}$ *(and so* C^1*) admitting a parameterization (still denoted by* γ_i*)* $\gamma_i \in \mathrm{W}^{2,p}\left([0,1],\mathbb{R}^2\right)$ *with unit velocity. Moreover, every curve of* Γ *can have tangential self-contacts but without crossing and two curves of* Γ *can have tangential contacts but without crossing. In particular,* $\gamma_i'(t_1)$ *and* $\gamma_j'(t_2)$ *are parallel whenever* $\gamma_i(t_1) = \gamma_j(t_2)$ *for some* $i,j \in \{1,...,N\}$ *and* $t_1,t_2 \in [0,1]$.

The trace (Γ) *of* Γ *is the union of the traces* (γ_i). *We define the interior of the system* Γ *as*

$$\mathrm{Int}(\Gamma) = \{x \in \mathbb{R}^2 \setminus (\Gamma) : \mathrm{I}(x,\Gamma) = 1 \mod 2\}$$

where $\mathrm{I}(x,\Gamma) = \sum_{i=1}^N \mathrm{I}(x,\gamma_i)$.
The multiplicity function θ_Γ *of* Γ *is*

$$\theta_\Gamma : (\Gamma) \to \mathbb{N} \qquad \theta(z) = \sharp\{\Gamma^{-1}(z)\},$$

where $\sharp$ *is the counting measure.*
If the system of curves is the boundary of a set E *with* $\partial E \in \mathrm{C}^2$*, we simply denote it as* ∂E.

Remark 2.2 Remark that, by previous definition, every $|\gamma_i'(t)|$ is constant for every $t \in [0,1]$ so the arc-length parameter is given by $s(t) = tL_i$ where L_i in the length of γ_i. Denoting by $\tilde{\gamma}_i$ the curve parameterized with respect to the arc-length parameter we have

$$s \in [0,L_i]\,,\ \tilde{\gamma}_i(s) = \gamma_i(s/L_i)\,,\ \tilde{\gamma}_i''(s) = \frac{\gamma_i''(s)}{L_i^2}.$$

Now, the curvature $\mathbf{k}$ as a functions of s, verifies

$$\mathbf{k} = \tilde{\gamma}_i''(s)$$

which implies

$$\int_0^{L_i} \left(1 + |\tilde{\gamma}_i''(s)|^p\right) \mathrm{d}s = \int_0^{L_i} (1 + |\mathbf{k}|^p)\,\mathrm{d}s = \int_0^1 \left(|\gamma_i'(t)| + L_i^{1-2p}|\gamma_i''(t)|^p\right) \mathrm{d}t.$$

Then, the condition $\gamma_i \in \mathrm{W}^{2,p}\left([0,1],\mathbb{R}^2\right)$ implies that $\tilde{\gamma}_i \in \mathrm{W}^{2,p}\left([0,L_i],\mathbb{R}^2\right)$ and, for simplicity, in the sequel we denote by γ_i the curve parameterized with respect to the arc-length parameter.

In dimension 2, the Willmore functional for a system of curves of class $W^{2,p}$, Γ, is defined by

$$W(\Gamma) = \sum_{i=1}^{N} W(\gamma_i) = \sum_{i=1}^{N} \int_{(\gamma_i)} \left(1 + |\mathbf{k}_{\gamma_i}|^p\right) \, d\mathcal{H}^1.$$

Definition 2.3 *We say that Γ is a limit system of curves of class $\mathrm{W}^{2,p}$ if Γ is the weak limit of a sequence (Γ_h) of boundaries of bounded open sets with $\mathrm{W}^{2,p}$ parameterizations.*

The following class of curve-valued functions will be used for covering the level lines of a real function.

Definition 2.4 *Let $\mathscr{A}$ denote the class of functions*

$$\Phi : t \in \mathbb{R} \to \Phi(t)$$

where for almost every $t \in \mathbb{R}$, $\Phi(t) = \{\gamma_t^1, ..., \gamma_t^N\}$ is a limit system of curves of class $\mathrm{W}^{2,p}$ and such that, for almost every $\underline{t}, \bar{t} \in \mathbb{R}$, $\underline{t} < \bar{t}$, the following conditions are satisfied:

(i) $\Phi(\underline{t})$ *and* $\Phi(\bar{t})$ *do not cross but may intersect tangentially;*

(ii) $\mathrm{Int}(\Phi(\bar{t})) \subseteq \mathrm{Int}(\Phi(\underline{t}))$ *(pointwisely);*

(iii) if, for some i, $\mathcal{H}^1\left((\gamma_{\bar{t}}^i) \setminus \overline{\mathrm{Int}(\Phi(\underline{t}))}\right) \neq 0$ *then*

$$\mathcal{H}^1\left([(\gamma_{\bar{t}}^i) \setminus \overline{\mathrm{Int}(\Phi(\underline{t}))}] \setminus (\Phi(\underline{t}))\right) = 0.$$

Remark 2.5 One may remark that, from condition (ii) of Definition 2.4, for every curve $\gamma \in \Phi(\underline{t})$

$$(\gamma) \cap \mathrm{Int}(\Phi(\bar{t})) = \emptyset.$$

In fact if $x \in (\gamma) \cap \mathrm{Int}(\Phi(\bar{t}))$ then $x \in \mathrm{Int}(\Phi(\bar{t}))$ and $x \notin \mathrm{Int}(\Phi(\underline{t}))$ which gives a contradiction with condition (ii).

Let us now introduce an important subset of $\mathscr{A}$.

Definition 2.6 (The class $\mathscr{A}(u)$) *Let $u \in \mathrm{BV}(\mathbb{R}^2)$. We define $\mathscr{A}(u)$ as the set of functions $\Phi \in \mathscr{A}$ such that, for almost every $t \in \mathbb{R}$, we have*

$$(\Phi(t)) \supseteq \partial^*\{u > t\} \quad \textit{(up to a } \mathcal{H}^1\textit{-negligible set)}$$

and

$$\{u > t\} = \mathrm{Int}(\Phi(t)) \quad \textit{(up to a } \mathcal{L}^2\textit{- negligible set)}.$$

In particular, if $u \in \mathrm{C}^2(\mathbb{R}^2)$, we will denote as $\Phi[u]$ the function of $\mathscr{A}(u)$ defined as

$$t \mapsto \partial\{u > t\}.$$

In [24] we proved the following representation result for the relaxation problem for the Willmore functional on $\mathbb{R}^2$

Theorem 2.7 *Let $u \in \mathrm{BV}(\mathbb{R}^2)$ with $\overline{F}(u, \mathbb{R}^2) < \infty$. Then*

$$\overline{F}(u, \mathbb{R}^2) = \operatorname*{Min}_{\Phi \in \mathscr{A}(u)} G(\Phi).$$

The next proposition points out the relationship between the relaxation problem on $\mathbb{R}^2$ for a function with compact support and the relaxation problem on a suitable $\Omega \subset \mathbb{R}^2$:

Proposition 2.8 *Let $u \in \mathrm{BV}(\mathbb{R}^2)$ with compact support and such that $\overline{F}(u,\mathbb{R}^2) < \infty$. There exists an open bounded domain Ω such that*

$$\overline{F}(u,\mathbb{R}^2) = \overline{F}_B^0(u,\Omega) := \inf\left\{ \liminf_{h\to\infty} \int_\Omega |\nabla u_h|(1+|\operatorname{div}\frac{\nabla u_h}{|\nabla u_h|}|^p)\,dx : \{u_h\} \in \mathrm{C}_\mathrm{c}^2(\Omega),\ u_h \xrightarrow{\mathrm{L}^1(\Omega)} u \right\}.$$

As pointed out in [24] such a proposition is not true for the relaxation problem defined with C^2 instead of C_c^2.

3 Functionals defined on measures

Let μ, ν be Radon measures on $\Omega \subset \mathbb{R}^n$, μ positive, ν $\mathbb{R}^m$-valued and let $f : \mathbb{R}^m \to [0,\infty]$ be convex. We set

$$G(\nu,\mu) = \int_\Omega f\left(\frac{\nu}{\mu}(x)\right)\,d\mu(x) + \int_\Omega f^\infty\left(\frac{\nu^s}{|\nu^s|}(x)\right)\,d|\nu^s|(x)$$

where ν^s is the singular part of ν with respect to μ and $f^\infty : \mathbb{R}^m \to \mathbb{R}\cup\{\infty\}$ is the recession function of f defined by

$$f^\infty(z) = \lim_{t\to\infty} \frac{f(z_0+tz)-f(z_0)}{t} \tag{2}$$

where $z_0 \in \mathbb{R}^m$ is any vector such that $f(z_0) < \infty$.

As stated in the theorem below, G is lower semicontinuous under suitable assumptions.

Theorem 3.1 ([3], Thm 2.34) *Let Ω be an open subset of $\mathbb{R}^n$ and ν, ν_h be $\mathbb{R}^m$-valued Radon measures on Ω, μ, μ_h positive Radon measures on Ω. Let $f : \mathbb{R}^m \to [0,\infty]$ be a convex lower semicontinuous function. If $\nu_h \overset{*}{\rightharpoonup} \nu$ and $\mu_h \overset{*}{\rightharpoonup} \mu$ in Ω then*

$$G(\nu,\mu) \le \liminf_{h\to\infty} G(\nu_h,\mu_h).$$

Notice that if f has superlinear growth (i.e. $f^\infty(z) < \infty$ only if $z = 0$) then $G(\nu,\mu) < \infty$ only if $\nu << \mu$ and in this case

$$G(\nu,\mu) = \int_\Omega f\left(\frac{\nu}{\mu}(x)\right)\,d\mu(x).$$

The next theorem will be useful in the sequel and is a direct consequence of Theorem 3.1:

Theorem 3.2 ([3], Example 2.36) *Let Ω be an open subset of $\mathbb{R}^n$, $\nu, (\nu_h)$ $\mathbb{R}^m$-valued Radon measures on Ω, and $\mu, (\mu_h)$ positive Radon measures on Ω. Let $f : \mathbb{R}^m \to [0,\infty]$ be a convex lower semicontinuous function with superlinear growth. If $\nu_h \overset{*}{\rightharpoonup} \nu$, $\mu_h \overset{*}{\rightharpoonup} \mu$ in Ω, $\nu_h << \mu_h$ and $\int_\Omega f(\nu_h/\mu_h)\,d\mu_h$ is bounded then $\nu << \mu$ and*

$$G(\nu,\mu) \le \liminf_{h\to\infty} G(\nu_h,\mu_h).$$

4 Varifolds

We collect below a few facts about varifolds. More details can be found in [28, 2].

4.1 Definitions

We consider $G(n,k)$, $k \leq n$, the set of all k-dimensional subspaces of $\mathbb{R}^n$ equipped with the metric

$$\|S-T\| = \left(\sum_{i,j=1}^{n} |P_S(i,j) - P_T(i,j)|^2 \right)^{1/2}$$

where P_S and P_T are the matrix of orthogonal projections of $\mathbb{R}^n$ on S and T written with respect to standard orthonormal basis. $G(n,k)$ is called the Grassmannian of all unoriented k-subspaces of $\mathbb{R}^n$.

For a subset Ω of $\mathbb{R}^n$ we define

$$G_k(\Omega) = \Omega \times G(n,k)$$

equipped with the product metric.

Definition 4.1 (Varifolds) *A k-varifold V on Ω is a Radon measure on $G_k(\Omega)$. The weight measure of V is the Radon measure on Ω defined by*

$$\mu_V(U) = V(\pi^{-1}(U))$$

where π is the projection $(x,S) \mapsto x$ of $G_k(\Omega)$ onto $\mathbb{R}^n$.

A very important class of varifolds is obtained from rectifiable sets.

Definition 4.2 (Countably C^r-k-rectifiable sets) *$M \subseteq \mathbb{R}^n$ is a countably C^r-k-rectifiable set ($r \geq 1$) if*

$$M = M_0 \cup \left(\bigcup_{i=1}^{+\infty} K_i \right)$$

where $\mathscr{H}^k(M_0) = 0$, $K_i \cap K_j = \emptyset$ if $i \neq j$ and for all $i \geq 1$ K_i is a subset of a C^r-k-manifold of $\mathbb{R}^n$.
M is C^r-k-rectifiable if M is countably C^r-k-rectifiable and $\mathscr{H}^k(M) \leq +\infty$.
M is (countably) C^r-k-rectifiable in $\Omega \subseteq \mathbb{R}^n$ if $M \cap \Omega$ is (countably) C^r-k-rectifiable.

Remark that if M is a countably C^r-k-rectifiable set then for every $x \in K_i$ we can consider the tangent plan to K_i at x, denoted by T_x, and, by the previous definition, the function

$$x \mapsto T_x$$

is defined for $\mathscr{H}^k$-a.e. $x \in M$.

Definition 4.3 (Rectifiable and integral varifolds) *V is a rectifiable k-varifold if there exists a C^r-k-rectifiable subset M of Ω ($r \geq 1$) such that*

$$V = \theta \mathscr{H}^k \llcorner M \otimes \delta_{T_x}$$

where θ is a positive $\mathscr{H}^k$-locally integrable function on M called multiplicity of V. In the following we denote V by $\mathbf{v}(M,\theta)$. If $\theta(M) \subset \mathbb{N}$ then V is called an integral varifold.
The weight measure of a rectifiable k-varifold is given by

$$\mu_V(U) = \int_{M \cap U} \theta(x) \, d\mathscr{H}^k(x).$$

4.2 First variation and generalized mean curvature

The first variation of a k-varifold V is the functional given by

$$\delta V : \mathrm{C_c}(\Omega, \mathbb{R}^n) \to \mathbb{R}$$

$$\delta V(X) = \int_{G_k(\Omega)} \mathrm{div}_S X(x)\, dV(x,S),$$

where div_S is the tangential divergence with respect to S.

Definition 4.4 *A k-varifold is called a Allard's varifold if it has locally bounded first variation in Ω, i.e. for each $W \subset\subset \Omega$ there exists a constant $0 < C < \infty$ such that*

$$|\delta V(X)| \le C \|X\|_{\mathrm{L}^\infty(W)} \quad \forall X \in \mathrm{C_c}(W, \mathbb{R}^n).$$

If V is a Allard's varifold then

$$\|\delta V\|(W) = \sup\{|\delta V(X)| : X \in \mathrm{C_c}(W,\mathbb{R}^n), \|X\|_{\mathrm{L}^\infty(W)} \le 1\} < \infty$$

for each $W \subset\subset \Omega$ so, by the Riesz representation theorem,

$$\delta V(X) = -\int_\Omega \langle X, \nu\rangle \, d\|\delta V\|$$

where $\|\delta V\|$ is the total variation measure of δV and ν is a $\|\delta V\|$-measurable $\mathbb{R}^n$-valued function with $|\nu| = 1$ $\|\delta V\|$-a.e in Ω.

