

Locality of the mean curvature of rectifiable varifolds

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Abstract. The aim of this paper is to investigate whether, given two rectifiable k -varifolds in \mathbb{R}^n with locally bounded first variations and integer-valued multiplicities, their mean curvatures coincide \mathcal{H}^k -almost everywhere on the intersection of the supports of their weight measures. This so-called *locality property*, which is well-known for classical C^2 surfaces, is far from being obvious in the context of varifolds. We prove that the locality property holds true for integral 1-varifolds, while for k -varifolds, $k > 1$, we are able to prove that it is verified under some additional assumptions (local inclusion of the supports and locally constant multiplicities on their intersection). We also discuss a couple of applications in elasticity and computer vision.

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Introduction

Let M be a k -dimensional rectifiable subset of \mathbb{R}^n , θ a positive function which is locally summable with respect to $\mathcal{H}^k \llcorner M$, and $T_x M$ the tangent space at \mathcal{H}^k -almost every $x \in M$. The Radon measure $V = \theta \mathcal{H}^k \llcorner M \otimes \delta_{T_x M}$ on the product space

$$G_k(\mathbb{R}^n) = \mathbb{R}^n \times \{k\text{-dim. subspaces of } \mathbb{R}^n\}$$

is an example of a rectifiable k -varifold.

Varifolds can be loosely described as generalized surfaces endowed with multiplicity (θ in the example above) and were initially considered by F. Almgren [2] and W. Allard [1] for studying critical points of the area functional.

Unlike currents, they do not carry information on the positive or negative orientation of tangent planes, hence cancellation phenomena typically occurring with currents do not arise in this context. A weak (variational) concept of mean curvature naturally stems from the definition of the *first variation* δV of a varifold V , which represents, as in the smooth case, the initial rate of change of the area with respect to smooth perturbations. This explains why it is often natural, as well as useful, to represent minimizers of area-type functionals as varifolds.

One of the main difficulties when dealing with varifolds is the lack of a boundary operator like the distributional one acting on currents. In several situations, one can circumvent this problem by considering varifolds that are associated to currents, or that are limits (in the sense of varifolds) of sequences of currents (see [11, 17, 18]).

This paper focuses on varifolds with locally bounded first variation. In this setting, the mean curvature vector H_V of a varifold V is defined as the Radon-Nikodým derivative of the first variation δV (which can be seen as a vector-valued Radon measure) with respect to the weight measure $\|V\|$ (see Section 1 for the precise definitions). In the smooth case, i.e. when V represents a smooth k -surface S and θ is constant, H_V coincides with the classical mean curvature vector defined on S .

However, it is not clear at all whether this generalized mean curvature satisfies the same basic properties of the classical one. In particular, it is well-known that if two smooth, k -dimensional surfaces have an intersection with positive \mathcal{H}^k measure, then their mean curvatures coincide \mathcal{H}^k -almost everywhere on that intersection. Thus it is reasonable to expect that the same property holds for two integral k -varifolds having a non-negligible intersection. The importance of assuming that the varifolds are integral (i.e., with integer-valued multiplicities) is clear, as one can build easy examples of varifolds with smoothly varying multiplicities, such that the corresponding mean curvatures are not even orthogonal to the tangent planes (see also the orthogonality result for the mean curvature of integral varifolds obtained by K. Brakke [9]).

This *locality property* of the generalized mean curvature is, however, far from being obvious, since varifolds, even the rectifiable ones, need not be regular at all. A famous example due to K. Brakke [9] consists of a varifold with integer-valued multiplicity and bounded mean curvature, that cannot even locally be represented as a union of graphs.

Previous contributions to the locality problem are the papers [4] and [18]. In [4], the locality is proved for integral $(n - 1)$ -varifolds in \mathbb{R}^n with mean curvature in L^p , where $p > n - 1$ and $p \geq 2$. The result is strongly based on a quadratic tilt-excess decay lemma due to R. Schätzle [17]. Taking two varifolds that locally coincide and whose mean curvatures satisfy the integrability condition above, the locality property is proved in [4] via the following steps:

- (i) calculate the difference between the two mean curvatures in terms of the local behavior of the tangent spaces;
- (ii) remove all points where both varifolds have same tangent space;
- (iii) finally, show that the rest goes to zero in density, thanks to the decay lemma [17].

The limitation to the case of varifolds of codimension 1, whose mean curvature is in L^p with $p > n - 1$, $p \geq 2$, is not inherent to the locality problem itself, but rather to the techniques used in R. Schätzle's paper [17] for proving the decay lemma.

A major improvement has been obtained by R. Schätzle himself in [18]. Indeed, he shows that, in any dimension and codimension, and assuming only the L^2_{loc} summability of the mean curvature, the quadratic decays of both tilt-excess and height-excess are equivalent to the C^2 -rectifiability of the varifold. Consequently, the locality property is shown to hold for C^2 -rectifiable k -varifolds in \mathbb{R}^n with mean curvature in L^2 , as stated in Corollary 4.2 in [18]: *let V_1, V_2 be integral k -varifolds in $U \subset \mathbb{R}^n$ open, with $H_{V_i} \in L^2_{\text{loc}}(\|V_i\|)$ for $i = 1, 2$. If the intersection of the supports of the varifolds is C^2 -rectifiable, then $H_{V_1} = H_{V_2}$ for \mathcal{H}^k -almost every point of the intersection.*

A careful inspection of the proof of the locality property in [4] and [18] shows the necessity of controlling only those parts of the varifolds that do not contribute to the weight density, but possibly to the curvature. However, the tilt-excess decay provides a local control of the variation of tangent planes on the whole varifold, which seems to be slightly more than what is actually needed for the locality to hold. This observation has led us to tackle this problem by means of different techniques, in order to weaken the requirement on the integrability of the mean curvature down to L^1_{loc} . Our main results in this direction are:

- (i) in the case of two integral 1-varifolds (in any codimension) with locally bounded first variations, we prove that the two generalized curvature vectors coincide \mathcal{H}^1 -almost everywhere on the intersection of the supports (Theorem 2.1);
- (ii) in the general case of rectifiable k -varifolds, $k > 1$, we prove that if $V_1 = \mathbf{v}(M_1, \theta_1)$, $V_2 = \mathbf{v}(M_2, \theta_2)$ are two rectifiable k -varifolds with locally bounded first variations, and if there exists an open set A such that $M_1 \cap A \subset M_2$ and both θ_1, θ_2 are constant on $M_1 \cap A$, then the two generalized mean curvatures coincide \mathcal{H}^k -almost everywhere on $M_1 \cap A$ (Theorem 3.4).

The strategy of proof consists of writing the total variation in a ball B in terms of a $(k - 1)$ -dimensional integral over the sphere ∂B and showing that this integral can be well controlled, at least for a suitable sequence of nested spheres whose radii decrease toward zero.

The 1-dimensional result is somehow optimal, as the only required hypothesis is the local boundedness of the first variation. Under this minimal assumption, we can prove that there exists a sequence of nested spheres that meet only the intersection of the two varifolds, i.e. essentially the part that counts for the weight density. In other words, the parts of the varifolds that do not contribute to the weight density do not either intersect these spheres. This is a key argument to prove that the curvature is essentially not altered by the presence of these "bad" parts.

In the general k -dimensional case it is no more possible to prove the existence of nested spheres that do not intersect at all the bad parts. But we are able to prove, under the extra assumptions cited above, that the integral over the $(k - 1)$ -dimensional sections of the bad parts with a suitable sequence of spheres is so small, that it does not contribute to the mean curvature, and thus the locality holds true in this case.

The plan of the paper is as follows: in Section 1 we recall basic notations and main facts about varifolds. Section 2 is devoted to the proof of the locality property for integral 1-varifolds in \mathbb{R}^n with locally bounded first variation (Theorem 2.1), whose immediate consequence is the fact that for any such varifold, the generalized curvature $\kappa(x)$ coincides with the classical curvature of any C^2 curve that intersects the support of the varifold, for \mathcal{H}^1 -almost all x in the intersection (Corollary 2.2). We also provide an example of a 1-varifold in \mathbb{R}^2 whose generalized curvature belongs to $L^1 \setminus \bigcup_{p>1} L^p$. In Section 3 we first derive two useful, local forms of the isoperimetric inequality for varifolds due to W.K. Allard [1]. Then, we prove that for almost every $r > 0$, the integral of the mean curvature vector in B_r coincides with the integral of a *conormal* vector field along the sphere ∂B_r , up to an error due to the singular part of the first variation. These preliminary results are then combined to show that an *improved decay* of the $(n-1)$ -weight of the “bad” parts contained in ∂B_r holds true, at least for a suitable sequence of radii $(r_h)_h$ converging to 0. This decay argument is the core of the proof of our locality result for k -varifolds in \mathbb{R}^n (Theorem 3.4).

Finally, we discuss in Section 4 some applications of the locality property for varifolds, in particular to lower semicontinuity results for the Euler’s elastica energy and for Willmore-type functionals that appear in elasticity and in computer vision.

Note to the reader: the preprint version of this paper contains an appendix where we have collected, for the reader’s convenience, the statements and proofs due to W.K. Allard [1] of both the fundamental monotonicity identity and the isoperimetric inequality for varifolds with locally bounded first variation.