Therefore, by the Radon-Nikodym decomposition theorem, we can decompose $\|\delta V\|$ as

$$\|\delta V\| = \frac{\|\delta V\|}{\mu_V}\mu_V + \sigma$$

where the derivative of $\|\delta V\|$ with respect to μ_V exists μ_V-a.e. and the measure σ, the singular part of $\|\delta V\|$ with respect to μ_V , is supported on Z such that

$$Z = \left\{ x \in \Omega : \frac{\|\delta V\|}{\mu_V}(x) = +\infty \right\}, \qquad \mu_V(Z) = 0.$$

So, defining $\mathbf{H}_V(x) = \frac{\|\delta V\|}{\mu_V}(x)\nu(x) = -\frac{\delta V}{\mu_V}(x)$, we can write

$$\delta V(X) = -\int_\Omega \langle X, \mathbf{H}_V\rangle \, d\mu_V - \int_Z \langle X, \nu\rangle \, d\sigma$$

for all $X \in \mathrm{C_c}(\Omega, \mathbb{R}^n)$.

Definition 4.5 *We call $\mathbf{H}_V$ the generalized mean curvature of V, Z the generalized boundary of V, σ the generalized boundary measure of V, and $\nu_{|Z}$ the generalized unit conormal of V.*

Remark 4.6 Notice that if V is a rectifiable k-varifold $\mathbf{v}(M,1)$ associated with M a C^2-manifold without boundary then, from the divergence theorem for manifolds (see [3]: Theorem 7.34), $\|\delta V\| << \mu_V$ and the mean curvature $\mathbf{H}_V$ coincides everywhere out of a $\mathscr{H}^k$-negligible set with the classical mean curvature of M.

Remark 4.7 (2-Varifolds supported on $\mathrm{W}^{2,p}$-curves) Let $V = \mathbf{v}(M,\theta)$ be the varifold on $\mathbb{R}^2$ associated with M a closed curve in $\mathbb{R}^2$ of class $\mathrm{W}^{2,p}$ with $p > 1$, and with the density function θ. M admits a parametrization (still denoted by M) $M \in \mathrm{W}^{2,p}\left([0,L],\mathbb{R}^2\right)$,

$$M(s) = (f(s), g(s)) \ , \ \ f,g \in \mathrm{W}^{2,p}\left([0,L],\mathbb{R}\right)$$

where s is the arc-length parameter and L is the length of the curve M. Then, by direct calculation, we will show that $\|\delta V\| << \mu_V$ and the mean curvature of V is a function of the weak second derivatives of f and g. This fact can be generalized using Hutchinson's varifolds [17, 16].

Consider $X \in \mathrm{C}^1_c(\mathbb{R}^2,\mathbb{R}^2)$, $X(x) = (X^1(x), X^2(x))$, $\{e_1,e_2\}$ the canonical orthonormal basis of $\mathbb{R}^2$ and denote by $\langle\cdot,\cdot\rangle$ the usual scalar product in $\mathbb{R}^2$. Then

$$\begin{aligned} \mathrm{div}_M X(M(s)) &= \langle e_1, \langle \nabla X^1(M(s)), M'(s)\rangle M'(s)\rangle + \langle e_2, \langle \nabla X^2(M(s)), M'(s)\rangle M'(s)\rangle = \\ &= f'(s)\langle \nabla X^1(M(s)), M'(s)\rangle + g'(s)\langle \nabla X^2(M(s)), M'(s)\rangle \end{aligned}$$

and

$$\begin{aligned} \delta V(X) = \int_M \theta \,\mathrm{div}_M X \, d\mathscr{H}^1 &= \int_0^L \left[f'(s)\langle \nabla X^1(M(s)), M'(s)\rangle + g'(s)\langle \nabla X^2(M(s)), M'(s)\rangle \right] ds \\ &= \int_0^L \left[f'(s)\frac{d}{ds}\left[X^1(M(s))\right] + g'(s)\frac{d}{ds}\left[X^2(M(s))\right] \right] ds. \end{aligned}$$

Now, integrating by parts and using the facts that X has compact support, M is closed and $f,g \in \mathrm{W}^{2,p}\left([0,L],\mathbb{R}\right)$, we get

$$\delta V(X) = -\int_0^L \langle X(M(s)), (f''(s), g''(s))\rangle \, ds$$

where f'', g'' are the weak second derivatives. It follows that

$$\delta V(X) = -\int_M \langle X, \mathbf{H}_V\rangle \theta \, d\mathscr{H}^1$$

where the curvature of varifold V is given by

$$\mathbf{H}_V(p) = M''(M^{-1}(p)) = (f''(M^{-1}(p)), g''(M^{-1}(p))) \ \forall \, p \in (M).$$

Clearly $\|\delta V\| << \mu_V$. By a similar calculation we can generalize this remark to the varifolds $V = \mathbf{v}(M,\theta)$ where M is a system of curves of class $\mathrm{W}^{2,p}$ and θ the density function on M.

5 Young measures

In this section we recall the theory of Young measures following [20].

5.1 Definitions and general results

Let Ω be a bounded Lipschitz domain of $\mathbb{R}^n$ and let $f \in C(\Omega \times \mathbb{R}^n)$. By $\mathbb{B}^n$ we denote the open unit ball in $\mathbb{R}^n$ and by $\mathbb{S}^{n-1}$ we denote $\partial\mathbb{B}^n$. We consider the following operator

$$T : C(\Omega \times \mathbb{R}^n) \to C(\Omega \times \mathbb{B}^n)$$

$$Tf(x,z) := (1-|z|) f\left(x, \frac{z}{1-|z|}\right)$$

and the property

$$\mathrm{T}f \text{ extends into a bounded continuous function on } \overline{\Omega \times \mathbb{B}^n}. \tag{3}$$

We can define the Banach space $(\mathbf{E}(\Omega;\mathbb{R}^n), \|\cdot\|_{\mathbf{E}})$, where

$$\mathbf{E}(\Omega;\mathbb{R}^n) = \{f \in C(\Omega \times \mathbb{R}^n) : f \ \text{ satisfies } \ (3)\}$$

$$\|f\|_{\mathbf{E}} = \|\mathrm{T}f\|_{\mathrm{L}^\infty(\overline{\Omega\times\mathbb{B}^n})}.$$

For example, a continuous function which is either uniformly bounded or positively 1-homogeneous in its second argument (i.e. $f(x,sz) = sf(x,z)$, for all $s \geq 0$) belongs to $\mathbf{E}(\Omega \times \mathbb{R}^n)$. Moreover, every $f \in \mathbf{E}(\Omega;\mathbb{R}^n)$ has linear growth to infinity since

$$|f(x,z)| = (1+|z|)Tf\left(x, \frac{z}{1+|z|}\right) \leq \|f\|_{\mathbf{E}}(1+|z|) \qquad \text{for all } x \in \Omega, z \in \mathbb{R}^n.$$

For all $f \in \mathbf{E}(\Omega \times \mathbb{R}^n)$ we define the recession function $f^\infty : \overline{\Omega} \times \mathbb{S}^{n-1} \to \mathbb{R}$ by

$$f^\infty(x,z) := \lim_{\substack{x' \to x \\ z' \to z \\ t \to \infty}} \frac{f(x',tz')}{t}.$$

Remark that for every convex function $f = f(z)$ belonging to $\mathbf{E}(\Omega;\mathbb{R}^n)$ the previous definition coincides with (2) (this follows from continuity for convex functions and taking $z_0 = 0$ in (2)).

Before defining generalized Young measures it is convenient to recall some notations about parametrized measures. For sets $E \subset \mathbb{R}^k$, $F \subset \mathbb{R}^l$ open or closed, a parametrized measure $(\nu_x)_{x\in E}$ is a mapping from E to $\mathscr{M}(F)$, the set of Radon measures on F. It is said to be weakly* μ-measurable, for some $\mu \in \mathscr{M}^+(E)$, if the function $x \mapsto \nu_x(B)$ is μ-measurable for all Borel sets $B \subset F$. Here μ-mesurability is the mesurability with respect to the μ-completion of the Borel σ-algebra on E.

In the following we denote by $\mathrm{L}^\infty_{w^*}(E,\mu,\mathscr{M}(F))$ the set of weakly* μ-measurable parametrized measures $(\nu_x)_{x\in E} \subset \mathscr{M}(F)$ such that $\sup\limits_{x\in E} |\nu_x|(F) < \infty$ (the essential supremum with respect to μ). We omit μ in the notation if μ is the Lebesgue measure.

Following [20], we define the generalized Young measures:

Definition 5.1 *The set of all generalized Young measures $\mathbf{Y}(\Omega,\mathbb{R}^n)$ is defined to be the set of all triplets $(\nu_x, \lambda_\nu, \nu_x^\infty)$, simply written ν, such that :*

- *(i) $\nu_x \in \mathrm{L}^\infty_{w^*}(\Omega, \mathscr{M}^1(\mathbb{R}^n))$ where the map $x \mapsto \nu_x$ is defined up to a $\mathscr{L}^n$-negligible set and with $x \mapsto \langle \nu_x, |\cdot|\rangle \in \mathrm{L}^1(\Omega)$. ν_x is called oscillation measure.*
- *(ii) $\lambda_\nu \in \mathscr{M}^+(\overline{\Omega})$. λ_ν is called concentration measure.*
- *(iii) $\nu_x^\infty \in \mathrm{L}^\infty_{w^*}(\overline{\Omega}, \lambda_\nu; \mathscr{M}^1(\mathbb{S}^{n-1}))$ where the map $x \mapsto \nu_x^\infty$ is defined up to a λ_ν- negligible set. ν_x^∞ is called concentration-angle measure.*

Therefore we can see $\mathbf{Y}(\Omega,\mathbb{R}^n)$ as a subset of $\mathbf{E}(\Omega \times \mathbb{R}^n)^*$ through the following duality pairing :

$$\langle\langle \nu, f \rangle\rangle := \int_\Omega \int_{\mathbb{R}^n} f(x,z)\, d\nu_x(z)\, dx + \int_{\overline{\Omega}} \int_{\mathbb{S}^{n-1}} f^\infty(x,z)\, d\nu_x^\infty(z)\, d\lambda_\nu(x)$$

Then we can define the convergence for Young measures in the sense of duality:

Definition 5.2 (Y-convergence) *A sequence* $\{\nu_h\} \subset \mathbf{Y}(\Omega,\mathbb{R}^n)$ *converges weakly* to* ν *in* $\mathbf{Y}(\Omega,\mathbb{R}^n)$*, written* $\nu_h \overset{\mathbf{Y}}{\to} \nu$*, if* $\langle\langle \nu_h, f\rangle\rangle \to \langle\langle \nu, f\rangle\rangle$ *for all* $f \in \mathbf{E}(\Omega\times\mathbb{R}^n)$.

Moreover, we have the following properties :

Theorem 5.3 (Closedness, [20], Cor. 1) *The set* $\mathbf{Y}(\Omega,\mathbb{R}^n)$ *is weakly* closed (as a subset of* $\mathbf{E}(\Omega\times\mathbb{R}^n)^*$*).*

Theorem 5.4 (Compactness, [20], Cor. 2) *Let* $\{\nu_h\} \subset \mathbf{Y}(\Omega,\mathbb{R}^n)$ *be a sequence such that :*
(i) the functions $x \mapsto \int_{\mathbb{R}^n} |\cdot|\, d\{\nu_h\}_x$ *are uniformly bounded in* $\mathrm{L}^1(\Omega)$*;*
(ii) the sequence $\{\lambda_{\nu_h}(\overline{\Omega})\}$ *is uniformly bounded.*
Then $\{\nu_h\}$ *is weakly* sequentially relatively compact in* $\mathbf{Y}(\Omega,\mathbb{R}^n)$.

The next definition allows us to associate every Radon measure on $\overline{\Omega}$ with a Young measure.

Definition 5.5 *Let* $\mu \in \mathscr{M}(\overline{\Omega},\mathbb{R}^n)$ *with Radon-Nikodym decomposition* $\mu = \alpha\mathscr{L}^n\llcorner\Omega + \mu_s$*. The Young measure* ν_μ *associated with* μ *is defined by :*

$$\nu_x = \delta_{\alpha(x)}, \quad \lambda_\nu = |\mu_s|, \quad \nu_x^\infty = \delta_{\frac{\mu_s}{|\mu_s|}}$$

Theorem 5.6 (Fundamental Theorem, [20], Thm 7) *Let* $\{\mu_h\} \subset \mathscr{M}(\overline{\Omega},\mathbb{R}^n)$ *be a sequence of Radon measures such that* $\sup_h |\mu_h|(\overline{\Omega}) < \infty$*. Then, there exists a subsequence (not relabeled) such that* $\nu_{\mu_h} \overset{\mathbf{Y}}{\to} \nu$ *for some* $\nu \in \mathbf{Y}(\Omega,\mathbb{R}^n)$.

Theorem 5.7 (Reshetnyak's Continuity, [20], Prop. 1) *Let* $\{\mu_h\} \subset \mathscr{M}(\overline{\Omega},\mathbb{R}^n)$ *be a sequence of Radon measures such that* $\mu_h \overset{*}{\rightharpoonup} \mu$ *in* $\mathscr{M}(\overline{\Omega},\mathbb{R}^n)$ *and* $|\mu_h|(\overline{\Omega}) \to |\mu|(\overline{\Omega})$*. Then* $\nu_{\mu_h} \overset{\mathbf{Y}}{\to} \nu_\mu$ *in* $\mathbf{Y}(\Omega,\mathbb{R}^n)$.

Finally, a useful notion of barycenter measure can be associated with a Young measure:

Definition 5.8 (Barycenter) *The barycenter of* $\nu \in \mathbf{Y}(\Omega,\mathbb{R}^n)$ *is the measure* $\xi \in \mathscr{M}(\overline{\Omega},\mathbb{R}^n)$ *given by*

$$\xi_\nu := \left(\int_{\mathbb{R}^n} z\, d\nu_x\right)\mathscr{L}^n\llcorner\Omega + \left(\int_{\mathbb{S}^{n-1}} z\, d\nu_x^\infty\right)\lambda_\nu.$$

5.2 Gradient Young measures

Definition 5.9 *The Young measure associated with* $u \in \mathrm{BV}(\Omega)$ *is the measure* $\nu_{Du(x)} = (\nu_x, \lambda_\nu, \nu_x^\infty)$ *with*

$$\nu_x = \delta_{\nabla u(x)}, \quad \lambda_\nu = |D^s u|, \quad \nu_x^\infty = \delta_{\frac{D^s u}{|D^s u|}(x)}.$$

Gradient Young measures can now be defined, see [20].

Definition 5.10 *We call* $\nu \in \mathbf{Y}(\Omega,\mathbb{R}^n)$ *a gradient Young measure if there exists a bounded sequence* $\{u_h\} \subset \mathrm{BV}(\Omega)$ *such that* $\nu_{Du_h} \overset{\mathbf{Y}}{\to} \nu$*. We denote the set of gradient Young measures by* $\mathbf{GY}(\Omega,\mathbb{R}^n)$ *and we say also that* ν *is generated by* $\{u_h\}$*.*

Remark in particular that if ν is generated by $\{u_h\} \subset \mathrm{BV}(\Omega)$, i.e. $\nu_{Du_h} \xrightarrow{\mathbf{Y}} \nu$, then for all $f \in \mathbf{E}(\Omega \times \mathbb{R}^n)$ we have

$$\begin{aligned}\lim_{h\to\infty} \langle\langle \nu_{Du_h}, f\rangle\rangle &= \lim_{h\to\infty} \left[\int_\Omega f(\nabla u(x))\, dx + \int_{\overline{\Omega}} f^\infty \left(x, \frac{D^s u}{|D^s u|}(x) \right) \, d|D^s u|(x) \right] = \\ = \langle\langle \nu, f\rangle\rangle &= \int_\Omega \int_{\mathbb{R}^n} f(x,z)\, d\nu_x(z)\, dx + \int_{\overline{\Omega}} \int_{\mathbb{S}^{n-1}} f^\infty(x,z)\, d\nu_x^\infty(z)\, d\lambda_\nu(x).\end{aligned}$$

The following propositions follow directly from Theorems 5.7 and 5.6:

Proposition 5.11 ([20], Prop. 3) *If $u_h \to u$ strictly in $\mathrm{BV}(\Omega)$ then $\nu_{Du_h} \xrightarrow{\mathbf{Y}} \nu_{Du}$.*

Proposition 5.12 ([20], Prop. 8) *Let $\{u_h\} \subset \mathrm{BV}(\Omega, \mathbb{R})$ be uniformly bounded in the BV-norm. Then, there exists a subsequence (not relabeled) such that $\nu_{Du_h} \xrightarrow{\mathbf{Y}} \nu$ for some $\nu \in \mathbf{GY}(\Omega, \mathbb{R}^n)$.*

5.3 Gradient Young measures and BV functions

In this section we point out the relationship, shown in [20], between a general gradient Young measure and the Young measures associated with BV-functions.

For every $\nu \in \mathbf{GY}(\Omega, \mathbb{R}^n)$ there exists a bounded sequence $\{u_h\} \subset \mathrm{BV}(\Omega)$ such that $\nu_{Du_h} \xrightarrow{\mathbf{Y}} \nu$. Moreover, since $\{u_h\}$ is bounded in the BV-norm, there exists $u \in \mathrm{BV}(\Omega)$ and a subsequence (not relabeled) such that $u_h \overset{*}{\rightharpoonup} u$ in $\mathrm{BV}(\Omega)$ and $\nu_{Du_h} \xrightarrow{\mathbf{Y}} \nu$.

So, if we test the Young convergence using $f(x,z) = \langle g(x), z\rangle$ with $g \in C_c^\infty(\Omega, \mathbb{R}^n)$, we have ($f^\infty = f$)

$$\begin{aligned}\lim_{h\to\infty} \langle\langle \nu_{Du_h}, f\rangle\rangle &= \lim_{h\to\infty} \left[\int_\Omega \langle g(x), \nabla u(x)\rangle\, dx + \int_\Omega \langle g(x), \frac{D^s u}{|D^s u|}(x)\rangle \, d|D^s u|(x) \right] \\ &= \lim_{h\to\infty} \int_\Omega \langle g(x),\, dDu_h(x)\rangle\end{aligned}$$

and then, taking the limit, we get

$$\langle\langle \nu, f(x,z)\rangle\rangle = \int_\Omega \langle g(x),\, dDu(x)\rangle.$$

Now, because of the choice of f, we have

$$\langle\langle \nu, f(x,z)\rangle\rangle = \int_\Omega \langle g(x),\, d\xi_\nu\rangle$$

so

$$Du = \xi_\nu \llcorner \Omega = \left(\int_{\mathbb{R}^n} z\, d\nu_x \right) \mathscr{L}^n \llcorner \Omega + \left(\int_{\mathbb{S}^{n-1}} z\, d\nu_x^\infty \right) \lambda_\nu \llcorner \Omega. \tag{4}$$

This shows that if $u_h \overset{*}{\rightharpoonup} u$ in $\mathrm{BV}(\Omega)$ then $\xi_\nu = Du$. In particular, as every gradient Young measure is generated by a sequence of functions weakly converging in BV to some function $u \in \mathrm{BV}(\Omega)$, the barycenter of a gradient Young measure always coincides (in the sense of measures) with Du for some function $u \in \mathrm{BV}(\Omega)$.

The Radon-Nikodym decomposition for λ_ν with respect to $\mathscr{L}^n$ implies

$$\lambda_\nu \llcorner \Omega = \frac{\lambda_\nu}{\mathscr{L}^n}(x) \mathscr{L}^n \llcorner \Omega + \lambda_\nu^s \llcorner \Omega \tag{5}$$

so, from (4), we can write

$$D^a u = \left(\int_{\mathbb{R}^n} z \, d\nu_x + \frac{\lambda_\nu}{\mathscr{L}^n}(x) \int_{\mathbb{S}^{n-1}} z \, d\nu_x^\infty \right) \mathscr{L}^n \llcorner \Omega \tag{6}$$

$$D^s u = \left(\int_{\mathbb{S}^{n-1}} z \, d\nu_x^\infty \right) \lambda_\nu^s \llcorner \Omega. \tag{7}$$

In particular, from (6),

$$\nabla u(x) = \int_{\mathbb{R}^n} z \, d\nu_x + \frac{\lambda_\nu}{\mathscr{L}^n}(x) \int_{\mathbb{S}^{n-1}} z \, d\nu_x^\infty \qquad \mathscr{L}^n\text{-a.e.} x \in \Omega \tag{8}$$

From the Radon-Nikodym decomposition of $\lambda_\nu^s \llcorner \Omega$ with respect to $|D^s u|$ we have also

$$\lambda_\nu^s \llcorner \Omega = \frac{\lambda_\nu^s}{|D^s u|}(x) |D^s u| + \lambda_\nu^* \tag{9}$$

so

$$D^s u = \left(\int_{\mathbb{S}^{n-1}} z \, d\nu_x^\infty \right) \left(\frac{\lambda_\nu^s}{|D^s u|}(x) |D^s u| + \lambda_\nu^* \right). \tag{10}$$

and in particular

$$\frac{D^s u}{|D^s u|}(x) = \frac{\lambda_\nu^s}{|D^s u|}(x) \left(\int_{\mathbb{S}^{n-1}} z \, d\nu_x^\infty \right) \quad |D^s u| - \text{a.e. } x \in \Omega. \tag{11}$$

So, from (10) and (11), we get

$$\int_{\mathbb{S}^{n-1}} z \, d\nu_x^\infty \neq 0 \quad |D^s u|\text{-a.e.} x \in \Omega$$

and

$$\int_{\mathbb{S}^{n-1}} z \, d\nu_x^\infty = 0 \quad \lambda_\nu^*\text{-a.e.} x \in \operatorname{spt} \lambda_\nu^*. \tag{12}$$

Therefore every $\nu \in \mathbf{GY}(\Omega, \mathbb{R}^n)$ is related by the previous relationships to some $u \in \mathrm{BV}(\Omega, \mathbb{R})$. However, ν does not coincide in general with ν_{Du} since a component λ_ν^* may appear at the limit. A stronger convergence is needed for the sequence generating ν to get $\nu = \nu_{Du}$. For instance, as shown in Proposition 5.11, if ν is generated by a sequence $\{u_h\} \subset \mathrm{BV}(\Omega)$ strictly converging in BV to some u, then $\nu = \nu_{Du}$.

Nevertheless, even if the limit Young measure does not coincide with the measure associated with u, its barycenter is exactly Du.

We end this section with the Characterization Theorem for gradient Young measures.

Theorem 5.13 (Characterization, [20]: Thm 9) *Let $\Omega \subset \mathbb{R}^n$ be an open bounded Lipschitz domain. Then, a Young measure $\nu \in \mathbf{Y}(\Omega, \mathbb{R}^n)$ satisfying*

$$\lambda_\nu(\partial\Omega) = 0$$

is a gradient Young measure, $\nu \in \mathbf{GY}(\Omega, \mathbb{R}^n)$, if and only if

$$\int_\Omega \int_{\mathbb{R}^n} |z| \, d\nu_x(z) \, dx + \lambda_\nu(\Omega) < \infty$$

and there exists $u \in \mathrm{BV}(\Omega)$ such that the barycenter of ν is Du.

5.4 Identification of gradient Young measures

In this section we provide some techniques to identify a gradient Young measure ν and we also give some examples illustrating that gradient Young measures are very useful to describe oscillation and concentration effects.

Let $\nu \in \mathbf{GY}(\Omega, \mathbb{R}^n)$ then there exists a bounded sequence $\{u_h\} \subset \mathrm{BV}(\Omega)$ such that $\nu_{Du_h} \xrightarrow{\mathbf{Y}} \nu$ and one can use this convergence to identify ν. The following techniques are presented in [1] and [20]:

1) identification of ν_x: test the Young convergence using
$$f(x,z) = \Phi(x)\varphi(z)$$
with $\Phi \in \mathrm{L}^\infty(\Omega)$ and $\varphi \in C_0(\mathbb{R}^n)$. Then $f^\infty = 0$ (because $\varphi^\infty = 0$) and we get
$$\varphi(\nabla u_h) \rightharpoonup \int_{\mathbb{R}^n} \varphi(z)\, d\nu_x(z) \ \text{ in } \ L^1(\Omega).$$
Using this fact for all such φ, we can identify ν_x.
In particular if $\nabla u_h \to v$ $\mathscr{L}^n$-a.e. for some $v \in L^1(\Omega)$ then, by the dominated convergence theorem, we get $\varphi(\nabla u_h) \to \varphi(v)$ in $L^1(\Omega)$ and therefore
$$\varphi(\nabla u_h) \rightharpoonup \varphi(v) \ \text{ in } \ L^1(\Omega).$$
Then we can state $\nu_x = \delta_{v(x)}$ for $\mathscr{L}^n$-a.e. $x \in \Omega$.

2) identification of λ_ν and ν_x^∞: test the Young convergence using the function
$$f(x,z) = \Phi(x)|z|\varphi\left(x, \frac{z}{|z|}\right)$$
with $\Phi \in C(\overline{\Omega})$ and $\varphi \in C(\mathbb{S}^{n-1})$. Then $f^\infty = \varphi$ and we have
$$\int_\Omega \Phi(x)|\nabla u_h(x)|\varphi\left(\frac{\nabla u_h(x)}{|\nabla u_h(x)|}\right)\, dx + \int_{\overline{\Omega}} \Phi(x)\varphi\left(\frac{D^s u_h}{|D^s u_h|}(x)\right)\, d|D^s u_h|(x) \to$$
$$\to \int_\Omega \int_{\mathbb{R}^n} \Phi(x)|z|\varphi\left(\frac{z}{|z|}\right)\, d\nu_x(z)\, dx + \int_{\overline{\Omega}} \int_{\mathbb{S}^{n-1}} \Phi(x)\varphi(z)\, d\nu_x^\infty(z)\, d\lambda_\nu(x)$$
The knowledge from 1) and testing with all such Φ, φ allows to identify λ_ν and ν_x^∞.
In particular taking $\varphi = 1$ so $f(x,z) = \Phi(x)|z|$, with $\Phi \in C(\overline{\Omega})$, we get
$$\int_\Omega |\nabla u_h(x)|\Phi(x)\, dx + \int_{\overline{\Omega}} \Phi(x)\, d|D^s u_h|(x) \to \int_\Omega \int_{\mathbb{R}^n} |z|\Phi(x)\, dx + \int_{\overline{\Omega}} \Phi(x)\lambda_\nu(x)$$
so
$$|Du_h| \overset{*}{\rightharpoonup} \left(\int_{\mathbb{R}^n} |z|\, d\nu_x(z)\right)\mathscr{L}^n \llcorner \Omega + \lambda_\nu, \ \text{ in } \ \mathscr{M}^+(\overline{\Omega}). \tag{13}$$

We now illustrate on a few examples what kind of information can be carried by gradient Young measures.

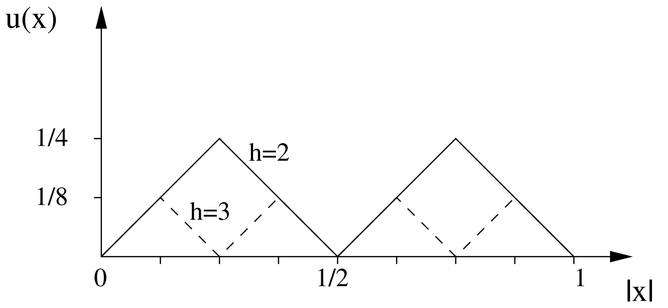

Figure 3: A radial section of the graph of u

Example 5.14 (Oscillations). Let $n = 2$, $\Omega = B(0,1)$ and

$$u_h(x) = \begin{cases} |x| - \frac{2k}{2^h} & \text{if } |x| \in \left[\frac{2k}{2^h}, \frac{2k+1}{2^h}\right] \\ -|x| + \frac{2k+2}{2^h} & \text{if } |x| \in \left[\frac{2k+1}{2^h}, \frac{2k+2}{2^h}\right] \end{cases}$$

for $k = 0, ..., 2^{h-1} - 1$.

We have

$$\nabla u_h(x) = \begin{cases} x/|x| & \text{if } |x| \in \left[\frac{2k}{2^h}, \frac{2k+1}{2^h}\right] \\ -x/|x| & \text{if } |x| \in \left[\frac{2k+1}{2^h}, \frac{2k+2}{2^h}\right] \end{cases}$$

$\{u_h\}$ is uniformly bounded in $\mathrm{BV}(\Omega)$ so, possibly after extraction of a subsequence (not relabeled), we obtain $\nu_{Du_h} \xrightarrow{\mathbf{Y}} \nu$ where ν_{Du_h} is defined by

$$\nu_x = \delta_{\nabla u_h(x)}, \quad \lambda_\nu = |D^s u_h|(x), \quad \nu_x^\infty = \delta_{\frac{D^s u_h}{|D^s u_h|}(x)}.$$

Let us prove that ν is defined by

$$\nu_x = \frac{1}{2}\delta_{\frac{x}{|x|}} + \frac{1}{2}\delta_{-\frac{x}{|x|}}, \quad \lambda_\nu = 0.$$

If we test the Young convergence using $f(x,z) = \Phi(x)\varphi(z)$, where $\Phi \in C(\overline{\Omega})$ and $\varphi \in C_0(\mathbb{R}^n)$, we have ($f^\infty = 0$)

$$\int_\Omega \int_{\mathbb{R}^n} \Phi(x)\varphi\left(\frac{z}{|z|}\right) \, d\nu_x(z) \, dx = \lim_{n \to +\infty} \langle\langle \nu_{Du_h}, f \rangle\rangle =$$

$$= \lim_{n\to\infty} \sum_{k=0}^{2^{h-1}-1} \left[\int_{\{\frac{2k}{2^h} \le |x| \le \frac{2k+1}{2^h}\}} \Phi(x)\varphi\left(\frac{x}{|x|}\right) \, dx + \int_{\{\frac{2k+1}{2^h} \le |x| \le \frac{2k+2}{2^h}\}} \Phi(x)\varphi\left(-\frac{x}{|x|}\right) \, dx \right]. \tag{14}$$