1 Notations and basic definitions

Let \mathbb{R}^n be equipped with its usual scalar product $\langle \cdot, \cdot \rangle$. Let $G_{n,k}$ be the Grassmannian of all unoriented k -subspaces of \mathbb{R}^n . We shall often identify in the sequel an unoriented k -subspace $S \in G_{n,k}$ with the orthogonal projection onto S , which is represented by the matrix $S^{ij} = \langle e_i, S(e_j) \rangle$, $\{e_1, \dots, e_n\}$ being the canonical basis of \mathbb{R}^n . $G_{n,k}$ is equipped with the metric

$$\|S - T\| := \left(\sum_{i,j=1}^n (S^{ij} - T^{ij})^2 \right)^{\frac{1}{2}}$$

For an open subset $U \subset \mathbb{R}^n$ we define $G_k(U) = U \times G_{n,k}$, equipped with the product metric.

By a k -varifold on U we mean any Radon measure V on $G_k(U)$. Given a varifold V on U , a Radon measure $\|V\|$ on U (called the *weight* of V) is defined by

$$\|V\|(A) = V(\pi^{-1}(A)), \quad A \subset U \text{ Borel},$$

where π is the canonical projection $(x, S) \mapsto x$ of $G_k(U)$ onto U . We denote by $\Theta^k(\|V\|, x)$ the k -dimensional density of the measure $\|V\|$ at x , i.e.

$$\Theta^k(\|V\|, x) = \lim_{r \rightarrow 0} \frac{\|V\|(B_r(x))}{\omega_k r^k},$$

ω_k being the standard k -volume of the unit ball in \mathbb{R}^k . Recall that $\Theta^k(\|V\|, x)$ is well defined $\|V\|$ -almost everywhere [19, 10].

Given M , a countably (\mathcal{H}^k, k) -rectifiable subset of \mathbb{R}^n [10, 3.2.14] (from now on, we shall simply say *k -rectifiable*), and given θ , a positive and locally \mathcal{H}^k -integrable function on M , we define the k -rectifiable varifold $V \equiv \mathbf{v}(M, \theta)$ by

$$V(A) = \int_{\pi(TM \cap A)} \theta d\mathcal{H}^k, \quad A \subset G_n(U) \text{ Borel},$$

where $TM = \{(x, T_x M) : x \in M^*\}$ and M^* stands for the set of all $x \in M$ such that M has an approximate tangent space $T_x M$ with respect to θ at x , i.e. for all $f \in C_c^0(\mathbb{R}^n)$,

$$\lim_{\lambda \downarrow 0} \lambda^{-k} \int_M f(\lambda^{-1}(z - x)) \theta(z) d\mathcal{H}^k(z) = \theta(x) \int_{T_x M} f(y) d\mathcal{H}^k(y).$$

Remark that $\mathcal{H}^k(M \setminus M^*) = 0$ and the approximate tangent spaces of M with respect to two different positive \mathcal{H}^k -integrable functions $\theta, \tilde{\theta}$ coincide \mathcal{H}^k -a.e. on M (see [19], 11.5).

Finally, it is straightforward from the definition above that

$$\|V\| = \theta \mathcal{H}^k \llcorner M.$$

Whenever θ is integer valued, $V = \mathbf{v}(M, \theta)$ is called an *integral varifold*.

Before giving the definition of the mean curvature of a varifold, we recall that for a smooth k -manifold $M \subset \mathbb{R}^n$ with smooth boundary, the following equality holds for any $X \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$:

$$\int_M \operatorname{div}_M X \, d\mathcal{H}^k = - \int_M \langle H_M, X \rangle d\mathcal{H}^k - \int_{\partial M} \langle \eta, X \rangle d\mathcal{H}^{k-1}, \quad (1.1)$$

where H_M is the mean curvature vector of M , and η is the inner *conormal* of ∂M , i.e. the unit normal to ∂M which is tangent to M and points into M at each point of ∂M . The formula involves the tangential divergence of X at $x \in M$ which is defined by

$$\operatorname{div}_M X(x) := \sum_{i=1}^n \nabla_i^M X_i(x) = \sum_{i=1}^n \langle e_i, \nabla^M X_i(x) \rangle = \sum_{j=1}^k \langle \nabla X(x) \tau_j, \tau_j \rangle,$$

where $\{\tau_1, \dots, \tau_k\}$ is an orthonormal basis for $T_x M$, with $\nabla^M f(x) = T_x M(\nabla f(x))$ being the projection of $\nabla f(x)$ onto $T_x M$.

The *first variation* δV of a k -varifold V on U is the linear functional on $C_c^1(U, \mathbb{R}^n)$ defined by

$$\delta V(X) := \int_{G_k(U)} \operatorname{div}_S X \, dV(x, S), \quad (1.2)$$

where, for any $S \in G_{n,k}$, we have set $\nabla^S X_i = S(\nabla X_i)$ and

$$\operatorname{div}_S X = \sum_{i=1}^n \langle e_i, \nabla^S X_i \rangle.$$

In the case of a k -rectifiable varifold V , $\delta V(X)$ is actually the initial rate of change of the total weight $\|V\|(U)$ under the smooth flow generated by the vector field X . More precisely, let $X \in C_c^1(U, \mathbb{R}^n)$ and $\Phi(y, \epsilon) \in \mathbb{R}^n$ be defined as the flow generated by X , i.e. the unique solution to the Cauchy problem at each $y \in U$

$$\frac{\partial}{\partial \epsilon} \Phi(y, \epsilon) = X(\Phi(y, \epsilon)), \quad \Phi(y, 0) = y.$$

Then, one can consider the *push-forwarded varifold* $V_\epsilon = \Phi(\cdot, \epsilon)_\# V$, for which one obtains

$$\|V_\epsilon\|(U) = \int_U J_y^M \Phi(y, \epsilon) \, d\|V\|(y) = \int_U |1 + \epsilon \operatorname{div}_M X(y) + o(\epsilon)| \, d\|V\|(y),$$

where $J_y^M \Phi(y, \epsilon) = |\det(\nabla_y^M \Phi(y, \epsilon))|$ is the tangential Jacobian of $\Phi(\cdot, \epsilon)$ at y , and therefore

$$\delta V(X) = \int_U \operatorname{div}_M X(y) \, d\|V\|(y) = \frac{d}{d\epsilon} \|V_\epsilon\|(U)|_{\epsilon=0}$$

(see [19, §9 and §16] for more details).

A varifold V is said to have a locally bounded first variation in U if for each $W \subset\subset U$ there is a constant $c < \infty$ such that $|\delta V(X)| \leq c \operatorname{sup}_U |X|$ for any $X \in C_c^1(U, \mathbb{R}^n)$ with $\operatorname{spt}(X) \subset W$. By the Riesz Representation Theorem, there exist a Radon measure $\|\delta V\|$ on U - the total variation measure of δV - and a $\|\delta V\|$ -measurable function $\nu : U \rightarrow \mathbb{R}^n$ with $|\nu| = 1$ $\|\delta V\|$ -a.e. in U satisfying

$$\delta V(X) = - \int_U \langle \nu, X \rangle \, d\|\delta V\| \quad \forall X \in C_c^1(U, \mathbb{R}^n).$$

According to the Radon-Nikodým Theorem, the limit

$$D_{\|V\|}\|\delta V\|(x) := \lim_{r \rightarrow 0} \frac{\|\delta V\|(B_r(x))}{\|V\|(B_r(x))}$$

exists for $\|V\|$ -a.e. $x \in \mathbb{R}^n$. The *mean curvature* of V is defined for $\|V\|$ -almost every $x \in U$ as the vector

$$H_V(x) = D_{\|V\|}\|\delta V\|(x) \nu(x) \equiv |H_V(x)|\nu(x).$$

It follows that, for every $X \in C_c^1(U, \mathbb{R}^n)$,

$$\delta V(X) = - \int_U \langle H_V, X \rangle d\|V\| - \int_U \langle \nu, X \rangle d\|\delta V\|_s, \quad (1.3)$$

where $\|\delta V\|_s := \|\delta V\| \llcorner \mathcal{B}_V$, with $\mathcal{B}_V := \{x \in U : D_{\|V\|}\|\delta V\|(x) = +\infty\}$.

A varifold V is said to have mean curvature in L^p if $H_V \in L^p(\|V\|)$ and $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$. In other words,

$$H_V \in L^p \Leftrightarrow \begin{cases} H_V \in L^p(\|V\|) \\ \delta V(X) = - \int_U \langle H_V, X \rangle d\|V\| \end{cases} \quad \text{for every } X \in C_c^1(U, \mathbb{R}^n)$$

When M is a smooth k -dimensional submanifold of \mathbb{R}^n , with $(\overline{M} \setminus M) \cap U = \emptyset$, the divergence theorem on manifolds implies that the mean curvature of the varifold $\mathbf{v}(M, \theta_0)$ for any positive constant θ_0 is exactly the classical mean curvature of M , which can be calculated as

$$H(x) = - \sum_j \operatorname{div}_M \nu_j(x) \nu_j(x), \quad (1.4)$$

where $\{\nu_j(x)\}_j$ is an orthonormal frame for the orthogonal space $(T_x M)^\perp$.