Now, using polar coordinates, we get

$$\sum_{k=0}^{2^{h-1}-1} \int_{\{\frac{2k}{2^h} \le |x| \le \frac{2k+1}{2^h}\}} \Phi(x)\varphi\left(\frac{x}{|x|}\right) \, dx = \sum_{k=0}^{2^{h-1}-1} \left[\int_{\frac{2k}{2^h}}^{\frac{2k+1}{2^h}} \int_0^{2\pi} \Phi(r,\theta)\varphi(r) \, r \, d\theta \, dr \right]$$

and by the mean value theorem for the function

$$F(r) = \int_0^{2\pi} \Phi(r,\theta)\varphi(r) r \, d\theta$$

(remark that F is continuous with respect to r because of the regularity of Φ and φ) we can find $r_{k,h} \in [\frac{2k}{2^h}, \frac{2k+1}{2^h}]$ such that

$$\sum_{k=1}^{2^{h-1}-1} \left[\int_{\frac{2k}{2^h}}^{\frac{2k+1}{2^h}} \int_0^{2\pi} \Phi(r,\theta)\varphi(r) \, r \, d\theta \, dr \right] = \sum_{k=1}^{2^{h-1}-1} \left[\frac{1}{2^h} F(r_{k,h}) \right] = \frac{1}{2} \sum_{k=1}^{2^{h-1}-1} \left[\frac{1}{2^{h-1}} F(r_{k,h}) \right].$$

Remark that the sum in the right term of the previous equality is a Riemann sum of F over $[0,1]$ with partition $\{\left[\frac{2k}{2^h},\frac{2k+2}{2^h}\right] : k=0,...,2^{h-1}-1\}$ so we get

$$\lim_{h\to+\infty}\sum_{k=1}^{2^{h-1}-1}\int_{\{\frac{2k}{2^h}\le|x|\le\frac{2k+1}{2^h}\}}\Phi(x)\varphi\left(\frac{x}{|x|}\right)dx=\frac{1}{2}\int_0^1 F(r)\,dr=\frac{1}{2}\int_\Omega\Phi(x)\varphi\left(\frac{x}{|x|}\right)dx.$$

With a similar manipulation of the second term in (14) we get

$$\int_\Omega\int_{\mathbb{R}^n}\Phi(x)\varphi\left(\frac{z}{|z|}\right)d\nu_x(z)\,dx=\frac{1}{2}\int_\Omega\Phi(x)\varphi\left(\frac{x}{|x|}\right)dx+\frac{1}{2}\int_\Omega\Phi(x)\varphi\left(-\frac{x}{|x|}\right)dx$$

which implies

$$\nu_x=\frac{1}{2}\delta_{\frac{x}{|x|}}+\frac{1}{2}\delta_{-\frac{x}{|x|}}.$$

Now, if we test the Young convergence for $f(x,z)=\Phi(x)|z|$, where $\Phi\in C(\overline{\Omega})$ we have $(f^\infty=\Phi)$

$$\lim_{h\to+\infty}\langle\langle\nu_{Du_h},f\rangle\rangle=\int_\Omega\Phi\,dx=\langle\langle\nu,f\rangle\rangle$$

and, from the knowledge of ν_x, we have

$$\langle\langle\nu,f\rangle\rangle=\int_\Omega\Phi\,dx+\int_{\overline{\Omega}}\Phi\,d\lambda_\nu.$$

Comparing these two equalities we get

$$\lambda_\nu=0.$$

Example 5.15 (Concentration). Let $n=2$, $\Omega=B(0,2)$ and

$$u_h(x)=\begin{cases}h(|x|-1) & \text{if } |x|\in\left[1,1+\frac{1}{h}\right]\\ h(1-|x|) & \text{if } |x|\in\left[1-\frac{1}{h},1\right]\\ 0 & \text{otherwise}\end{cases}$$

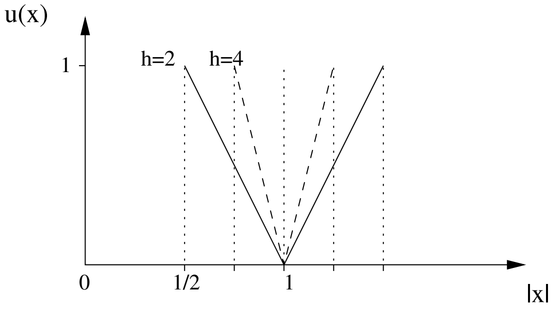

Figure 4: A radial section of the graph of u

Since

$$\nabla u_h(x)=\begin{cases}hx/|x| & \text{if } |x|\in\left[1,1+\frac{1}{h}\right]\\ -hx/|x| & \text{if } |x|\in\left[1-\frac{1}{h},1\right]\\ 0 & \text{otherwise,}\end{cases}$$

$\{u_h\}$ is uniformly bounded in $\mathrm{BV}(\Omega)$ thus, possibly after extracting a subsequence (not relabeled), $\nu_{Du_h} \xrightarrow{\mathbf{Y}} \nu$. Let us prove that ν is defined by

$$\nu_x = \delta_0, \quad \lambda_\nu = 4\mathscr{H}^1 \llcorner \partial B(0,1), \quad \nu_x^\infty = \frac{1}{2}\delta_{\frac{x}{|x|}} + \frac{1}{2}\delta_{-\frac{x}{|x|}}.$$

Since $u_h \to u = 0$ $\mathscr{L}^2$-a.e. and $\nabla u_h \to 0$ $\mathscr{L}^2$-a.e., we get

$$\nu_x = \delta_0.$$

To identify λ_ν we can test the Young convergence using $f(x,z) = |z|$ and, by (13) and the knowledge of ν_x, we get $|Du_h| \overset{*}{\rightharpoonup} \lambda_\nu$ in $\mathscr{M}^+(\overline{\Omega})$. Then for all $\varphi \in C(\overline{\Omega})$ we have

$$\begin{aligned} \int_\Omega \varphi \, d\lambda_\nu &= \lim_{h\to\infty} \left\{ \int_{B(0,1+\frac{1}{h})\setminus B(0,1-\frac{1}{h})} h\varphi(x)\, dx + \int_{\partial B(0,1+1/h)\cup \partial B(0,1-1/h)} \varphi(x)\, d\mathscr{H}^1 \right\} = \\ &= \lim_{h\to\infty} \left\{ h\int_{1-\frac{1}{h}}^{1+\frac{1}{h}} \int_0^{2\pi} \varphi(r,\theta) r\, d\theta\, dr + \int_{\partial B(0,1+1/h)\cup \partial B(0,1-1/h)} \varphi(x)\, d\mathscr{H}^1 \right\}. \end{aligned}$$

Now,

$$\lim_{h\to+\infty} \int_{\partial B(0,1+1/h)\cup \partial B(0,1-1/h)} \varphi(x)\, d\mathscr{H}^1 = 2\int_{\partial B(0,1)} \varphi(x)\, d\mathscr{H}^1.$$

Moreover, applying Lebesgue's Theorem to

$$h\int_{1-\frac{1}{h}}^{1+\frac{1}{h}} F(r)\, dr, \quad F(r) = \int_0^{2\pi} \varphi(r,\theta) r\, d\theta$$

we get

$$\lim_{h\to+\infty} h\int_{1-\frac{1}{h}}^{1+\frac{1}{h}} \int_0^{2\pi} \varphi(r,\theta) r\, d\theta\, dr = 2F(1) = 2\int_{\partial B(0,1)} \varphi(x)\, d\mathscr{H}^1.$$

Therefore

$$\int_\Omega \varphi\, d\lambda_\nu = 4\int_{\partial B(0,1)} \varphi(x)\, d\mathscr{H}^1$$

and

$$\lambda_\nu = 4\mathscr{H}^1 \llcorner \partial B(0,1).$$

Now we can identify ν_x^∞ testing the Young convergence with $f(x,z) = \Phi(x)|z||\varphi(z/|z|)$, where $\Phi \in C(\overline{\Omega})$ and $\varphi \in C(\mathbb{S}^1)$ ($f^\infty = \Phi\varphi$). There holds

$$\begin{aligned} \langle\langle \nu_{Du_h}, f\rangle\rangle = &\int_{B(0,1+1/h)\setminus B(0,1)} h\Phi(x)\varphi\left(\frac{x}{|x|}\right) dx + \int_{B(0,1)\setminus B(0,1-1/h)} h\Phi(x)\varphi\left(-\frac{x}{|x|}\right) dx + \\ &+ \int_{\partial B(0,1+\frac{1}{h})} \Phi(x)\varphi\left(\frac{x}{|x|}\right) d\mathscr{H}^1 + \int_{\partial B(0,1-\frac{1}{h})} \Phi(x)\varphi\left(-\frac{x}{|x|}\right) d\mathscr{H}^1 \end{aligned}$$

and, with the same argument as above and using Lebesgue's Theorem, we get

$$\lim_{h\to+\infty} \langle\langle \nu_{Du_h}, f\rangle\rangle = \int_{\partial B(0,1)} 2\Phi(x)\left(\varphi\left(\frac{x}{|x|}\right) + \varphi\left(-\frac{x}{|x|}\right)\right) d\mathscr{H}^1 = \langle\langle \nu, f\rangle\rangle.$$

Therefore, from the knowledge of ν_x and λ_ν, we have

$$\int_{\mathbb{S}^1} \varphi\, d\nu_x^\infty = \frac{1}{2}\varphi\left(-\frac{x}{|x|}\right) + \frac{1}{2}\varphi\left(\frac{x}{|x|}\right) \quad \forall \varphi \in C(\mathbb{S}^1)$$

which implies

$$\nu_x^\infty = \frac{1}{2}\delta_{\frac{x}{|x|}} + \frac{1}{2}\delta_{-\frac{x}{|x|}}.$$

Example 5.16 (Diffuse concentration). Let $n=2$, $\Omega = B(0,1)$ and

$$u_h(x) = \begin{cases} h\left(|x| - \dfrac{k}{h}\right) & \text{if } |x| \in \left[\dfrac{k}{h}, \dfrac{k}{h} + \dfrac{1}{2h^2}\right] \\ h\left(\dfrac{k}{h} + \dfrac{1}{h^2} - |x|\right) & \text{if } |x| \in \left[\dfrac{k}{h} + \dfrac{1}{2h^2}, \dfrac{k}{h} + \dfrac{1}{h^2}\right] \\ 0 & \text{otherwise} \end{cases}$$

where $k = 0, \dots, h-1$.

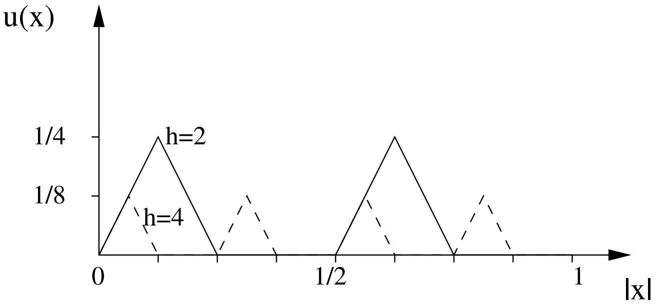

Figure 5: A radial section of the graph of u

Therefore,

$$\nabla u_h(x) = \begin{cases} hx/|x| & \text{if } x \in \left[\dfrac{k}{h}, \dfrac{k}{h} + \dfrac{1}{2h^2}\right] \\ -hx/|x| & \text{if } x \in \left[\dfrac{k}{h} + \dfrac{1}{2h^2}, \dfrac{k}{h} + \dfrac{1}{h^2}\right] \\ 0 & \text{otherwise} \end{cases}$$

Thus $\{u_h\}$ is bounded in $\mathrm{BV}(\Omega)$ so, possibly after the extraction of a subsequence (not relabeled), we have $\nu_{Du_h} \xrightarrow{\mathbf{Y}} \nu$ and we will prove that ν is defined by

$$\nu_x = \delta_0, \quad \lambda_\nu = \mathscr{L}^2 \llcorner \Omega, \quad \nu_x^\infty = \frac{1}{2}\delta_{\frac{x}{|x|}} + \frac{1}{2}\delta_{-\frac{x}{|x|}}.$$

As for the previous example $u_h \to u = 0$ $\mathscr{L}^2$-a.e. and $\nabla u_h \to 0$ $\mathscr{L}^2$-a.e. which implies

$$\nu_x = \delta_0.$$

To identify λ_ν we can test the Young convergence with $f(x,z) = |z|$ so, by (13) and the knowledge of ν_x, we get $|Du_h| \overset{*}{\rightharpoonup} \lambda_\nu$ in $\mathscr{M}^+(\overline{\Omega})$. Then for all $\varphi \in C(\overline{\Omega})$ we have

$$\int_\Omega \varphi \, d\lambda_\nu = \lim_{h\to\infty} \sum_{k=0}^{h-1} \int_{B\left(0, \frac{k}{h} + \frac{1}{h^2}\right) \setminus B\left(0, \frac{k}{h}\right)} h\varphi(x)\, dx = \lim_{h\to\infty} \sum_{k=0}^{h-1} h \int_{\frac{k}{h}}^{\frac{k}{h} + \frac{1}{h^2}} \int_0^{2\pi} r\varphi(r,\theta)\, d\theta\, dr$$

and using the mean value theorem for the function

$$F(r) = \int_0^{2\pi} r\varphi(r,\theta)\, d\theta$$

we can find $r_{k,h} \in \left[\frac{k}{h}, \frac{k}{h} + \frac{1}{h^2}\right]$ such that

$$\int_\Omega \varphi \, d\lambda_\nu = \lim_{h\to\infty} \sum_{k=0}^{h-1} h \frac{F(r_{k,h})}{h^2} = \lim_{h\to\infty} \sum_{k=0}^{h-1} \frac{F(r_{k,h})}{h}.$$

Remark that the sum in the last term of the previous equality is a Riemann sum of F over $[0,1]$ with partition $\{\left[\frac{k}{h},\frac{k+1}{h}\right] : k=0,...,h-1\}$ so we get

$$\int_\Omega \varphi \, d\lambda_\nu = \int_0^1 F(r)\, dr = \int_\Omega \varphi \, dx.$$

It follows that

$$\lambda_\nu = \mathscr{L}^2 \llcorner \Omega.$$

Now we can identify ν_x^∞ testing the Young convergence with $f(x,z) = \Phi(x)|z||\varphi(z/|z|)$, where $\Phi \in C(\overline{\Omega})$ and $\varphi \in C(\mathbb{S}^1)$ ($f^\infty = \Phi\varphi$). So we have

$$\langle\langle \nu_{Du_h}, f\rangle\rangle = \int_\Omega \Phi(x)|\nabla u_h(x)|\varphi\left(\frac{\nabla u_h}{|\nabla u_h|}(x)\right) dx =$$

$$= \sum_{k=0}^{h-1} \int_{B\left(0,\frac{k}{h}+\frac{1}{2h^2}\right)\setminus B\left(0,\frac{k}{h}\right)} h\Phi(x)\varphi\left(\frac{x}{|x|}\right) dx + \sum_{k=0}^{h-1} \int_{B\left(0,\frac{k}{h}+\frac{1}{h^2}\right)\setminus B\left(0,\frac{k}{h}+\frac{1}{2h^2}\right)} h\Phi(x)\varphi\left(-\frac{x}{|x|}\right) dx.$$

By the same arguments used above to identify λ_ν we have

$$\lim_{h\to+\infty} \langle\langle \nu_{Du_h}, f\rangle\rangle = \int_\Omega \Phi(x) \int_{\mathbb{S}^1} \varphi(z)\, d\nu_x^\infty(z)\, dx = \int_\Omega \Phi(x) \frac{1}{2}\left(\varphi\left(\frac{x}{|x|}\right) + \varphi\left(-\frac{x}{|x|}\right)\right) dx$$

and from the knowledge of $\lambda_\nu = \mathscr{L}^2 \llcorner \Omega$ we get

$$\nu_x^\infty = \frac{1}{2}\delta_{\frac{x}{|x|}} + \frac{1}{2}\delta_{-\frac{x}{|x|}}.$$

This example shows that diffusion phenomena can also be generated by sequences converging to zero.