We recall the *coarea formula* (see [19, 10]) for rectifiable sets in \mathbb{R}^n and mappings from \mathbb{R}^n to \mathbb{R}^m , $m < n$. Let M be a k -rectifiable set in \mathbb{R}^n with $k \geq m$, $\theta : M \rightarrow [0, +\infty]$ a Borel function, and $f : U \rightarrow \mathbb{R}^m$ a Lipschitz mapping defined on an open set $U \subset \mathbb{R}^n$. Then,

$$\int_{x \in M \cap U} J_M f(x) \theta(x) d\mathcal{H}^k(x) = \int_{\mathbb{R}^m} \int_{y \in f^{-1}(t) \cap M} \theta(y) d\mathcal{H}^{k-m}(y) d\mathcal{H}^m(t), \quad (1.5)$$

where $J_M f(x)$ denotes the tangential coarea factor of f at $x \in M$, defined for \mathcal{H}^k -almost every $x \in M$ by

$$J_M f(x) = \sqrt{\det(\nabla^M f(x) \cdot \nabla^M f(x)^t)}.$$

We also recall Allard's isoperimetric inequality for varifolds (see [1])

Theorem 1.1 (Isoperimetric inequality for varifolds). *There exists a constant $C > 0$ such that, for every k -varifold V with locally bounded first variation and for every smooth function $\varphi \geq 0$ with compact support in \mathbb{R}^n ,*

$$\int_{E_\varphi} \varphi d\|V\| \leq C \left(\int_{\mathbb{R}^n} \varphi d\|V\| \right)^{\frac{1}{k}} \left(\int_{\mathbb{R}^n} \varphi d\|\delta V\| + \int_{\mathbb{R}^n \times G_{n,k}} |\nabla^S \varphi| dV \right), \quad (1.6)$$

where $E_\varphi = \{x : \varphi(x)\Theta^k(\|V\|, x) \geq 1\}$.

2 Integral 1-varifolds with locally bounded first variation

2.1 Locality property of the generalized curvature

We consider integral 1-varifolds of type $V = \mathbf{v}(M, \theta)$ in $U \subset \mathbb{R}^n$, where $M \subset U$ is a 1-rectifiable set and $\theta \geq 1$ is an integer-valued Borel function on M . Thus, $\|V\| = \theta \mathcal{H}^1 \llcorner M$ is a Radon measure on U and we assume in addition that V has a locally bounded first variation, that is, for any smooth vectorfield $X \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$

$$\delta V(X) = \int_M \operatorname{div}_M X \, d\|V\| = - \int_M \langle \kappa, X \rangle \, d\|V\| + \delta V_s(X),$$

where δV_s denotes the singular part of the first variation with respect to the weight measure $\|V\|$. We now prove the following

Theorem 2.1. *Let $V_1 = \mathbf{v}(M_1, \theta_1)$, $V_2 = \mathbf{v}(M_2, \theta_2)$ be two integral 1-varifolds with locally bounded first variation. Then, denoting by κ_1, κ_2 their respective curvatures, one has $\kappa_1(x) = \kappa_2(x)$ for \mathcal{H}^1 -almost every $x \in S = M_1 \cap M_2$.*

Proof. Let $x \in S$ satisfy the following properties:

- (i) x is a point of density 1 for M_1, M_2 and S ;
- (ii) x is a Lebesgue point for θ_i and $\kappa_i \theta_i$ ($i = 1, 2$);
- (iii) $\lim_{r \rightarrow 0} \frac{\|\delta V_s\|(B_r(x))}{\|V\|(B_r(x))} = 0$ for $V = V_1, V_2$.

In particular, this means

$$\lim_{r \rightarrow 0} \frac{\mathcal{H}^1((M_i \setminus S) \cap B_r(x))}{2r} = 0 \quad (2.1)$$

$$\lim_{r \rightarrow 0} \frac{1}{2r} \int_{y \in M_i \cap B_r(x)} |\theta_i(y) - \theta_i(x)| \, d\mathcal{H}^1(y) = 0 \quad (2.2)$$

$$\lim_{r \rightarrow 0} \frac{1}{2r} \int_{y \in M_i \cap B_r(x)} |\kappa_i(y) \theta_i(y) - \kappa_i(x) \theta_i(x)| \, d\mathcal{H}^1(y) = 0 \quad (2.3)$$

for $i = 1, 2$ and with $B_r(x)$ denoting the ball of radius r and center x . Recall that \mathcal{H}^1 -a.e. $x \in S$ has such properties. Without loss of generality, we may assume that $x = 0$ and we shall denote in the sequel $B_r = B_r(0)$. In view of Property 3 above, we may also neglect the singular part, i.e. assume that the varifolds have curvatures in L^1_{loc} .

Let us write the coarea formula (1.5) with $f(x) = |x|$, $M = M_i \setminus S$ and $\theta = \theta_i$, also observing that

$$J_M f(x) = |\nabla^M f(x)| = \frac{|x_M|}{|x|} \leq 1$$

where x_M denotes the projection of x onto the tangent line $T_x M$. We obtain the inequality

$$\|V_i\|((M_i \setminus S) \cap B_r) \geq \int_0^r \int_{(M_i \setminus S) \cap \partial B_t} \theta_i \, d\mathcal{H}^0 \, dt \quad i = 1, 2. \quad (2.4)$$

By combining (2.1) and (2.2), one can show that

$$\frac{\|V_i\|((M_i \setminus S) \cap B_r)}{2r} = \frac{1}{2r} \int_{(M_i \setminus S) \cap B_r} \theta_i \, d\mathcal{H}^1 \longrightarrow 0 \quad \text{as } r \rightarrow 0, \quad i = 1, 2, \quad (2.5)$$

hence if we define

$$g_i(t) = \int_{(M_i \setminus S) \cap \partial B_t} \theta_i \, d\mathcal{H}^0, \quad i = 1, 2,$$

we find by (2.4) and (2.5) that 0 is a point of density 1 for the set $\mathcal{T}_i = \{t > 0 : g_i(t) = 0\}$, that is,

$$\lim_{r \rightarrow 0} \frac{|\mathcal{T}_i \cap [0, r]|}{r} = 1.$$

Therefore, by the fact that the measure \mathcal{H}^0 is integer-valued we can find a decreasing sequence $(r_k)_k$ converging to 0 and such that r_k is a Lebesgue point for both g_1 and g_2 , with $g_1(r_k) = g_2(r_k) = 0$, thus

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_{r_k - \epsilon}^{r_k} g_i(t) dt = 0 \quad \forall i = 1, 2. \quad (2.6)$$

By arguing exactly in the same way, we can also assume that

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_{r_k - \epsilon}^{r_k} h(t) dt = 0, \quad (2.7)$$

where

$$h(t) = \int_{y \in S \cap \partial B_t} |\theta_1(0)\theta_2(y) - \theta_2(0)\theta_1(y)| d\mathcal{H}^0(y).$$

Indeed, one can observe as before that the set

$$\mathcal{Q} = \{t > 0 : h(t) = 0\}$$

has density 1 at $t = 0$, as it follows from the integrality of the multiplicity functions combined with coarea formula and

$$\lim_{r \rightarrow 0} \frac{1}{2r} \int_{S \cap B_r} |\theta_1(0)\theta_2 - \theta_2(0)\theta_1| d\mathcal{H}^1 = 0,$$

this last equality being a consequence of (2.2). Therefore, r_k can be chosen in such a way that (2.7) holds, too. Now, for a given $\xi \in \mathbb{R}^n$ and $0 < \epsilon < r_k$, we define the vector field $X_{k,\epsilon}(x) = \eta_{r_k,\epsilon}(|x|)\xi$, where $\eta_{r,\epsilon}$ is a C^1 function defined on $[0, +\infty)$, with support contained in $[0, r)$ and such that

$$\eta_{r,\epsilon}(t) = 1 \quad \text{if } 0 \leq t \leq r - \epsilon, \quad \|\eta'_{r,\epsilon}\|_\infty \leq \frac{2}{\epsilon}.$$

By applying the coarea formula (1.5) and recalling that $\nabla^{M_i}|x| = x_{M_i}/|x|$, we get

$$\begin{aligned} \int_{M_i \setminus S} \left| \eta'_{r_k,\epsilon} \nabla^{M_i}|x| \right| \theta_i d\mathcal{H}^1 &\leq \frac{2}{\epsilon} \int_{r_k - \epsilon}^{r_k} \int_{\partial B_t \cap (M_i \setminus S)} \theta_i d\mathcal{H}^0 dt \\ &= \frac{2}{\epsilon} \int_{r_k - \epsilon}^{r_k} g_i(t) dt. \end{aligned}$$

Combining this last inequality with (2.6) and

$$\operatorname{div}_{M_i} X_{k,\epsilon}(x) = \eta'_{r_k,\epsilon}(x) \langle \xi, \frac{x_{M_i}}{|x|} \rangle$$

implies

$$\lim_{\epsilon \rightarrow 0} \int_{(M_i \setminus S) \cap B_{r_k}} \operatorname{div}_{M_i} X_{k,\epsilon} d\|V_i\| = 0, \quad \forall i = 1, 2, \quad \forall k. \quad (2.8)$$

At this point, we only need to show that the scalar product $\Delta = \langle \kappa_1(0) - \kappa_2(0), \xi \rangle$ cannot be positive, thus it has to be zero by the arbitrary choice of ξ . First, thanks to (2.3) we get

$$\begin{aligned} \Delta = \frac{1}{\theta_1(0)\theta_2(0)} \lim_k \left(\frac{\theta_2(0)}{2r_k} \int_{M_1 \cap B_{r_k}} \langle \xi, \kappa_1 \rangle d\|V_1\| \right. \\ \left. - \frac{\theta_1(0)}{2r_k} \int_{M_2 \cap B_{r_k}} \langle \xi, \kappa_2 \rangle d\|V_2\| \right), \end{aligned}$$

and, owing to the Dominated Convergence Theorem,

$$\Delta = \frac{1}{\theta_1(0)\theta_2(0)} \lim_k \lim_{\epsilon \rightarrow 0} \left(\frac{\theta_2(0)}{2r_k} \int_{M_1 \cap B_{r_k}} \langle X_{k,\epsilon}, \kappa_1 \rangle d\|V_1\| - \frac{\theta_1(0)}{2r_k} \int_{M_2 \cap B_{r_k}} \langle X_{k,\epsilon}, \kappa_2 \rangle d\|V_2\| \right),$$

Therefore, by the definition of the generalized curvature we immediately infer that

$$\Delta = \frac{1}{\theta_1(0)\theta_2(0)} \lim_k \lim_{\epsilon \rightarrow 0} \left(-\frac{\theta_2(0)}{2r_k} \int_{M_1 \cap B_{r_k}} \operatorname{div}_{M_1} X_{k,\epsilon} d\|V_1\| + \frac{\theta_1(0)}{2r_k} \int_{M_2 \cap B_{r_k}} \operatorname{div}_{M_2} X_{k,\epsilon} d\|V_2\| \right). \quad (2.9)$$

Noticing that $\operatorname{div}_S G(x) = \operatorname{div}_{M_1} G(x) = \operatorname{div}_{M_2} G(x)$ for \mathcal{H}^1 -almost all $x \in S$, and thanks to (2.8), one can rewrite (2.9) as

$$\Delta = \frac{1}{\theta_1(0)\theta_2(0)} \lim_k \lim_{\epsilon \rightarrow 0} \left(\frac{1}{2r_k} \int_{S \cap B_{r_k}} \operatorname{div}_S X_{k,\epsilon} (\theta_1(0)\theta_2 - \theta_2(0)\theta_1) d\mathcal{H}^1 \right). \quad (2.10)$$

Computing the tangential divergence of $X_{k,\epsilon}$ and, then, using the coarea formula (1.5) in (2.10), gives

$$\Delta \leq \frac{|\xi|}{\theta_1(0)\theta_2(0)} \lim_k \frac{1}{r_k} \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_{r_k - \epsilon}^{r_k} h(t) dt = 0.$$

We conclude that $\Delta = 0$, hence $\kappa_1(0) = \kappa_2(0)$, as wanted. \square

A straightforward consequence of Theorem 2.1 is the following

Corollary 2.2. *Let $V = \mathbf{v}(M, \theta)$ be an integral 1-varifold in $U \subset \mathbb{R}^n$, with locally bounded first variation. Then the vector $\kappa(x)$ coincides with the classical curvature of any C^2 curve γ , for \mathcal{H}^1 -almost all $x \in \gamma \cap \operatorname{spt} \|V\|$.*

2.2 A 1-varifold with curvature in $L^1 \setminus L^p$ for all $p > 1$

Here we construct an integral 1-varifold in \mathbb{R}^2 with curvature in $L^1 \setminus L^p$ for any $p > 1$. This varifold is obtained as the limit of a sequence of graphs of smooth functions, its support is C^2 -rectifiable (i.e., covered up to a negligible set by a countable union of C^2 curves, see [5, 18]) and, due to our Theorem 2.1, its curvature coincides \mathcal{H}^1 -almost everywhere with the classical one, as stated in Corollary 2.2 above.

Let $\zeta \in C^2([0, 1])$ with $\zeta \neq 0$ and

$$\zeta(0) = \zeta'(0) = \zeta(1) = \zeta'(1) = 0.$$

Given $\lambda > 0$ and $0 \leq a < b \leq 1$, define

$$\zeta_{a,b,\lambda}(t) = \begin{cases} \lambda \zeta\left(\frac{t-a}{b-a}\right) & \text{if } t \in [a, b], \\ 0 & \text{otherwise.} \end{cases}$$

Let $(a_n, b_n)_{n \geq 2}$ be a sequence of nonempty, open and mutually disjoint subintervals of $[0, 1]$, such that $b_n - a_n \leq 2^{-n}$ and

$$0 < \sum_{n \geq 2} (b_n - a_n) < 1.$$

In particular, the set $C = [0, 1] \setminus \bigcup_n (a_n, b_n)$ is closed and has positive \mathcal{L}^1 measure. We denote by $(\lambda_n)_n$ a sequence of positive real numbers, that will be chosen later, and we set

$$\zeta_n(t) = \zeta_{a_n, b_n, \lambda_n}(t)$$

for $t \in [0, 1]$ and $n \geq 2$. Then, we compute the integral of the p -th power of the curvature of the graph of ζ_n over the graph itself, that is,

$$K_n^p = \int_{a_n}^{b_n} \frac{|\zeta_n''(t)|^p}{\left[1 + \zeta_n'(t)^2\right]^{\frac{3p-1}{2}}} dt.$$

Since

$$\begin{aligned} \zeta_n'(t) &= \frac{\lambda_n}{b_n - a_n} \zeta' \left(\frac{t - a_n}{b_n - a_n} \right), \\ \zeta_n''(t) &= \frac{\lambda_n}{(b_n - a_n)^2} \zeta'' \left(\frac{t - a_n}{b_n - a_n} \right), \end{aligned}$$

and choosing $0 \leq \lambda_n \leq b_n - a_n$, we infer that the Lipschitz constant of ζ_n is bounded by that of ζ , for all $n \geq 2$. Therefore, there exists a uniform constant $c \geq 1$ such that

$$c^{-1} K_n^p \leq \int_{a_n}^{b_n} |\zeta_n''(t)|^p dt \leq c K_n^p,$$

and therefore

$$c^{-1} K_n^p \leq K \frac{\lambda_n^p}{(b_n - a_n)^{2p-1}} \leq c K_n^p,$$

where

$$K = \int_0^1 |\zeta''(t)|^p dt > 0.$$

At this point, we look for λ_n satisfying

- (i) $0 < \lambda_n \leq b_n - a_n$,
- (ii) $\sum_{n \geq 2} K_n^p < +\infty$ if and only if $p = 1$.

A possible choice for λ_n is given by

$$\lambda_n = \frac{b_n - a_n}{n^2}.$$

Indeed, up to multiplicative constants one gets

$$\sum_n K_n^1 = \sum_n \frac{1}{n^2} < +\infty \tag{2.11}$$

and

$$\sum_n K_n^p \geq \sum_n \frac{2^{n(p-1)}}{n^{2p}} = +\infty \tag{2.12}$$

for all $p > 1$. Now, define for $t \in \mathbb{R}$

$$\eta(t) = \sum_n \zeta_n(t).$$

Thanks to (2.11) and (2.12), the 1-varifold $V = \mathbf{v}(G, 1)$ associated to the graph G of η has curvature in $L^1 \setminus L^p$ for all $p > 1$. Indeed, setting $\eta_N = \sum_{n=2}^N \zeta_n$ and letting G_N be the graph of η_N , one can verify that the 1-rectifiable varifolds $V_N = \mathbf{v}(G_N, 1)$ weakly converge to V as $N \rightarrow \infty$, and the same happens for the respective first variations:

$$\delta V_N \xrightarrow{N} \delta V,$$

thus for any open set $A \subset (0, 1) \times \mathbb{R}$ one has

$$\begin{aligned} \|\delta V\|(A) &\leq \lim_N \|\delta V_N\|(A) \\ &= \lim_N \int_A |\kappa_N| \mathbb{1}_{G_N} d\|V\| \\ &= \int_A |\kappa| d\|V\|, \end{aligned}$$

where $\mathbb{1}_{G_N}$ is the characteristic function of G_N and for $(x, y) \in G$ we define $\kappa(x, y) = \kappa_N(x, y)$ for N large enough and $y > 0$ (the definition is well-posed, since the intervals (a_n, b_n) are pairwise disjoint) and $\kappa(x, y) = 0$ whenever $y = 0$. This shows that V has curvature in L^1 . It is also evident from (2.12) that the curvature of V cannot belong to L^p for $p > 1$. Lastly, the C^2 -rectifiability comes from $\mathcal{H}^1(\text{spt } V \setminus \bigcup_{N \geq 2} G_N) = 0$.

An example of the construction of such varifold V is illustrated in Figure 1.