More generally, all previous examples illustrate that the limit gradient Young measure is not determined by the BV limit function but rather by the kind of sequence that generates it.

6 Young varifolds

In this section we define the Young varifolds and we study their basic properties.

Definition 6.1 (Young varifolds) *Let Ω be a bounded Lipschitz domain and let V be a $(n-1)$-varifold. We say that V is a Young varifold if there exists $\nu \in \mathbf{Y}(\Omega,\mathbb{R}^n)$ such that*

$$V(E\times A) = \int_E \int_{\{z\in\mathbb{R}^n : z^\perp \in A\}} |z|\, d\nu_x(z)\, dx + \int_{\overline{E}} \nu_x^\infty(\{z\in\mathbb{S}^{n-1} : z^\perp \in A\})\, d\lambda_\nu$$

for every $E\times A \subseteq G_{n-1}(\Omega)$, where $z^\perp$ is the element of $G(n,n-1)$ perpendicular to the space spanned by z. In the following V is denoted by V_ν and is called varifold associated with ν.

The weight measure is given by

$$\mu_{V_\nu}(E) = V_\nu(E\times G(n,n-1)) = \int_E\int_{\mathbb{R}^n} |z|\, d\nu_x(z)\, dx + \lambda_\nu(\overline{E}) \quad \forall E\subseteq\Omega.$$

We denote by $\mathbf{YV}(\Omega,\mathbb{R}^n)$ the class of Young varifolds (i.e. varifolds associated with Young measures).

Remark that, for every $f \in \mathrm{C_c}(G_{n-1}(\Omega))$ the function $g(x,z) = |z| f\left(x, z^{\perp}\right)$ belongs to $\mathbf{E}(\Omega, \mathbb{R}^n)$. Moreover, as for every $s \in \mathbb{R}$ the linear spaces $(sz)^{\perp}$ and $z^{\perp}$ represent the same element of $G(n, n-1)$, g is continuous and positively 1-homogeneous in z and it is easy to check that $g^{\infty} = f$.

Thus we have

$$\int_{G_{n-1}(\Omega)} f(x,S)\, dV_{\nu}(x,S) = \langle\langle \nu, |z| f(x, z^{\perp}) \rangle\rangle \qquad \forall f \in \mathrm{C_c}(G_{n-1}(\Omega)).$$

Remark also that V_{ν} needs not be a rectifiable varifold since its projection on Ω might be a n-measure whereas the tangent measure lives in $G(n, n-1)$.

The following proposition shows that the convergence of Young measures implies the convergence of the associated Young varifolds:

Proposition 6.2 *If* $\nu_h \xrightarrow{\mathbf{Y}} \nu$ *then* $V_{\nu_h} \overset{*}{\rightharpoonup} V_{\nu}$.

Proof : For all $f \in \mathrm{C_c}(G_{n-1}(\Omega))$ we consider the function $g(x,z) = |z| f(x, z^{\perp}) \in \mathbf{E}(\Omega; \mathbb{R}^n)$ and we use the function g to test the Young convergence. Then if $\nu_h \xrightarrow{\mathbf{Y}} \nu$ we have

$$\int_{G_{n-1}(\Omega)} f\, dV_{\nu_h} = \langle\langle g, \nu_h \rangle\rangle \to \langle\langle g, \nu \rangle\rangle = \int_{G_{n-1}(\Omega)} f\, dV_{\nu} \quad \forall f \in \mathrm{C_c}(G_{n-1}(\Omega))$$

so

$$V_{\nu_h} \overset{*}{\rightharpoonup} V_{\nu}.$$

□

Next proposition provides a sufficient condition for compactness in $\mathbf{YV}(\Omega, \mathbb{R}^n)$.

Proposition 6.3 (Compactness) *Let* $\{V_h\} \subseteq \mathbf{YV}(\Omega, \mathbb{R}^n)$ *be a sequence of Young varifolds such that* $\sup_h \mu_{V_h}(\Omega) < \infty$. *Then, possibly extracting a subsequence,* $V_h \overset{*}{\rightharpoonup} V$ *with* $V \in \mathbf{YV}(\Omega, \mathbb{R}^n)$.

Proof : By definition of Young varifolds, there exists a sequence of Young measures $\{\nu_h\}$ such that $V_h = V_{\nu_h}$ and by the uniform bound on μ_{V_h} we get

$$\sup_h \left[\int_{\Omega} \int_{\mathbb{R}^n} |z|\, d(\nu_h)_x(z)\, dx + \lambda_{\nu_h}(\overline{\Omega}) \right] < \infty.$$

Then, by Theorem 5.4 , there exists a Young measure ν such that $\nu_h \xrightarrow{\mathbf{Y}} \nu$ (possibly extracting a subsequence) and we get

$$\lim_{h\to\infty} \int_{G_{n-1}(\Omega)} f(x,S)\, dV_{\nu_h}(x,S) = \lim_{h\to\infty} \langle\langle g, \nu_h \rangle\rangle = \langle\langle g, \nu \rangle\rangle \qquad \forall f \in \mathrm{C_c}(G_{n-1}(\Omega)) \tag{15}$$

where $g(x,z) = |z| f(x, z^{\perp})$. Then considering the Young varifold associated with ν, (15) proves that $V_h \overset{*}{\rightharpoonup} V_{\nu}$ and the proposition ensues. □

The **first variation of a Young varifold** V_{ν} is given by:

$$\begin{aligned} \delta V_{\nu} : \quad X \in \mathrm{C_c}(\Omega, \mathbb{R}^n) \quad \mapsto \quad & \int_{G_{n-1}(\Omega)} \mathrm{div}_S X(x)\;\, dV_{\nu}(x,S) \\ & = \int_{\Omega} \int_{\mathbb{R}^n} |z| \mathrm{div}_{z^{\perp}} X\;\, d\nu_x(z)\, dx + \int_{\overline{\Omega}} \int_{\mathbb{S}^{n-1}} \mathrm{div}_{z^{\perp}} X\;\, d\nu_x^{\infty}(z)\, d\lambda_{\nu}. \end{aligned}$$

Example 6.4 If $u \in \mathrm{BV}(\Omega)$ then the Young varifold $V_{\nu_{Du}}$ associated with the gradient Young measure ν_{Du} is defined as:

$$\int_{G_{n-1}(\Omega)} f(x,S)\, dV_{\nu_{Du}}(x,S) = \int_\Omega |\nabla u| f\left(x, \nabla u^\perp\right)\, dx + \int_\Omega f\left(x, \frac{D^s u}{|D^s u|}^\perp\right)\, d|D^s u|$$

for all $f \in \mathrm{C_c}(G_{n-1}(\Omega))$. The weight measure is

$$\mu_{V_\nu} = |Du|$$

and the first variation is

$$\delta V_{\nu_{Du}}(X) = \int_\Omega |\nabla u| \mathrm{div}_{\nabla u^\perp} X\, dx + \int_\Omega \mathrm{div}_{\frac{D^s u}{|D^s u|}^\perp} X\, d|D^s u|$$

for all $X \in \mathrm{C_c}(\Omega, \mathbb{R}^n)$.

We can observe that, if $\{u_h\}$ is bounded in $\mathrm{BV}(\Omega)$, then, by Proposition 5.12, there exists a subsequence (not relabeled) such that $\nu_{Du_h} \xrightarrow{\mathbf{Y}} \nu$ thus $V_{\nu_{Du_h}} \overset{*}{\rightharpoonup} V_\nu$. Furthermore, if $u_h \to u$ strictly in BV, Proposition 5.11 implies that $V_{\nu_{Du_h}} \overset{*}{\rightharpoonup} V_{\nu_{Du}}$.

Remark 6.5 (Smooth functions) If $u \in \mathrm{C}^2(\Omega)$, $\Omega \subset \mathbb{R}^n$, then for all $f \in \mathrm{C_c}(G_{n-1}(\Omega))$ we get

$$\int_{G_{n-1}(\Omega)} f(x,z)\, dV_{\nu_{Du}}(x,z) = \int_\Omega f(x, \nabla u^\perp) |\nabla u|\, dx.$$

The weight measure is

$$\mu_{V_{\nu_{Du}}}(A) = |Du|(A) = \int_A |\nabla u|\, dx, \qquad \forall A \subseteq \Omega.$$

Moreover, for all $X \in \mathrm{C_c}(\Omega, \mathbb{R}^n)$ the first variation is

$$\delta V_{\nu_{Du}}(X) = \int_\Omega |\nabla u| \mathrm{div}_{\nabla u^\perp} X\, dx.$$

Since ∇u is regular, the coarea formula yields

$$\delta V_{\nu_{Du}}(X) = \int_\mathbb{R} \int_{\partial\{u>t\}\cap\Omega} \mathrm{div}_{\partial\{u>t\}} X\, d\mathscr{H}^{n-1}\, dt$$

because $\nabla u^\perp(x)$ is the tangent space at x to the isolevel surface $\{y, u(y) = u(x)\}$. Moreover, for a.e. $t \in \mathbb{R}$ the generalized mean curvature of the varifold $V_t = \mathbf{v}(\partial\{u > t\}, 1)$ coincides with the mean curvature vector of the C^2 $(n-1)$-manifold $\partial\{u > t\}$.

Now, a normal unit vector to $\partial\{u > t\}$ at $x \in \partial\{u > t\}$ is $\nabla u(x)/|\nabla u(x)|$ and, denoting by $\mathbf{H}_t(x)$ the mean curvature (in the manifold sense !) of $\partial\{u > t\}$ at x, we get

$$\mathbf{H}_t(x) = -(\mathrm{div} \frac{\nabla u}{|\nabla u|}(x)) \frac{\nabla u}{|\nabla u|}(x).$$

Then, using the representation formula for the first variation of rectifiable varifolds and the coarea formula, it follows that

$$\delta V_{\nu_{Du}}(X) = -\int_\mathbb{R} \int_{\partial\{u>t\}\cap\Omega} \langle X, \mathbf{H}_t\rangle\, d\mathscr{H}^{n-1}\, dt = \int_\Omega |\nabla u| \langle X, (\mathrm{div} \frac{\nabla u}{|\nabla u|}) \frac{\nabla u}{|\nabla u|}\rangle\, dx.$$

Then the mean curvature vector of the varifold $V_{\nu_{Du}}$ is given by

$$\mathbf{H}_{V_{\nu_{Du}}}(x) = -\frac{\delta V_{\nu_{Du}}}{\mu_{V_{\nu_{Du}}}}(x) = \mathbf{H}_t(x) = -(\operatorname{div}\frac{\nabla u}{|\nabla u|})\frac{\nabla u}{|\nabla u|}(x) \quad \mathscr{L}^n - a.e \text{ in } \Omega.$$

Thus, for all $p > 1$,

$$F(u,\Omega) = \int_\Omega |\nabla u| \left(1 + \left|\operatorname{div}\frac{\nabla u}{|\nabla u|}\right|^p\right) dx = \int_\Omega \left[1 + \left|\mathbf{H}_{V_{\nu_{Du}}}(x)\right|^p\right] d\mu_{V_{\nu_{Du}}},$$

This formula shows the relationship, in the case of a regular function, between the generalized Willmore functional and the Young varifold associated with the Young measure ν_{Du} and it motivated our interest for the Willmore functional for Young varifolds studied in the next section.

7 The Willmore functional for Young varifolds

In this section we extend the Willmore functional to Young varifolds. We consider the class

$$\mathbf{GY}(u) = \{\nu \in \mathbf{GY}(\Omega,\mathbb{R}^n) : \xi_\nu \llcorner \Omega = Du\}.$$

Proposition 7.1 *The set* $\mathbf{GY}(u)$ *is weakly* closed* (*as a subset of* $(\mathbf{E}(\Omega,\mathbb{R}^n))^*$).

Proof : Take ν from the weak closure of $\mathbf{GY}(u)$. By Theorem 5.3, $\nu \in \mathbf{Y}(\Omega,\mathbb{R}^n)$ and there exists a sequence $\{\nu_h\} \subset \mathbf{GY}(u)$ such that, for every $h \in \mathbb{N}$ and for every $f \in \mathbf{E}(\Omega,\mathbb{R}^n)$,

$$|\langle\langle \nu_h, f(x,z)\rangle\rangle - \langle\langle \nu, f(x,z)\rangle\rangle| \leq \frac{1}{h} \quad \text{and} \quad |\langle\langle \nu_h, |z|\rangle\rangle - \langle\langle \nu, |z|\rangle\rangle| \leq \frac{1}{h}.$$

Now, for every h $\nu_h \in \mathbf{GY}(u)$, and since $\mathrm{W}^{1,1}$-functions are dense in BV with respect to the strict convergence, it follows from Proposition 5.11 that for every $h \in \mathbb{N}$ there exists $u_h \in \mathrm{W}^{1,1}(\Omega)$ with

$$\|u_h - u\|_{BV} \leq 1/h$$

and such that

$$|\int_\Omega f(x,\nabla u_h)dx - \langle\langle \nu_h, f(x,z)\rangle\rangle| \leq \frac{1}{h}$$

and

$$|\|\nabla u_h\|_{\mathrm{L}^1} - \langle\langle \nu_h, |z|\rangle\rangle| \leq \frac{1}{h}.$$

Being $\{u_h\}$ uniformly bounded in $\mathrm{BV}(\Omega)$, there exists a subsequence (not relabeled) such that $u_h \overset{*}{\rightharpoonup} u$ in BV and $\nu_{Du_h} \xrightarrow{\mathbf{Y}} \mu \in \mathbf{GY}(\Omega,\mathbb{R}^n)$. Now, the two estimates above show that necessarily $\mu = \nu$ and we get $\xi_\nu \llcorner \Omega = \xi_\mu \llcorner \Omega = Du$, hence $\nu \in \mathbf{GY}(u)$. □

We can now define a class of Young varifolds that is suitable in the Willmore context.

Definition 7.2 *The class* $\mathbb{V}(u)$ *of Young varifolds associated with* $\nu \in \mathbf{GY}(u)$ *is defined as*

$$\mathbb{V}(u) = \{V_\nu \in \mathbf{YV}(\Omega,\mathbb{R}^n) : \nu \in \mathbf{GY}(u),\ \|\delta\mu_{V_\nu}\| << \mu_{V_\nu}\}.$$

The following theorem shows a property of the weight measures of Young varifolds that are associated with Young measures belonging to $\mathbf{GY}(u)$.