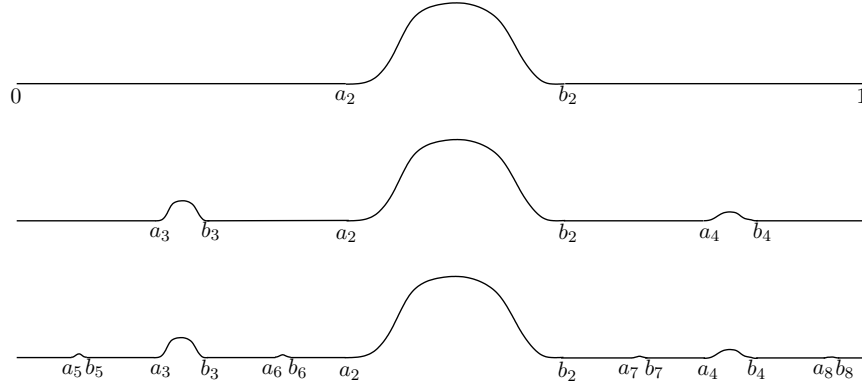


Figure 1. As a particular example, we take the sequence (a_n, b_n) of all middle intervals in $[0, 1]$ of size 2^{-2p-2} whenever $2^p < n \leq 2^{p+1}$, $p = 0, 1, 2, \dots$. The union of these intervals is the complement of a Cantor-type set C with positive measure $\mathcal{H}^1(C) = \frac{1}{2}$. We have represented from top to bottom the functions ζ_2 , $\sum_{n=2}^4 \zeta_n$ and $\sum_{n=2}^8 \zeta_n$.

3 Rectifiable k -varifolds with locally bounded first variation

3.1 Relative isoperimetric inequalities for k -varifolds

The isoperimetric inequality for varifolds due to W.K. Allard [1] is recalled in Theorem 1.1. We derive from it the following *differential inequalities*, that will be useful for studying the locality of rectifiable k -varifolds.

Proposition 3.1 (Relative isoperimetric inequalities). *Let V be a k -varifold in \mathbb{R}^n , and let $A \subset \mathbb{R}^n$ be an open, bounded set with Lipschitz boundary. Then,*

$$\|V\|(A)^{\frac{k-1}{k}} \leq C (\|\delta V\|(A) - D_+ \|V\|(A \setminus A_\epsilon)|_{\epsilon=0}), \quad (3.1)$$

where A_ϵ is the set of points of A whose distance from $\mathbb{R}^n \setminus A$ is less than ϵ , and $D_+ \|V\|(A \setminus A_\epsilon)|_{\epsilon=0}$ denotes the lower right derivative of the non-increasing function $\epsilon \rightarrow \|V\|(A \setminus A_\epsilon)$ at $\epsilon = 0$.

Moreover, if we define $g(r) = \|V\|(B_r)$, then g is a non-decreasing (thus almost everywhere differentiable) function, and it holds

$$g(r)^{\frac{k-1}{k}} \leq C (\|\delta V\|(B_r) + g'(r)) \quad \text{for almost all } r > 0. \quad (3.2)$$

Proof. Let $\epsilon > 0$ and let $\varphi_\epsilon : A \rightarrow \mathbb{R}$ be defined as

$$\varphi_\epsilon(x) = \min(\epsilon^{-1} d(x, \mathbb{R}^n \setminus A), 1).$$

Clearly, φ_ϵ is a Lipschitz function with compact support in \mathbb{R}^n . Approximating φ_ϵ by a sequence of non-negative, C^1 functions with compact support in \mathbb{R}^n , it follows from Allard's isoperimetric inequality (1.6) that

$$\int_{E_{\varphi_\epsilon}} \varphi_\epsilon d\|V\| \leq C \left(\int_{\mathbb{R}^n} \varphi_\epsilon d\|V\| \right)^{\frac{1}{k}} \left(\int_{\mathbb{R}^n} \varphi_\epsilon d\|\delta V\| + \int_{\mathbb{R}^n \times G(n,k)} |\nabla^S \varphi_\epsilon| dV \right), \quad (3.3)$$

where $E_{\varphi_\epsilon} = \{x : \varphi_\epsilon(x)\Theta^k(\|V\|, x) \geq 1\}$. Moreover, we have

$$|\nabla \varphi_\epsilon(x)| \leq \frac{1}{\epsilon} \quad \text{on } A_\epsilon := \{x \in A : d(x, \mathbb{R}^n \setminus A) \leq \epsilon\},$$

and therefore (3.3) can be rewritten as

$$\int_{A \setminus A_\epsilon} d\|V\| \leq \int_{E_{\varphi_\epsilon}} \varphi_\epsilon d\|V\| \leq C \left(\int_A d\|V\| \right)^{\frac{1}{k}} \left(\int_A d\|\delta V\| + \frac{1}{\epsilon} \int_{A_\epsilon} d\|V\| \right) \quad (3.4)$$

Now, since

$$\lim_{\epsilon \rightarrow 0} \int_{A \setminus A_\epsilon} d\|V\| = \int_A d\|V\|$$

and

$$\frac{1}{\epsilon} \int_{A_\epsilon} d\|V\| = \frac{\|V\|(A_\epsilon)}{\epsilon} = -\frac{\|V\|(A \setminus A_\epsilon) - \|V\|(A)}{\epsilon},$$

the Dominated Convergence Theorem allows us to take the limit in (3.4) as $\epsilon \rightarrow 0$, yielding

$$\|V\|(A)^{\frac{k-1}{k}} \leq C(\|\delta V\|(A) - D_+ \|V\|(A \setminus A_\epsilon)|_{\epsilon=0}).$$

Take now $A \equiv B_r$ and remark that $A \setminus A_\epsilon = B_{r-\epsilon}$. Denoting $g(r) = \|V\|(B_r)$, we deduce from the monotonicity of g that it is almost everywhere differentiable. In particular, for almost every $r > 0$, and using the fact that $g(r-\epsilon) - g(r) = -\|V\|(A_\epsilon)$ for almost every $r > 0$ (and for every $\epsilon > 0$), we get

$$g'(r) = -\lim_{\epsilon \rightarrow 0} \frac{g(r-\epsilon) - g(r)}{\epsilon} = \lim_{\epsilon \rightarrow 0} \frac{\|V\|(A_\epsilon)}{\epsilon}. \quad (3.5)$$

Then, (3.2) immediately follows from (3.5) and (3.1). \square

3.2 A locality result for rectifiable k -varifolds

First, we derive a useful formula for computing the mean curvature of a rectifiable k -varifold. This formula will be crucial in the proof of our second locality result (Theorem 3.4). More precisely, given a rectifiable k -varifold $V = \mathbf{v}(M, \theta)$ with locally bounded first variation, we show in the next proposition that the integral of the mean curvature on a ball B_r essentially coincides with the integral on the sphere ∂B_r of the conormal η to M , up to an error term due to the singular part of the first variation. Therefore, we obtain an equivalent expression for the curvature at a Lebesgue point $x_0 \in M$. Recall that x_M denotes the orthogonal projection of x onto $T_x M$.

Proposition 3.2. *Let $x_0 \in \mathbb{R}^n$ and $V = \mathbf{v}(M, \theta)$ be a rectifiable k -varifold with locally bounded first variation. Then, setting $\sigma = \theta \mathcal{H}^{k-1} \llcorner M$, we get for almost every $r > 0$*

$$\left| \int_{B_r(x_0)} H d\|V\| + \int_{\partial B_r(x_0)} \eta d\sigma \right| \leq \|\delta V\|_s(B_r(x_0)), \quad (3.6)$$

where $\eta(x) = \begin{cases} -\frac{x_M}{|x_M|} & \text{if } |x_M| \neq 0 \\ 0 & \text{elsewhere} \end{cases}$ is the inner conormal to $M \cap B_r(x_0)$ at $x \in M \cap \partial B_r(x_0)$. Consequently, if $x_0 \in M$ is a Lebesgue point for H , then

$$H(x_0) = -\lim_{r \rightarrow 0^+} \frac{1}{\|V\|(B_r(x_0))} \int_{\partial B_r(x_0)} \eta d\sigma. \quad (3.7)$$

Proof. For simplicity, we assume that $x_0 = 0$. Let us consider a Lipschitz cutoff function $\beta_\epsilon : [0, +\infty) \rightarrow \mathbb{R}$ such that $\beta_\epsilon(t) = 1$ for $t \in [0, r - \epsilon]$, $\beta_\epsilon(t) = 1 - \frac{t-r+\epsilon}{\epsilon}$ for $t \in (r - \epsilon, r]$ and $\beta_\epsilon(t) = 0$ elsewhere. Then, choose a unit vector $w \in \mathbb{R}^n$ and define the vector field $X_\epsilon = \beta_\epsilon(|x|)w$. The definition of the generalized mean curvature yields

$$\int_{B_r} \operatorname{div}_M X_\epsilon d\|V\| = - \int_{B_r} \beta_\epsilon \langle H, w \rangle d\|V\| + \delta V_s(X_\epsilon),$$

and, thanks to our assumptions, we also have

$$\int_{B_r} \operatorname{div}_M X_\epsilon d\|V\| = -\frac{1}{\epsilon} \int_{B_r \setminus B_{r-\epsilon}} \frac{\langle x_M, w \rangle}{|x|} d\|V\|$$

By the Dominated Convergence Theorem,

$$\lim_{\epsilon \rightarrow 0} \int_{B_r} \langle H, w \rangle \beta_\epsilon d\|V\| = \int_{B_r} \langle H, w \rangle d\|V\|, \quad \forall r > 0.$$

Therefore, the derivative

$$\frac{d}{dr} \int_{B_r} \frac{\langle x_M, w \rangle}{|x|} d\|V\|$$

exists for almost all $r > 0$ as the limit of the difference quotient

$$\frac{1}{\epsilon} \int_{B_r \setminus B_{r-\epsilon}} \frac{\langle x_M, w \rangle}{|x|} d\|V\|$$

and, in view of (1.3), one has

$$\frac{d}{dr} \int_{B_r} \frac{\langle x_M, w \rangle}{|x|} d\|V\| = \int_{B_r} \langle H, w \rangle d\|V\| + \int_{B_r} \langle \nu, w \rangle d\|\delta V_s\|.$$