Proposition 7.3 *For all $V \in \mathbb{V}(u)$ we have $\mu_{V_\nu}(\Omega) \geq |Du|(\Omega)$.*

Proof : For every $\nu \in \mathbf{GY}(u)$, $\xi_\nu = Du \llcorner \Omega$ and, using the Radon-Nikodym decomposition (5),

$$\mu_{V_\nu}(\Omega) = \int_\Omega \int_{\mathbb{R}^n} |z|\, d\nu_x(z)\, dx + \lambda_\nu(\overline{\Omega}) \geq \int_\Omega \left[\int_{\mathbb{R}^n} |z|\, d\nu_x(z) + \frac{\lambda_\nu}{\mathscr{L}^n}(x)\right] dx + \lambda_\nu^s(\Omega)$$

Then, using (8), and reminding that ν_x^∞ is a probability measure, it follows that

$$\begin{aligned} \mu_{V_\nu}(A) \quad &\geq \int_A \Big| \int_{\mathbb{R}^n} z\, d\nu_x + \frac{\lambda_\nu}{\mathscr{L}^n}(x) \int_{\mathbb{S}^{n-1}} z\, d\nu_x^\infty \Big| dx + \int_A \Big| \int_{\mathbb{S}^{n-1}} z\, d\nu_x^\infty \Big| \lambda_\nu^s \\ &\geq \Big| \int_A \int_{\mathbb{R}^n} z\, d\nu_x\, dx + \int_A \int_{\mathbb{S}^{n-1}} z\, d\nu_x^\infty\, d\lambda_\nu \Big| = |Du(A)| \end{aligned}$$

for every $A \subseteq \Omega$. From the definition of total variation (see [3]: Definition 1.4), we get

$$\mu_{V_\nu}(\Omega) \geq |Du|(\Omega).$$

□

Remark that, since every $u \in \mathrm{BV}(\Omega)$ can be approximated by a sequence $\{u_h\} \subset \mathrm{W}^{1,1}(\Omega)$ strictly converging to u in BV, Proposition 5.11 implies that $\nu_{Du} \in \mathbf{GY}(u)$ for all $u \in \mathrm{BV}(\Omega)$.

However $V_{\nu_{Du}} \notin \mathbb{V}(u)$ in general (see Example 7.4) because it depends on the absolute continuity of $\|\delta\mu_{V_{Du}}\|$ with respect to $\mu_{V_{Du}}$.

Example 7.4 (A case where $\mathbf{V}_{\nu_{\mathbf{Du}}} \notin \mathbb{V}(\mathbf{u})$) We consider $E \subset \mathbb{R}^2$ like in Fig. 6, Ω an open set such that $E \subset\subset \Omega$ and $u = \mathbb{1}_E$.

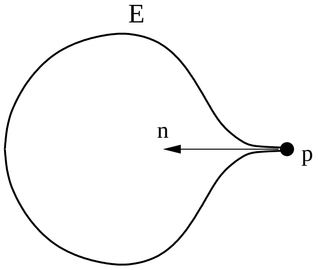

Figure 6: $\delta V_{\nu_{Du}}$ has a singular component

Remark that $V_{\nu_{Du}} = \mathbf{v}(\partial E, 1)$ and $\mu_{V_{\nu_{Du}}} = |D^s u|$. Nevertheless, using the theory of rectifiable varifolds, it is easy to check that $\|\delta V_{\nu_{Du}}\|$ is not absolutely continuous with respect to $\mu_{V_{\nu_{Du}}}$. In particular, denoting by σ the generalized boundary measure of $V_{\nu_{Du}}$, we have

$$\sigma = 2\mathbf{n}\delta_{\mathbf{p}}$$

where $\mathbf{n}$ is the unit vector drawn in Fig. 6.

The mean curvature vector and the Willmore functional associated with a Young varifold are defined as follows

Definition 7.5 (Mean curvature of a Young varifold) *The generalized mean curvature vector of $V \in \mathbb{V}(u)$ is defined as the Radon-Nikodym derivative*

$$\mathbf{H}_V = -\frac{\delta\mu_V}{\mu_V}$$

Definition 7.6 (Willmore energy of a Young varifold) *The Willmore energy of a Young varifold is defined as:*

$$W : \mathbb{V}(u) \longrightarrow \mathbb{R}$$

$$W(V) = \int_{\Omega} (1 + |\mathbf{H}_V|^p) \, d\mu_V \quad p > 1.$$

Remark that in general the class $\mathbb{V}(u)$ is not closed with respect to varifold convergence because, to preserve the condition $\|\delta\mu_V\| << \mu_V$, one needs a uniform bound on (see Theorem 3.2)

$$\int_{\Omega} |\mathbf{H}_V|^p \, d\mu_V.$$

Remark 7.7 (Regular case) If $u \in C^2(\Omega)$ and $F(u,\Omega) < \infty$ then $\nu_{Du} \in \mathbf{GY}(u)$ and from Example 6.5 we have $\|\delta\mu_{V_{\nu_{Du}}}\| << \mu_{V_{\nu_{Du}}}$. Then $V_{\nu_{Du}} \in \mathbb{V}(u)$ and

$$W(V_{\nu_{Du}}) = F(u,\Omega) = \overline{F}(u,\Omega).$$

Moreover, by the coarea formula,

$$\int_{G_{n-1}(\Omega)} f(x,S) \, dV_{\nu_{Du}}(x,S) = \int_{\mathbb{R}} \int_{G_{n-1}(\Omega)} f(x,S) \, d\mathbf{v}(\partial\{u > t\}, 1) dt$$

so, in the regular case, the Young varifold $V_{\nu_{Du}}$ satisfies a slicing formula that involves the unit-density varifolds supported on the boundaries of the level sets of u.

We give now some examples where we can explicitly calculate the Willmore functional for Young varifolds and even show the continuity of the energy for the provided approximating sequences. These examples illustrate that Young varifolds are suitable for catching the limit energy.

Example 7.8 We consider the sequence of Young measures studied in Example 5.14 and we will show that for all $p \in (1,2)$ and for all h we have

$$W(V_{\nu_h}) = W(V_\nu).$$

The limit varifold satisfies

$$\mu_{V_\nu} = \mathscr{L}^2 \llcorner B(0,1)$$

and

$$\begin{aligned} \delta V_{\nu_h}(X) &= \int_{B(0,1)} \operatorname{div}_{\frac{x}{|x|}^\perp} X \, dx = \int_0^1 \int_{\partial B(0,r)} \operatorname{div}_{\frac{x}{|x|}^\perp} X \, d\mathscr{H}^1 \, dr = \\ &= -\int_0^1 \int_{\partial B(0,r)} \langle X, \frac{x}{|x|^2} \rangle \, d\mathscr{H}^1 \, dr = -\int_{B(0,1)} \langle X, \frac{x}{|x|^2} \rangle \, dx \end{aligned}$$

using the coarea formula and the representation of the first variation for 1-rectifiable varifolds. Then we get

$$\mathbf{H}_{V_\nu}(x) = \frac{x}{|x|^2} \qquad \forall x \in B(0,1)$$

and

$$W(V_\nu) = \int_{B(0,1)} \left(1 + \frac{1}{|x|^p}\right) dx.$$

Moreover, for every h,

$$\mu_{V_{\nu_h}} = \mathscr{L}^2 \llcorner B(0,1)$$

and, using the same arguments as above,

$$\mathbf{H}_{V_{\nu_h}}(x) = \frac{x}{|x|^2} \qquad\qquad \forall x \in B(0,1).$$

Thus $W(V_{\nu_h}) = W(V_\nu)$.

Example 7.9 We consider the sequence of Young measures studied in Example 5.15 and we will show that for all $p > 1$

$$W(V_{\nu_h}) \to W(V_\nu).$$

For the limit varifold we have

$$\mu_{V_\nu} = 4\mathscr{H}^1(\partial B(0,1)) = 8\pi$$

and

$$\delta V_\nu(X) = 4\int_{\partial B(0,1)} \operatorname{div}_{\frac{x}{|x|}^\perp} X \, d\mathscr{H}^1 = -4\int_{\partial B(0,1)} \langle X, x\rangle \, d\mathscr{H}^1$$

where we used the representation for the first variation of 1-rectifiable varifolds. Then we have

$$\mathbf{H}_{V_\nu}(x) = x \qquad\qquad \forall x \in \partial B(0,1)$$

and

$$W(V_\nu) = 16\pi.$$

For every V_{ν_h},

$$\mu_{V_{\nu_h}} = h\mathscr{L}^2(\Omega_h) + \mathscr{H}^1(\partial\Omega_h) = 8\pi$$

where $\Omega_h = B\left(0, 1+\frac{1}{h}\right) \setminus B\left(0, 1-\frac{1}{h}\right)$ and

$$\delta V_\nu(X) = \int_{\Omega_h} h\operatorname{div}_{\frac{x}{|x|}^\perp} X \, dx + \int_{\partial B\left(0,1+\frac{1}{h}\right)} \operatorname{div}_{\frac{x}{|x|}^\perp} X \, d\mathscr{H}^1 + \int_{\partial B\left(0,1-\frac{1}{h}\right)} \operatorname{div}_{\frac{x}{|x|}^\perp} X \, d\mathscr{H}^1 =$$

$$= -\int_{\Omega_h} h\langle X, \frac{x}{|x|^2}\rangle \, dx - \int_{\partial B\left(0,1+\frac{1}{h}\right)} \langle X, \frac{x}{|x|}\frac{h}{h+1}\rangle \, d\mathscr{H}^1 - \int_{\partial B\left(0,1-\frac{1}{h}\right)} \langle X, \frac{x}{|x|}\frac{h}{h-1}\rangle \, d\mathscr{H}^1$$

where we used the coarea formula and the properties of 1-rectifiable varifolds. So we have

$$W(V_{\nu_h}) = 8\pi + 2\pi h \int_{1-1/h}^{1+1/h} \frac{1}{r^{p-1}} \, dr + 2\pi\left[\left(\frac{h}{h+1}\right)^{p-1} + \left(\frac{h}{h-1}\right)^{p-1}\right]$$

and by Lebesgue theorem we get

$$W(V_{\nu_h}) \to 8\pi + 4\pi + 4\pi = W(V_\nu).$$

Example 7.10 We finally consider the sequence of Young measures studied in Example 5.16 and we will show that for all $p \in (1,2)$

$$W(V_{\nu_h}) \to W(V_\nu).$$

For the Young varifold associated with ν we have $\mu_{V_\nu} = \mathscr{L}^2 \llcorner \Omega$ and

$$\delta V_\nu(X) = \int_{B(0,1)} \int_{\mathbb{S}^1} \operatorname{div}_{z^\perp} X \, d\nu_x^\infty(z) \, d\lambda_\nu = \int_{B(0,1)} \operatorname{div}_{\frac{x}{|x|}^\perp} X \, dx$$

and, by the coarea formula,

$$\begin{aligned} \delta V_{\nu_h}(X) &= \int_0^1 \int_{\partial B(0,r)} \operatorname{div}_{\frac{x}{|x|}^\perp} X \, d\mathscr{H}^1 \, dr = \\ &= -\int_0^1 \int_{\partial B(0,r)} \langle X, \frac{x}{|x|^2} \rangle \, d\mathscr{H}^1 \, dr = -\int_{B(0,1)} \langle X, \frac{x}{|x|^2} \rangle \, dx. \end{aligned}$$

Then we get

$$\mathbf{H}_{V_\nu}(x) = \frac{x}{|x|^2} \qquad\qquad \forall x \in B(0,1)$$

and

$$W(V_\nu) = \pi + \int_{B(0,1)} \frac{dx}{|x|^p} = \pi + \frac{2\pi}{2-p} = \frac{4-p}{2-p}\pi.$$

Moreover,

$$W(V_{\nu_h}) = \pi \sum_{k=0}^{h-1} \frac{2kh+1}{h^3} + 2\pi \sum_{k=0}^{h-1} h \int_{\frac{k}{h}}^{\frac{k}{h}+\frac{1}{h^2}} \frac{dr}{r^{p-1}} \to \pi + 2\pi \int_0^1 \frac{dr}{r^{p-1}} = \frac{4-p}{2-p}\pi$$

so

$$W(V_{\nu_h}) \to \frac{4-p}{2-p}\pi = W(V_\nu).$$

Remark 7.11 Remark that, in all these examples, we have convergence of the Willmore energy, but the limit varifold is not the varifold associated with Du, i.e. $V_\nu \neq V_{\nu_{Du}}$. In fact the sequence $\{u_h\}$ converges to $u = 0$ weakly* in BV but ν_{Du_h} does not converge (in the sense of Young measures) to the gradient Young measure associated with 0. For instance, in Example 7.10, the sequence of gradients creates some curvature at the limit that is captured by the diffuse part λ_ν.

Moreover, since u is identically 0, its level sets are empty for all positive levels but $\mu_{V_\nu} \neq 0$. This means that the level lines of *u do not provide any* information about the Young measure generated by $\{u_h\}$ so it will not be possible in general to write a slicing formula (like in Remark 7.7) that links the measures belonging to $\mathbf{GY}(u)$ and the level sets of u.

8 Relaxation of the generalized Willmore functional and Young varifolds

This section is devoted to the minimum problem associated with the Willmore functional for Young varifolds and its relationship with the relaxation problem for F.

Theorem 8.1 *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded Lipschitz domain and let $u \in \mathrm{BV}(\Omega)$ with $\overline{F}(u,\Omega) < \infty$. Then $\mathbb{V}(u) \neq \emptyset$, the problem*

$$\mathrm{Min}\,\{W(V) : V \in \mathbb{V}(u)\}$$

has a solution, and

$$\overline{F}(u,\Omega) \geq \mathrm{Min}\{W(V) : V \in \mathbb{V}(u)\}.$$

Proof : We can suppose $W \neq \infty$. Let $\{V_h\} \subset \mathbb{V}(u)$ be a minimizing sequence such that $W(V_h)$ is uniformly bounded, therefore $\{\mu_{V_h}(\Omega)\}$ is uniformly bounded. By Theorem 5.4 and Proposition 7.1, there exists $\nu \in \mathbf{GY}(u)$ such that (possibly extracting a subsequence) $\nu_h \xrightarrow{\mathbf{Y}} \nu$ and, by Proposition 6.2, $V_h \overset{*}{\rightharpoonup} V_\nu$.

Moreover, by Theorem 3.2, $\|\delta V_\nu\| << \mu_{V_\nu}$, $V_\nu \in \mathbb{V}(u)$ and

$$W(V_\nu) \leq \liminf_{n\to\infty} W(V_h)$$

which proves that

$$W(V_\nu) = \mathrm{Min}\,\{W(V) : V \in \mathbb{V}(u)\}.$$

Let $\{u_h\} \subset \mathrm{C}^2(\Omega)$ be a sequence converging to u in $\mathrm{L}^1(\Omega)$ and such that $\overline{F}(u,\Omega) = \lim_{h\to\infty} F(u_h,\Omega)$. We can also suppose that $F(u_h,\Omega)$ is uniformly bounded so, possibly taking a subsequence, $u_h \overset{*}{\rightharpoonup} u$ in $\mathrm{BV}(\Omega)$. Then, by Remark 6.5, the Young varifolds $V_{\nu_{Du_h}}$ associated with the gradient Young measures ν_{Du_h} satisfy

$$F(u_h,\Omega) = W(V_{\nu_{Du_h}}), \quad \forall h.$$

Moreover, by Proposition 5.12 and possibly taking a subsequence, we can assume that $\nu_{Du_h} \xrightarrow{\mathbf{Y}} \tilde{\nu} \in \mathbf{GY}(\Omega,\mathbb{R}^n)$ and hence $V_{\nu_{Du_h}} \overset{*}{\rightharpoonup} V_{\tilde{\nu}}$. Clearly, as $u_h \overset{*}{\rightharpoonup} u$ in $\mathrm{BV}(\Omega)$, we have $\xi_{\tilde{\nu}} \llcorner \Omega = Du$ which implies that $\tilde{\nu} \in \mathbf{GY}(u)$.