Observe now that, denoting $N := \{x : |x_M| = 0\}$, the coarea formula (1.5) gives

$$\begin{aligned} \int_{B_r} \frac{\langle x_M, w \rangle}{|x|} d\|V\| &= \int_{B_r \setminus N} \frac{\langle x_M, w \rangle}{|x_M|} \frac{|x_M|}{|x|} d\|V\| \\ &= \int_0^r \int_{\partial B_t \setminus N} \frac{\langle x_M, w \rangle}{|x_M|} d\sigma dt. \end{aligned}$$

We deduce that, for every Lebesgue point of the integrable function

$$t \mapsto \int_{\partial B_t \setminus N} \frac{\langle x_M, w \rangle}{|x_M|} d\sigma,$$

one gets

$$\frac{d}{dr} \int_{B_r} \frac{\langle x_M, w \rangle}{|x|} d\|V\| = \int_{\partial B_r \setminus N} \frac{\langle x_M, w \rangle}{|x_M|} d\sigma.$$

By the definition of the conormal η , we conclude that, for every vector $w \in \mathbb{R}^n$,

$$\left| \int_{B_r} \langle H, w \rangle d\|V\| + \int_{\partial B_r} \langle \eta, w \rangle d\sigma \right| \leq |w| \|\delta V_s\|(B_r), \quad \text{for a.e. } r > 0$$

or, equivalently,

$$\left| \int_{B_r} H d\|V\| + \int_{\partial B_r} \eta d\sigma \right| \leq \|\delta V_s\|(B_r), \quad \text{for a.e. } r > 0.$$

This proves (3.6) and, since

$$\frac{\|\delta V_s\|(B_r)}{\|V\|(B_r)} \rightarrow 0 \quad \text{as } r \rightarrow 0,$$

also (3.7) follows. \square

Remark 3.3. In case δV has no singular part with respect to $\|V\|$, (3.6) becomes

$$\int_{B_r(x_0)} H d\|V\| = - \int_{\partial B_r(x_0)} \eta d\sigma, \quad \text{for almost every } r > 0.$$

Below we prove a locality property for k -varifolds in \mathbb{R}^n , $k \geq 2$, requiring some extra hypotheses on the varifolds under consideration. The proof is quite different from that of Theorem 2.1, mainly because the Hausdorff measure \mathcal{H}^{k-1} is no more a discrete (counting) measure. Our result gives a positive answer to the locality problem in any dimension $k \geq 2$ and any codimension, assuming that the support of one of the two varifolds is locally contained into the other, and also that the two multiplicities are locally constant on the intersection of the supports.

Theorem 3.4. *Let $V_i = \mathbf{v}(M_i, \theta_i)$, $i = 1, 2$ be two rectifiable k -varifolds in $U \subset \mathbb{R}^n$ with locally bounded first variations, and let H_1, H_2 denote their respective mean curvatures. Suppose that there exists an open set $A \subset U$ such that*

- (i) $M_1 \cap A \subset M_2$,
- (ii) $\theta_1(x)$ and $\theta_2(x)$ are \mathcal{H}^k -a.e. constant on $M_1 \cap A$.

Then, $H_1(x) = H_2(x)$ for \mathcal{H}^k -a.e. $x \in M_1 \cap A$.

Proof. Up to multiplication by suitable constants, we may assume without loss of generality that $\theta_1(x) = \theta_2(x) = \theta_0$ constant, for \mathcal{H}^k -almost every $x \in M_1 \cap A$. Moreover, the theory of rectifiable sets and of rectifiable measures ensures that \mathcal{H}^k -a.e. point $x \in M_1 \cap A$ is *generic* in the sense that it satisfies

- (i) $\Theta^k(\|V_i\| \llcorner (M_2 \setminus M_1), x) = 0$ and $\Theta^k(\|V_i\|, x) = \theta_0$ for $i = 1, 2$;
- (ii) x is a Lebesgue point for H_1 and H_2 ;
- (iii) $\|\delta V_s\|(B_r(x)) = o(\|V\|(B_r(x)))$ for $V = V_1, V_2$.

Suppose, without loss of generality, that $x = 0$ is a generic point of $M_1 \cap A$. Let r_0 be such that $B_{r_0} := B_{r_0}(0) \subset A$, let $\widetilde{M}_2 = M_2 \setminus M_1$ and $\widetilde{V}_2 = \mathbf{v}(\widetilde{M}_2, \theta_2)$. Obviously, \widetilde{V}_2 is a rectifiable k -varifold, but possibly $\delta \widetilde{V}_2$ has an extra singular part with respect to $\|\widetilde{V}_2\|$. By (3.6), for almost every $0 < r < r_0$

$$\begin{aligned} \int_{B_r} H_2 d\|V_2\| + o(\|V_2\|(B_r)) &= - \int_{\partial B_r} \eta_2 d\sigma_2 \\ &= - \int_{\partial B_r \cap M_1} \eta_2 d\sigma_2 - \int_{\partial B_r \cap \widetilde{M}_2} \eta_2 d\sigma_2, \end{aligned} \tag{3.8}$$

where $\sigma_2 = \theta_2 \mathcal{H}^{k-1} \llcorner M_2$. Since both M_1 and M_2 are rectifiable, they have the same tangent space at \mathcal{H}^k -almost every point of $M_1 \cap A$, thus $\eta_1 = \eta_2$ for \mathcal{H}^k -a.e. $x \in M_1 \cap A$. Then, observe that the coarea formula and the assumption $\theta_2(x) = \theta_1(x) = \theta_0$ \mathcal{H}^k -almost everywhere on $M_1 \cap A$ yield, for almost every $0 < r < r_0$,

$$\int_{\partial B_r \cap M_1} (\theta_2 - \theta_1) d\mathcal{H}^{k-1} = 0$$

that is,

$$\sigma_2(\partial B_r \cap M_1) = \sigma_1(\partial B_r) = \theta_0 \mathcal{H}^{k-1}(\partial B_r \cap M_1),$$

where $\sigma_i = \theta_i \mathcal{H}^{k-1} \llcorner M_i$, $i = 1, 2$. We deduce by (3.6) that, for a.e. $0 < r < r_0$,

$$\int_{\partial B_r \cap M_1} \eta_2 d\sigma_2 = \int_{\partial B_r} \eta_1 d\sigma_1 = - \int_{B_r} H_1 d\|V_1\| - o(\|V_1\|(B_r)).$$

Being $x = 0$ generic, and as $r \rightarrow 0^+$, we have

$$\frac{1}{\omega_k r^k} \int_{B_r} H_2 d\|V_2\| \longrightarrow \theta_0 H_2(0) \tag{3.9}$$

and

$$-\frac{1}{\omega_k r^k} \int_{\partial B_r \cap M_1} \eta_2 d\sigma_2 = \frac{1}{\omega_k r^k} \int_{B_r} H_1 d\|V_1\| + o(1) \longrightarrow \theta_0 H_1(0), \quad (3.10)$$

thus, in view of (3.8), it remains to prove that $\int_{\partial B_r \cap \widetilde{M}_2} \eta_2 d\sigma_2 = o(r^k)$ – at least for a suitable sequence of radii – to get the locality property at $x = 0$, i.e. that $H_1(0) = H_2(0)$.

For every $X \in C_c^1(A, \mathbb{R}^n)$, we observe that, by the definition of the first variation, and thanks to the inclusion $M_1 \cap A \subset M_2$,

$$\begin{aligned} \delta V_2(X) &= \int_{M_2} \operatorname{div}_{M_2} X \theta_2 d\mathcal{H}^k \\ &= \int_{\widetilde{M}_2} \operatorname{div}_{\widetilde{M}_2} X \theta_2 d\mathcal{H}^k + \int_{M_1} \operatorname{div}_{M_1} X \theta_1 d\mathcal{H}^k \\ &= \delta \widetilde{V}_2(X) + \delta V_1(X), \end{aligned}$$

hence

$$\|\delta \widetilde{V}_2\|(A) \leq \|\delta V_1\|(A) + \|\delta V_2\|(A).$$

Therefore, \widetilde{V}_2 has locally bounded first variation in A , like V_1 and V_2 . Furthermore, using the genericity of 0, one gets

$$\frac{\|\delta V_1\|(B_r)}{\omega_k r^k} \rightarrow \theta_0 |H_1(0)|$$

and

$$\frac{\|\delta V_2\|(B_r)}{\omega_k r^k} \rightarrow \theta_0 |H_2(0)|,$$

as $r \rightarrow 0$, whence

$$\|\delta V_1\|(B_r) + \|\delta V_2\|(B_r) = O(r^k),$$

and finally

$$\|\delta \widetilde{V}_2\|(B_r) = O(r^k). \quad (3.11)$$

Let $g(r) := \|\widetilde{V}_2\|(B_r)$. Since $g(0) = 0$ and g is non-decreasing on $[0, +\infty)$ – thus g has locally bounded variation – it holds for every $0 \leq \alpha < \beta < r_0$

$$g(\beta) - g(\alpha) = \int_{\alpha}^{\beta} g'(t) dt + |D^s g|((\alpha, \beta])$$

where $g'(t) dt$ and $D^s g$ are, respectively, the absolutely continuous part and the singular part of the distributional derivative Dg . Besides, the coarea formula (1.5) yields

$$g(\beta) - g(\alpha) = \int_{B_\beta \setminus B_\alpha \cap \widetilde{M}_2} d\|V_2\| \geq \int_{B_\beta \setminus B_\alpha \cap \widetilde{M}_2} \frac{|x_{M_2}|}{|x|} d\|V_2\| = \int_{\alpha}^{\beta} \int_{\partial B_t \cap \widetilde{M}_2} d\sigma_2 dt.$$

Since $D^s g$ and $g'(t) dt$ are mutually singular, it follows that

$$\int_{\alpha}^{\beta} g'(t) dt \geq \int_{\alpha}^{\beta} \int_{\partial B_t \cap \widetilde{M}_2} d\sigma_2 dt,$$

for almost every $0 \leq \alpha, \beta < r_0$. Therefore, by the Radon-Nikodým Theorem,

$$g'(r) \geq \sigma_2(\partial B_r \cap \widetilde{M}_2), \quad \text{for a.e. } 0 < r < r_0.$$

We deduce that for almost every $r \in (0, r_0)$

$$\left| \int_{\partial B_r \cap \widetilde{M}_2} \eta_2 d\sigma_2 \right| \leq \sigma_2(\partial B_r \cap \widetilde{M}_2) \leq g'(r). \quad (3.12)$$

Then, it follows from the relative isoperimetric inequality (3.2) that for almost every $0 < r < r_0$

$$g(r)^{\frac{k-1}{k}} \leq C \left(\|\delta \widetilde{V}_2\|(B_r(x)) + g'(r) \right),$$

thus, by (3.11), for another suitable constant still denoted by C ,

$$g(r)^{\frac{k-1}{k}} \leq C(r^k + g'(r)), \quad (3.13)$$

At the same time, the genericity of $x = 0$ and the assumption $\Theta^k(\mathcal{H}^k \llcorner \widetilde{M}_2, x) = 0$ give

$$g(r) = o(r^k). \quad (3.14)$$

Let N be the set of real numbers in $(0, r_0)$ such that (3.12) and (3.13) hold. Clearly, N has full measure in $(0, r_0)$. To conclude, we need to show that there exists a sequence of radii $(r_h)_{h \in \mathbb{N}} \in N$ decreasing to 0, such that

$$g'(r_h) = o(r_h^k). \quad (3.15)$$

By contradiction, suppose that there exist a constant $C_1 > 0$ and a radius $0 < r_1 < r_0$, such that $g'(r) \geq C_1 r^k$ for every $r \in N \cap (0, r_1)$. Then, by (3.13) and for an appropriate constant $C_2 > 0$,

$$g(r)^{\frac{k-1}{k}} \leq C_2 g'(r)$$

thus, for a.e. $0 < r < r_1$,

$$g(r)^{\frac{1-k}{k}} g'(r) \geq \frac{1}{C_2}.$$

Observing that $g(r)$ is non-decreasing and $g(0) = 0$, we can integrate both sides of the inequality between 0 and r , to obtain

$$k g(r)^{\frac{1}{k}} \geq \frac{r}{C_2},$$

i.e. $g(r) \geq \frac{r^k}{(C_2 k)^k}$, in contradiction with the fact that $g(r) = o(r^k)$. In conclusion, by (3.12) and (3.15), there exists a sequence of radii $(r_h)_{h \in \mathbb{N}}$ decreasing to 0 such that (3.14) holds and

$$\int_{\partial B_{r_h} \cap \widetilde{M}_2} \eta_2 d\sigma_2 = o(r_h^k).$$

Combining with (3.8), (3.9) and (3.10), we conclude the proof. \square

Corollary 3.5. *Let $V_M = \mathbf{v}(M, \theta_M)$, be a rectifiable k -varifold with positive density and locally bounded first variation, such that*

- (i) *there exist an open set $A \subset \mathbb{R}^n$ and a C^2 k -manifold S such that $S \cap A \subset M$,*
- (ii) *$\theta_M(x) \equiv \theta_0$ constant for \mathcal{H}^k -a.e. $x \in S \cap A$*

Then, $H_M(x) = H_S(x)$ for \mathcal{H}^k -almost every $x \in S \cap A$, where H_M and H_S denote, respectively, the generalized mean curvature of V_M and the classical mean curvature of S .

Proof. It is an obvious consequence of the previous theorem by simply observing that, thanks to the divergence theorem for smooth sets, the classical mean curvature H_S of S coincides with the mean curvature of the varifold $\mathbf{v}(S, \theta_0)$. \square

Conjecture 3.6. We would expect that the locality property of the mean curvature of k -varifolds, $k > 1$, holds true under the sole hypothesis of locally bounded first variation. However, we have not been able to prove this assertion in full generality.

4 Applications

4.1 Lower semicontinuity of the elastica energy for curves in \mathbb{R}^n

Let E be an open subset of \mathbb{R}^2 with smooth boundary ∂E and let us consider the functional

$$\mathcal{F}(E) = \int_{\partial E} (\alpha + \beta |\kappa_{\partial E}(y)|^p) d\mathcal{H}^1(y),$$

where $p \geq 1$, $\kappa_{\partial E}(y)$ denotes the curvature at $y \in \partial E$ and α, β are positive constants. This functional is an extension to boundaries of smooth sets and to different curvature exponents of the celebrated elastica energy

$$\int_{\gamma} (\alpha + \beta \kappa^2) d\mathcal{H}^1$$

that was proposed in 1744 by Euler to study the equilibrium configurations of a thin, flexible beam γ subjected to end forces. This energy, mainly used in elasticity theory, has also appeared to be of interest for a shape completion model in computer vision [15, 16].

Let $\overline{\mathcal{F}}$ denote the lower semicontinuous envelope – the relaxation – of \mathcal{F} with respect to L^1 convergence, i.e. for any measurable bounded subset $E \subset \mathbb{R}^2$,

$$\overline{\mathcal{F}}(E) = \inf \left\{ \liminf_{h \rightarrow \infty} \mathcal{F}(E_h), E_h \subset \mathbb{R}^2 \text{ open}, \partial E_h \in C^2, |E_h \Delta E| \rightarrow 0 \right\},$$

where $|E_h \Delta E|$ denotes the Lebesgue 2-dimensional outer measure of the symmetric difference of the sets E_h and E .

Many properties of \mathcal{F} and $\overline{\mathcal{F}}$ have been carefully studied in [6, 7, 8]. In particular, it has been proved in [6] that, whenever $E, (E_h)_h \subset \mathbb{R}^2$, $\partial E, (\partial E_h)_h \in C^2$ and $|E_h \Delta E| \rightarrow 0$ as $h \rightarrow \infty$, then

$$\int_{\partial E} (\alpha + \beta |\kappa_{\partial E}|^p) d\mathcal{H}^1 \leq \liminf_{h \rightarrow \infty} \int_{\partial E_h} (\alpha + \beta |\kappa_{\partial E_h}|^p) d\mathcal{H}^1 \quad \text{for any } p > 1.$$

This lower semicontinuity result is proved through a parameterization procedure that can be extended to the case of sets whose boundaries can be decomposed as a union of non crossing $W^{2,p}$ curves. As a consequence, $\mathcal{F}(E) = \overline{\mathcal{F}}(E)$ for any E in this class [6].

Thanks to Theorem 2.1, we can easily prove the lower semicontinuity of the p -elastica energy for curves in \mathbb{R}^n , $n \geq 2$, and for $p \geq 1$, thus getting an affirmative answer also for the case $p = 1$. In this context, it is more appropriate to use the convergence in the sense of currents (see [19, 10] for the definitions and properties of currents), and the following result ensues:

Theorem 4.1. *Let $(\mathcal{C}_k)_{k \in \mathbb{N}}$ with $\mathcal{C}_k = \bigcup_{i \in I(k)} C_{k,i}$ be a sequence of countable collections of disjoint, closed and uniformly bounded C^2 curves in \mathbb{R}^n , converging in the sense of currents to a countable collection of disjoint, closed C^2 curves $\mathcal{C} = \bigcup_{i \in I} C_i$, and satisfying*

$$\sup_{k \in \mathbb{N}} \sum_{i \in I(k)} \int_{C_{k,i}} (1 + |\kappa_{C_{k,i}}|^p) d\mathcal{H}^1 < +\infty.$$

Then, for $\alpha, \beta \geq 0$,

$$\sum_{i \in I} \int_{C_i} (\alpha + \beta |\kappa_{C_i}|^p) d\mathcal{H}^1 \leq \liminf_{k \rightarrow \infty} \sum_{i \in I(k)} \int_{C_{k,i}} (\alpha + \beta |\kappa_{C_{k,i}}|^p) d\mathcal{H}^1$$

for every $p \geq 1$.