Then, we deduce from Theorem 3.2 that $\|\delta V_{\tilde{\nu}}\| << \mu_{V_{\tilde{\nu}}}$ and

$$W(V_{\tilde{\nu}}) \leq \liminf_{h\to\infty} W(V_{\nu_{Du_h}}),$$

Therefore $V_{\tilde{\nu}} \in \mathbb{V}(u)$ and,

$$\overline{F}(u,\Omega) \geq \mathrm{Min}\{W(V) : V \in \mathbb{V}(u)\}.$$

□

The previous theorem states a partial result only. In order to find a full characterization of $\overline{F}$, a preliminary question is the following: can we associate with every BV function u such that $\overline{F}(u) < \infty$ a Young varifold $V \in \mathbb{V}(u)$ such that $\overline{F}(u) = W(V)$? We believe that this can be reasonably conjectured.

Conjecture 8.2 *For every $u \in \mathrm{BV}(\Omega)$ with $\overline{F}(u) < \infty$, there exists $V \in \mathbb{V}(u)$ such that*

$$\overline{F}(u) = W(V).$$

We do not know so far how to prove it in any dimension. We do not know either if it implies the equality in Theorem 8.1. At least we can prove the conjecture in dimension 2, see Theorem 8.10. To this aim, we shall use Theorem 8.4 below that links, in dimension 2, systems of curves belonging to $\mathscr{A}(u)$ and Young varifolds. In particular, given a system of curves, one can define a Young varifold with a coarea structure similar to that in Remark 7.7. The proof uses the following theorem on the locality of curvature for integral 1-varifolds.

Theorem 8.3 ([21], Thm 2.1) *Let $V_1 = \mathbf{v}(M_1,\theta_1)$, $V_2 = \mathbf{v}(M_2,\theta_2)$ be two integral 1-varifolds in $\mathbb{R}^n$ and let $\mathbf{k}_1,\mathbf{k}_2$ be their generalized curvature vectors. Then*

$$\mathbf{k}_1 = \mathbf{k}_2 \;\; \mathscr{H}^1\text{-a.e. on } M_1 \cap M_2.$$

Theorem 8.4 *Let $u \in \mathrm{SBV}(\mathbb{R}^2)$ with compact support (denoted by spt u). Let $\Phi \in \mathscr{A}(u)$ with $G(\Phi) < \infty$ and such that there exists a bounded Lipschitz domain Ω such that*

$$\bigcup_{t\in\mathbb{R}} (\Phi(t)) \subset\subset \Omega. \tag{16}$$

Then, there exists a Young measure $\nu \in \mathbf{GY}(\Omega, \mathbb{R}^2)$ such that $V_\nu \in \mathbb{V}(u)$ and

$$G(\Phi) = W(V_\nu).$$

Proof : We first remark that, since $\Phi(t)$ is a system of curves of class $\mathrm{W}^{2,p}$ for a.e. t, the 1-rectifiable varifold $V_t = \mathbf{v}(\Phi(t), \theta_{\Phi(t)})$ ($\theta_{\Phi(t)}$ is the density of the system $\Phi(t)$, see Definition 2.1) is such that

$$\|\delta V_t\| << \mu_{V_t}$$

(see Remark 4.7) so for every $X \in \mathrm{C_c}(\Omega, \mathbb{R}^2)$ we have

$$\delta V_t(X) = -\int_\Omega \langle X, \mathbf{H}_{V_t}\rangle \, d\mu_{V_t} \tag{17}$$

where $\mathbf{H}_{V_t}$ is the mean curvature of the varifold $\mathbf{v}(\Phi(t), \theta_{\Phi(t)})$.

Let $\Phi \in \mathscr{A}(u)$ and consider the varifold V, supported on $\bigcup_{t\in\mathbb{R}}(\Phi(t)) \times G(2,1)$, defined as

$$V(E\times A) = \int_{\mathbb{R}} \mathbf{v}(\Phi(t), \theta_{\Phi(t)})(E\times A)\, dt\ , \ \ \forall\, E\times A \subset G_1(\Omega) \tag{18}$$

where $\theta_{\Phi(t)}$ is the density of the system $\Phi(t)$.

Then for every $f \in \mathrm{C_c}(G_1(\Omega))$ we get

$$\int_{G_1(\Omega)} f(x,S)\, dV(x,S) = \int_{\mathbb{R}}\int_{\Phi(t)} \theta_{\Phi(t)}(x) f(x, n_t(x)^\perp)\, d\mathscr{H}^1\, dt \tag{19}$$

where $n_t(x)$ denotes a unit normal to the system $\Phi(t)$ at x and $\theta_t(x)$ denotes the density of the system $\Phi(t)$ at x.

By definition of $\mathscr{A}(u)$, for $\mathscr{L}^1$-a.e. t $(\Phi(t)) \supseteq \partial^*\{u>t\}$ (up to a $\mathscr{H}^1$-negligible set) so

$$\begin{aligned}\int_{G_1(\Omega)} f(x,S)\, dV(x,S) &= \int_{\mathbb{R}}\int_{\partial^*\{u>t\}} f(x,n_t(x)^\perp)\, d\mathscr{H}^1 dt + \\ &+ \int_{\mathbb{R}}\int_{\partial^*\{u>t\}} (\theta_{\Phi(t)}(x)-1) f(x,n_t(x)^\perp)\, d\mathscr{H}^1 dt + \int_{\mathbb{R}}\int_{\Phi(t)\setminus\partial^*\{u>t\}} \theta_{\Phi(t)}(x) f(x,n_t(x)^\perp)\, d\mathscr{H}^1 dt.\end{aligned} \tag{20}$$

The coarea formula in BV yields that

$$\int_{\mathbb{R}}\int_{\partial^*\{u>t\}} f(x,n_t(x)^\perp)\, d\mathscr{H}^1 dt = \int_{\mathbb{R}^2} f(x, Du(x)^\perp)\, d|Du|. \tag{21}$$

The decomposition theorem for the derivative of SBV functions implies

$$|Du| = |\nabla u|\mathscr{L}^2 + |D^s u|$$

where ∇u is the approximate gradient of u and $|D^s u| = |u^+ - u^-|\mathscr{H}^1 \llcorner J_u$ where J_u is the set of approximate jump points of u, so

$$\int_{\mathbb{R}^2} f(x,Du(x)^\perp)\, d|Du| = \int_{\mathbb{R}^2} |\nabla u|(x) f\left(x, \nabla u(x)^\perp\right)\, dx + \int_\Omega f\left(x, D^s u(x)^\perp\right)\, d|D^s u|. \tag{22}$$

Moreover we can consider the measure m defined as:

$$m(A) = \int_{\mathbb{R}} \int_{[\Phi(t)\setminus\partial^*\{u>t\}]\cap A} \theta_{\Phi(t)}(x)\, d\mathscr{H}^1\, dt + \int_{\mathbb{R}} \int_{\partial^*\{u>t\}\cap A} (\theta_{\Phi(t)}(x) - 1)\, d\mathscr{H}^1\, dt \tag{23}$$

for every measurable set $A \subset \Omega$ and, from (16), we get

$$m(\partial\Omega) = 0 \tag{24}$$

Thus, by (20), (21), (22) and (23), we can write

$$\begin{aligned} \int_{G_1(\Omega)} f(x,S)\, dV(x,S) = & \int_{\Omega} |\nabla u|(x) f\left(x, \nabla u(x)^{\perp}\right)\, dx + \int_{\Omega} f\left(x, D^s u(x)^{\perp}\right)\, d|D^s u| + \\ & + \int_{\Omega} f(x, n_t(x)^{\perp})\, dm(x). \end{aligned} \tag{25}$$

We deduce from the previous equality that V is a Young varifold. In fact, by (25), we get

$$\int_{G_1(\Omega)} f(x,S)\, dV(x,S) = \langle\langle \nu, |z| f\left(x, z^{\perp}\right) \rangle\rangle \tag{26}$$

where $\nu = (\nu_x, \nu_x^{\infty}, \lambda_\nu)$ is the Young measure defined as:

$$\nu_x = \delta_{\nabla u(x)}, \qquad \lambda_\nu = |D^s u| + m, \qquad \nu_x^{\infty} = \begin{cases} \delta_{\frac{D^s u}{|D^s u|}(x)} & \text{if } x \in J_u \\ \dfrac{1}{2}\left(\delta_{n_t(x)} + \delta_{-n_t(x)}\right) & \text{if } x \in \operatorname{spt} m \end{cases} \tag{27}$$

where $n_t(x)$ is a unit normal vector to the system $\Phi(t)$ at x. Remark that, as different curves belonging either to the same system or to different systems may intersect only tangentially, the previous measure is well defined.

By (16) and (24) we have $\lambda_\nu(\partial\Omega) = 0$ and in addition

$$\int_{\Omega} \int_{\mathbb{R}^n} |z|\, d\nu_x(z)\, dx + \lambda_\nu(\overline{\Omega}) = \int_{\mathbb{R}} \int_{\Phi(t)} \theta_{\Phi(t)}\, d\mathscr{H}^1\, dt \le G(\Phi) < \infty$$

so, using Theorem 5.13, we get $\nu \in \mathbf{GY}(\Omega, \mathbb{R}^2)$. In addition it is easy to check that $\xi_\nu = Du$ so $\nu \in \mathbf{GY}(u)$. To end the proof we have to show that $V = V_\nu \in \mathbb{V}(u)$ and

$$W(V) = \int_{\mathbb{R}} W(\Phi(t))\, dt = G(\Phi).$$

For every measurable set A,

$$\mu_V(A) = \int_{\mathbb{R}} \int_{\Phi(t)\cap A} \theta_{\Phi(t)}\, d\mathscr{H}^1\, dt \tag{28}$$

and for every $X \in \mathrm{C}_0^1(\Omega, \mathbb{R}^2)$, by (19) and (17), we get

$$\begin{aligned} \delta V(X) &= \int_{\mathbb{R}} \int_{\Phi(t)} \theta_{\Phi(t)} \operatorname{div}_{\Phi(t)} X\, d\mathscr{H}^1\, dt = \\ &= -\int_{\mathbb{R}} \int_{\Phi(t)} \theta_{\Phi(t)} \langle X, \mathbf{H}_{V_t} \rangle\, d\mathscr{H}^1\, dt = -\int_{\Omega} \langle X, \mathbf{H}_{V_t} \rangle\, d\mu_V \end{aligned}$$

where, by the locality of the curvature (see Theorem 8.3), $\mathbf{H}_V(x)$ coincides with the mean curvature of the varifold $V_t = \mathbf{v}(\Phi(t), \theta_{\Phi(t)})(x)$ for every $x \in (\Phi(t))$.

Then

$$\mathbf{H}_V(x) = \mathbf{H}_{V_t}(x) \quad \mu_V - a.e. \tag{29}$$

where t is such that $x \in (\Phi(t))$. Thus $\|\delta V\| << \mu_V$ so $V = V_\nu \in \mathbb{V}(u)$.

Moreover, by (28), (29), and the coarea formula, we obtain

$$\begin{aligned} G(\Phi) &= \int_{\mathbb{R}} W(\Phi(t))\, dt = \int_{\mathbb{R}} \int_{\Phi(t)} \theta_{\Phi(t)}\, d\mathscr{H}^1\, dt + \int_{\mathbb{R}} \int_{\Phi(t)} \theta_{\Phi(t)} |\mathbf{H}_{V_t}|^p\, d\mathscr{H}^1\, dt = \\ &= \mu_V(\Omega) + \int_{\Omega} |\mathbf{H}_V|^p\, d\mu_V = W(V). \end{aligned}$$

□

The following subclass of gradient Young measures can be defined in $\mathbb{R}^n$.

Definition 8.5 *Given $u \in \mathrm{BV}(\Omega)$, $\Omega \subset \mathbb{R}^n$, we denote by $\nu_{Du} + \mathbf{GY}(0)$ the class of gradient Young measures for which there exists $\tilde{\nu} \in \mathbf{GY}(0)$ such that*

$$\langle\langle \nu, f \rangle\rangle = \langle\langle \nu_{Du}, f \rangle\rangle + \langle\langle \tilde{\nu}, f \rangle\rangle \quad \forall f \in \mathbf{E}(\Omega; \mathbb{R}^2).$$

It is easy to check that $\nu_{Du} + \mathbf{GY}(0) \subseteq \mathbf{GY}(u)$. Remark that if $\nu \in \nu_{Du} + \mathbf{GY}(0)$ then

$$\nu = \nu_{Du} + \tilde{\nu}\,, \qquad \tilde{\nu} \in \mathbf{GY}(0)$$

so

$$V_\nu = V_{\nu_{Du}} + V_{\tilde{\nu}}\,, \quad \text{with} \quad \tilde{\nu} \in \mathbf{GY}(0).$$

Proposition 8.6 *The Young measure ν defined in (27) belongs to $\nu_{Du} + \mathbf{GY}(0)$.*

Proof : For every $f \in \mathbf{E}(\Omega; \mathbb{R}^2)$ we have

$$\begin{aligned} \langle\langle \nu, f \rangle\rangle = \int_{\Omega} f(x, \nabla u(x))\, dx + \int_{\Omega} f^\infty\left(x, \frac{D^s u}{|D^s u|}\right) d|D^s u| + \int_{\Omega} f^\infty(x, n(x))\, dm = \\ = \langle\langle \nu_{Du}, f \rangle\rangle + \langle\langle \tilde{\nu}, f \rangle\rangle \end{aligned}$$

where $\tilde{\nu}$ is defined by the following triplet:

$$\tilde{\nu}_x = \delta_0, \qquad \lambda_{\tilde{\nu}} = m, \qquad \tilde{\nu}_x^\infty = \frac{1}{2}\left(\delta_{n(x)} + \delta_{-n(x)}\right). \tag{30}$$

Then $\nu = \nu_{Du} + \tilde{\nu}$ and it is sufficient to prove that $\tilde{\nu} \in \mathbf{GY}(0)$. Now, $m(\Omega) < \infty$ and $m(\partial\Omega) = 0$ so, by Theorem 5.13, $\tilde{\nu} \in \mathbf{GY}(\Omega, \mathbb{R}^2)$. Moreover $\xi_{\tilde{\nu}} = 0$ thus $\tilde{\nu} \in \mathbf{GY}(0)$. □

We now define a useful subclass of $\mathbb{V}(u)$.

Definition 8.7 *We denote by $\mathbb{V}_0(u)$ the class of Young varifolds $V_\nu \in \mathbb{V}(u)$ such that $\nu \in \nu_{Du} + \mathbf{GY}(0)$. Thus for every $V \in \mathbb{V}_0(u)$ we have:*

$$V = V_{\nu_{Du}} + V_{\tilde{\nu}}\,, \quad \textit{with} \quad \tilde{\nu} \in \mathbf{GY}(0).$$

Remark 8.8 The definition of $\mathbb{V}_0(u)$ is motivated by the relationship between a general $\nu \in \mathbf{GY}(u)$ and ν_{Du}. Remark 6.5 shows that ν_{Du} represents $\overline{F}(u)$ for every $u \in \mathrm{C}^2(\Omega)$ and we have

$$\overline{F}(u) = W(\nu_{Du}).$$

However, Remark 7.4 implies that we cannot expect such a relation for every $u \in \mathrm{BV}(\Omega)$ because in general $V_{\nu_{Du}} \notin \mathbb{V}(u)$. Thus, a natural question is the following: given $u \in \mathrm{BV}(\Omega)$, how different are the measures $\nu \in \mathbf{GY}(u)$ and ν_{Du}?

This leads to characterizing the measures belonging to $\mathbb{V}_0(u)$ in order to estimate $\tilde{\nu}$. Several questions arise naturally. Do $\mathbb{V}_0(u)$ and $\mathbb{V}(u)$ coincide? Does the solution of the minimum problem in Proposition 8.1 live in $\mathbb{V}_0(u)$? This would mean that, in order to reach the minimum of W, one should take an "economic" $\tilde{\nu}$, which is very expectable. But, so far, all these questions remain open.

In the following, the set $\mathbb{V}_0(u)$ shall be used only in the proof of Theorem 8.10.

Theorem 8.9 *Let $u \in \mathrm{BV}(\Omega)$, $\Omega \subset \mathbb{R}^n$, such that $\overline{F}(u,\Omega) < \infty$. The problem*

$$\mathrm{Min}\,\{W(V) : V \in \mathbb{V}_0(u)\}$$

has at least one solution.

Proof : We can suppose $W \neq \infty$. Let $\{V_h\} \subset \mathbb{V}_0(u)$ be a minimizing sequence. By definition of $\mathbb{V}_0(u)$ we can take a sequence $\{\tilde{\nu}_h\} \subset \mathbf{GY}(0)$ such that

$$V_h = V_{\nu_{Du}} + V_{\tilde{\nu}_h} \quad \forall h.$$

We can suppose $W(V_h)$ uniformly bounded so we get

$$\sup_h \mu_{V_{\tilde{\nu}_h}}(\Omega) < \infty.$$

Then, by Theorems 5.4 and 7.1, we can take a subsequence (not relabeled) of $\{\tilde{\nu}_h\}$ and a Young measure $\tilde{\nu} \in \mathbf{GY}(0)$ such that

$$\tilde{\nu}_h \xrightarrow{\mathbf{Y}} \tilde{\nu}.$$

Then we get

$$\nu_h = \nu_{Du} + \tilde{\nu}_h \xrightarrow{\mathbf{Y}} \nu = \nu_{Du} + \tilde{\nu}$$

and

$$V_h \overset{*}{\rightharpoonup} V_\nu$$

where $V_\nu = V_{\nu_{Du}} + V_{\tilde{\nu}}$ is the Young varifold associated with the gradient Young measure $\nu = \nu_{Du} + \tilde{\nu}$, $\tilde{\nu} \in \mathbf{GY}(0)$.

In addition, by Theorem 3.2, we have $\|\delta V_\nu\| << \mu_{V_\nu}$ and

$$W(V_\nu) \leq \liminf_{n\to\infty} W(V_h).$$

Then $V_\nu \in \mathbb{V}_0(u)$ and

$$W(V_\nu) = \mathrm{Min}\,\{W(V) : V \in \mathbb{V}_0(u)\}$$

and the theorem ensues. □

The next theorem points out, in dimension 2, the relationship between Young varifolds and the relaxation problem on a bounded domain. This gives in particular a positive answer in dimension 2 to Conjecture 8.2.

Theorem 8.10 *Let $u \in \mathrm{SBV}(\mathbb{R}^2)$ with compact support and such that $\overline{F}(u,\mathbb{R}^2) < \infty$. Then there exists a bounded open domain Ω and a Young measure $\nu \in \mathbf{GY}(\Omega,\mathbb{R}^2)$ such that $V_\nu \in \mathbb{V}(u)$ and*

$$\overline{F}(u,\mathbb{R}^2) = \inf\left\{ \liminf_{h\to\infty} F(u_h,\Omega) : \{u_h\} \in \mathrm{C}^2_{\mathrm{c}}(\Omega),\ u_h \xrightarrow{\mathrm{L}^1(\Omega)} u \right\} = W(V_\nu).$$

Moreover $V_\nu \in \mathbb{V}_0(u)$ and

$$\mathrm{Min}\{W(V) : V \in \mathbb{V}_0(u)\} \leq \overline{F}(u,\mathbb{R}^2).$$

Proof : By Theorem 2.7 there exists $\Phi \in \mathscr{A}(u)$ such that $\overline{F}(u,\mathbb{R}^2) = G(\Phi)$ and, by Proposition 2.8, there exists an open bounded domain Ω with

$$\bigcup_{t\in\mathbb{R}} (\Phi(t)) \subset\subset \Omega$$

and such that $u \in \mathrm{BV}(\Omega)$ and

$$\inf\left\{ \liminf_{h\to\infty} F(u_h,\Omega) : \{u_h\} \in \mathrm{C}^2_{\mathrm{c}}(\Omega),\ u_h \overset{\mathrm{L}^1(\Omega)}{\longrightarrow} u \right\} = \overline{F}(u,\mathbb{R}^2) = G(\Phi)$$

.

Then, using Theorem 8.4, we can define a suitable Young measure $\nu \in \mathbf{GY}(\Omega,\mathbb{R}^2)$ such that $V_\nu \in \mathbb{V}(u)$ and

$$\inf\left\{ \liminf_{h\to\infty} F(u_h,\Omega) : \{u_h\} \in \mathrm{C}^2_{\mathrm{c}}(\Omega),\ u_h \overset{\mathrm{L}^1(\Omega)}{\longrightarrow} u \right\} = G(\Phi) = W(V_\nu).$$

Moreover, by Proposition 8.6, $V_\nu \in \mathbb{V}_0(u)$ and, by Theorem 8.9, we have

$$\mathrm{Min}\{W(V) : V \in \mathbb{V}_0(u)\} \le \inf\left\{ \liminf_{h\to\infty} F(u_h,\Omega) : \{u_h\} \in \mathrm{C}^2_{\mathrm{c}}(\Omega),\ u_h \overset{\mathrm{L}^1(\Omega)}{\longrightarrow} u \right\}.$$

□

Remark 8.11 In the previous theorem, the quantity

$$\inf\left\{ \liminf_{h\to\infty} F(u_h,\Omega) : \{u_h\} \in \mathrm{C}^2_{\mathrm{c}}(\Omega),\ u_h \overset{\mathrm{L}^1(\Omega)}{\longrightarrow} u \right\}$$

corresponds to the relaxation of F using approximating functions in $\mathrm{C}^2_{\mathrm{c}}$ instead of C^2. This definition is well posed but induces different properties for the relaxed functional, as discussed in [23] where some examples are also provided. We shall adopt this definition over this remark to show the link between Young varifolds, Young measures and the Willmore functional.

In the regular planar case (i.e. $u \in \mathrm{C}^2_{\mathrm{c}}(\Omega)$, $\Omega \subset \mathbb{R}^2$) there are two natural frameworks to represent $\overline{F}$:

- by a coarea-type formula, using the representation by systems of curves of class $\mathrm{W}^{2,p}$:

$$\overline{F}(u,\Omega) = G(\Phi[u]) \qquad \forall u \in \mathrm{C}^2_{\mathrm{c}}(\Omega);$$

- by the varifold theory (see Remark 6.5), using Young varifolds:

$$\overline{F}(u,\Omega) = W(V_{\nu_{Du}}) \qquad \forall u \in \mathrm{C}^2_{\mathrm{c}}(\Omega).$$

Theorem 8.10 states that, as in the regular case, Young varifolds provide a natural framework to represent $\overline{F}$, at least in dimension 2. Moreover, the definition of the Young measure ν in (27) shows the relationship between the Young varifold representing $\overline{F}(u,\Omega)$ and $V_{\nu_{Du}}$: the Young varifold involves an additional term that contains in particular all ”ghost” parts, as in Figure 2.

9 Conclusion

We have introduced in this paper a new framework to address the relaxation of a generalized Willmore functional. We believe that this combination of Young measures and varifolds is the right approach to track, in the limit of oscillations and concentration, the behavior of the energy. We were able to prove a representation result in dimension 2 (Theorem 8.10), but so far we can only relate through an inequality the relaxed Willmore energy with a minimum problem involving Young varifolds (Theorem 8.1).

There are several obstacles to get a full understanding of the problem:

- The class of Young varifolds associated with a given function is very rich. This is due to the fact that there are infinitely many Young measures that are the limits of sequences of gradient Young measures associated with smooth functions u_h that converge weakly-* to 0 in BV. It is reasonable to think, however, that minimizing W in $\mathbb{V}(u)$ reduces considerably the measures of interest.
- The proof of the representation result in dimension 2 (Theorem 8.10) strongly relies on Theorem 2.7 that involves a curve stretching technique. This technique can hardly be generalized to higher dimensions, in particular because the description of the boundaries of sets with finite relaxed energy is still an open problem. Another strategy, that is totally unexplored so far, requires understanding how a limit Young varifold can be regularized with a control of the energy, using in particular the direction of concentration indicated by ν_x^∞.

Among other problems that could be tackled with Young varifolds, the correct definition of a flow associated with the generalized Willmore functional could be a way to have a new look at the critical points of the Willmore functional. We believe that the versatility of Young varifolds makes them delicate but powerful tools.

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References

[1] J.J. Alibert and G. Bouchitté. Non-uniform integrability and generalized Young measures. *J. Convex Anal.*, 4:129–147, 1997.

[2] W.K. Allard. On the first variation of a varifold. *Ann. of Math.*, 95:417–491, 1972.

[3] L. Ambrosio, N. Fusco, and D. Pallara. Functions of bounded variation and free discontinuity problems. *Oxford Science Publications*, 2000.

[4] L. Ambrosio and S. Masnou. A direct variational approach to a problem arising in image reconstruction. *Interfaces and Free Boundaries*, 5:63–81, 2003.

[5] C. Ballester, M. Bertalmio, V. Caselles, G. Sapiro, and J. Verdera. Filling-in by joint interpolation of vector fields and gray levels. *IEEE Trans. On Image Processing*, 10(8):1200–1211, 2001.

[6] G. Bellettini, Novaga M., and Orlandi G. Lorentzian varifolds and applications to closed relativistic strings. Submitted, 2011.

[7] G. Bellettini, G. Dal Maso, and M. Paolini. Semicontinuity and relaxation properties of curvature depending functional in 2D. *Annali della Scuola Normale di Pisa, Classe di Scienze,* 4^e *série*, 20(2):247–297, 1993.

[8] G. Bellettini and L. Mugnai. Characterization and representation of the lower semicontinuous envelope of the elastica functional. *Ann. Inst. H. Poincaré, Anal. non Linéaire*, 21(6):839–880, 2004.

[9] G. Bellettini and L. Mugnai. A varifold representation of the relaxed elastica functional. *Journal of Convex Analysis*, 14(3):543–564, 2007.

[10] J. A. Boon, Christopher J Budd, and G. W. Hunt. Level set methods for the displacement of layered materials. *Proceedings of the Royal Society of London Series A - Mathematical Physical and Engineering Sciences*, 463(2082):1447–1466, 2007.

[11] C. Castaing, P.R. de Fitte, and M. Valadier. Young measures on topological spaces: With application in control theory and probability theory. *Mathematics and its Applications*, Kluwer, Dordrecht, 2004.

[12] T.F. Chan, S.H. Kang, and J. Shen. Euler's elastica and curvature based inpainting. *SIAM Journal of Applied Math.*, 63(2):564–592, 2002.

[13] G. Citti and A. Sarti. A cortical based model of perceptual completion in the roto-translation space. *J. Math. Imaging Vis.*, 24(3):307–326, 2006.

[14] R.J. DiPerna and A.J. Majda. Oscillations and concentrations in weak solutions of the incompressible fluid equations. *Commun. Math. Phys.*, 108:667–689, 1987.

[15] L.C. Evans and R.F. Gariepy. *Measure Theory and Fine Properties of Functions*. Studies in Advanced Math. C.R.C. Press, 1992.

[16] J. E. Hutch1inson. Multiple function regularity and tangent cone behaviour for varifold with second fundamental form in lp$C^{1,\alpha}$. *in: Geometricmeasure theory and the calculus of variations,W. K. Allard e F. J. Almgren ed., Proc. Symp. Pure Math.*, 44 Am.Math. Soc.:281–306, 1986.

[17] J. E. Hutchinson. Second fondamental form for varifold and the existence of surfaces minimising curvature. *Indiana Univ.Math. Jou.*, 35:45–71, 1986.

[18] D. Kinderlehrer and P. Pedregal. Characterizations of Young measures generated by gradients. *Arch. Rational Mech. Anal.*, 115(4):329–365, 1991.

[19] D. Kinderlehrer and P. Pedregal. Characterizations of Young measures generated by sequences in sobolev spaces. *J. Geom. Anal.*, 4:59–90, 1994.

[20] J. Kristensen and F. Rindler. Characterization of generalised gradient Young measures by sequences $W^{1,1}$ and BV. *Arch. Ration. Mech. Anal.*, 197:539–598, 2010.

[21] G.P. Leonardi and S. Masnou. Locality of the mean curvature of rectifiable varifolds. *Adv. Calc. of Var.*, 2(1):17–42, 2009.

[22] S. Masnou and J.-M. Morel. Level lines based disocclusion. In *5th IEEE Int. Conf. on Image Processing, Chicago, Illinois, October 4-7*, 1998.

[23] S. Masnou and J.M. Morel. On a variational theory of image amodal completion. *Rendiconti del Seminario Matematico della Università di Padova*, 116:211–252, 2006.

[24] S. Masnou and G. Nardi. A coarea-type formula for the relaxation of a generalized elastica functional. *submitted*, 2011.

[25] G. Dal Maso. An introduction to Γ-convergence. *Progress in Nonlinear Differential Equations and their Applications*, 8, 1993.

[26] U. Menne. Second order rectifiability of integral varifolds of locally bounded first variation. *arXiv:0808.3665v3 [math.DG].*, 2010.

[27] P. Pedregal. Parametrized measures and variational principles. *Progress in Non Linear Differential Analysis and Their Applications*, Vol. 30, 1997.

[28] L. Simon. Lectures on geometric measure theory. *Proceedings of the Centre for mathematical analysis, Australian National University*, 3, 1983.

[29] M. Valadier. A course on Young measures. *Workshop on Measure Theory and Real Analysis*, Grado, Prepublication 8-1994, Montpellier, 1993.

[30] L.C Young. Generalized curves and the existence of an attained absolute minimum in the calculus of variations. *C.R. Soc. Sci. Lett. Varsovie, CL III*, 30:212–234, 1937.

[31] L.C Young. Generalized surfaces in the calculus of variations. *Ann. Math.*, 43:84–103, 1942.

[32] L.C Young. Generalized surfaces in the calculus of variations II. *Ann. Math.*, 43:530–544, 1942.