Proof. With the notations of Section 1, we consider the sequence of varifolds $V_k = \mathbf{v}(\mathcal{C}_k, 1)$. As an obvious consequence of our assumptions, the V_k 's have uniformly bounded first variation and their curvatures are in $L^p(\|V_k\|)$. By Allard's Compactness Theorem for rectifiable varifolds [1, 19], and possibly taking a subsequence, we get that (V_k) converges in the sense of varifolds to an integral varifold V with locally bounded first variation. In addition, by Theorem 2.34 and Example 2.36 in [3]

- (i) if $p > 1$ then the absolute continuity of δV_k with respect to $\|V_k\|$ passes to the limit, i.e. V has curvature in L^p , and

$$\int_{\mathbb{R}^n} (\alpha + \beta|\kappa_V|^p) d\|V\| \leq \liminf_{k \rightarrow \infty} \sum_{i \in I(k)} \int_{C_{k,i}} (\alpha + \beta|\kappa_{C_{k,i}}|^p) d\mathcal{H}^1;$$

- (ii) if $p = 1$, then δV may not be absolutely continuous with respect to V , but the lower semicontinuity of both measures $\|\delta V\|$ and $\|V\|$ implies that

$$\begin{aligned} \int_{\mathbb{R}^n} (\alpha + \beta|\kappa_V|) d\|V\| &\leq \alpha\|V\|(\mathbb{R}^n) + \beta\|\delta V\|(\mathbb{R}^n) \\ &\leq \liminf_{k \rightarrow \infty} \sum_{i \in I(k)} \int_{C_{k,i}} (\alpha + \beta|\kappa_{C_{k,i}}|) d\mathcal{H}^1. \end{aligned}$$

Besides, as the convergence of the curves holds in the sense of currents, we know that $\mathcal{H}^1 \llcorner \mathcal{C} = \|V_{\mathcal{C}}\| \leq \|V\|$, where $V_{\mathcal{C}} = \mathbf{v}(\mathcal{C}, 1)$. Since both $V_{\mathcal{C}}$ and V have locally bounded first variation, it is a consequence of Theorem 2.1 that the curvatures of $V_{\mathcal{C}}$ and V coincide \mathcal{H}^1 -almost everywhere on \mathcal{C} . In conclusion, for every $p \geq 1$,

$$\begin{aligned} \sum_{i \in I} \int_{C_i} (\alpha + \beta|\kappa_{C_i}|^p) d\mathcal{H}^1 &\leq \int_{\mathbb{R}^n} (\alpha + \beta|\kappa_V|^p) d\|V\| \\ &\leq \liminf_{k \rightarrow \infty} \sum_{i \in I(k)} \int_{C_{k,i}} (\alpha + \beta|\kappa_{C_{k,i}}|^p) d\mathcal{H}^1 \end{aligned}$$

and the theorem ensues. \square

Remark 4.2. Using the same kind of arguments, the result can be extended to unions of $W^{2,p}$ curves in \mathbb{R}^n , $p \geq 1$.

Remark 4.3. In higher dimension, the elastica energy becomes the celebrated Willmore energy [20], that can also be generalized to arbitrary mean curvature exponent under the form

$$\int_S (\alpha + \beta|H_S|^p) d\mathcal{H}^k.$$

with S a smooth k -surface in \mathbb{R}^n and H_S its mean curvature vector. Our partial locality result for rectifiable k -varifolds in \mathbb{R}^n is not sufficient to prove the extension to smooth k -surfaces of the semicontinuity result for curves stated above. This is due to the fact that the limit varifold obtained in the proof of Theorem 4.1 might not have a locally constant multiplicity. For instance, consider the varifold \hat{V} obtained by adding the horizontal x -axis (with multiplicity 1) to the varifold V that we have built in section 2.2. Then, one immediately observes that the x -axis is contained in the support of $\|\hat{V}\|$, but the multiplicity $\hat{\theta}$ of \hat{V} is not locally constant at the points corresponding to the ‘‘fat’’ Cantor set ($\hat{\theta}$ takes both values 1 and 2 in any neighbourhood of such points). Therefore, Theorem 3.4 cannot be directly used in this situation.

Were the locality property true in general, one would obtain the lower semicontinuity result in any dimension k and codimension $n - k$, and for any $p \geq 1$. Currently, to our best knowledge, the most general lower semicontinuity result for the case $k > 1$ is due to R. Schätzle [18, Thm 5.1] and is valid when $p \geq 2$.

4.2 Relaxation of functionals for image reconstruction

Recall that for any smooth function $u : \mathbb{R}^n \rightarrow \mathbb{R}$ and for almost every $t \in \mathbb{R}$, $\partial\{u \geq t\}$ is a union of smooth hypersurfaces whose mean curvature at a point x is given by

$$H(x) = \operatorname{div} \frac{\nabla u}{|\nabla u|}(x).$$

Thus, for any open set $\Omega \subset \mathbb{R}^n$ and by application of the coarea formula, we get

$$\int_{-\infty}^{+\infty} \int_{\Omega \cap \partial\{u \geq t\}} (1 + |H_{\partial\{u \geq t\}}|^p) d\mathcal{H}^{n-1} dt = \int_{\Omega} |\nabla u| (1 + |\operatorname{div} \frac{\nabla u}{|\nabla u|}|^p) dx,$$

where the integrand of the right term is taken to be zero whenever $|\nabla u| = 0$. The minimization of the energy

$$\mathcal{F}(u) := \int_{\Omega} |\nabla u| (1 + |\operatorname{div} \frac{\nabla u}{|\nabla u|}|^p) dx$$

has been proposed in the context of digital image processing [13, 12, 14] as a variational criterion for the restoration of missing parts in an image. It is therefore natural to study the connections between $\mathcal{F}(u)$, and its relaxation $\overline{\mathcal{F}}(u)$ with respect to the convergence of functions in L^1 . In particular, the question whether $\mathcal{F}(u) = \overline{\mathcal{F}}(u)$ for smooth functions has been addressed in [4] and a positive answer has been given whenever $n \geq 2$ and $p > n - 1$. Following the same proof line combined with our Theorem 4.1 and with Schätzle's Theorem 5.1 in [18], one can prove the following :

Theorem 4.4. *Let $u \in C^2(\mathbb{R}^n)$. Then*

$$\overline{\mathcal{F}}(u) = \mathcal{F}(u) \quad \text{whenever} \quad \begin{cases} n = 2 & \text{and } p \geq 1 & \text{or} \\ n \geq 3 & \text{and } p \geq 2 \end{cases}$$

Proof. Let $(u_h)_{h \in \mathbb{N}} \subset L^1(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$ converge to u in $L^1(\mathbb{R}^n)$ and set $L := \liminf_{h \rightarrow \infty} \mathcal{F}(u_h)$, assuming with no loss of generality that $L < \infty$. Using Cavalieri's formula and possibly taking a subsequence, it follows that for almost every $t \in \mathbb{R}$,

$$\mathbb{1}_{\{u_h \geq t\}} \rightarrow \mathbb{1}_{\{u \geq t\}} \quad \text{in } L^1(\mathbb{R}^n).$$

Observing that, by Sard's Lemma, $\{u_h \geq t\}$, $h \in \mathbb{N}$, and $\{u \geq t\}$ have smooth boundaries for almost every $t \in \mathbb{R}$, we get that $\partial\{u_h \geq t\}$ converges to $\partial\{u \geq t\}$ in the sense of rectifiable currents for almost every $t \in \mathbb{R}$ [19]. Therefore, applying either Theorem 4.1 or Theorem 5.1 in [18], we obtain that for almost every $t \in \mathbb{R}$

$$\int_{\partial\{u \geq t\}} (1 + |H_{\{u \geq t\}}|^p) d\mathcal{H}^{n-1} \leq \liminf_{h \rightarrow \infty} \int_{\partial\{u_h \geq t\}} (1 + |H_{\{u_h \geq t\}}|^p) d\mathcal{H}^{n-1}$$

whenever $\begin{cases} n = 2 & \text{and } p \geq 1 & \text{or} \\ n \geq 3 & \text{and } p \geq 2 \end{cases}$

Integrating over \mathbb{R} and using Fatou's lemma, we get

$$\mathcal{F}(u) \leq \liminf_{h \rightarrow \infty} \mathcal{F}(u_h),$$

thus \mathcal{F} is lower semicontinuous in the class of C^2 functions and coincides with $\overline{\mathcal{F}}$ on that class. \square

A Monotonicity identity and isoperimetric inequality

See the note to the reader at the end of the introduction.

In this section we recall some fundamental results of the theory of varifolds, which can be found in [1] (see also [19, 10]). We also provide their proofs, for convenience of the reader. The first result is the following

Theorem A.1 (Monotonicity identity). *Let V be a k -varifold in \mathbb{R}^n with locally bounded first variation. Denoting $\mu(t) := \|V\|(\overline{B}_t)$ and*

$$Q(t) = \frac{k}{t} - \frac{1}{t\mu(t)} \frac{d}{dt} \int_{B_t \times G_{n,k}} \frac{|x_S|^2}{|x|} dV(x, S), \quad (\text{A.1})$$

it holds

$$\frac{\mu(r)}{r^k} \exp\left(\int_{\rho}^r Q(t) dt\right) - \frac{\mu(\rho)}{\rho^k} = \int_{(\overline{B}_r \setminus B_{\rho}) \times G_{n,k}} \frac{|x_{S^{\perp}}|}{|x|^{k+2}} \exp\left(\int_{\rho}^{|x|} Q(t) dt\right) dV(x, S) \quad (\text{A.2})$$

Formula (A.2) shows the interplay between some crucial quantities associated to a varifold V with locally bounded first variation δV . The local boundedness of δV is needed basically to apply Riemann-Stieltjes integration by parts, and is meaningful also in the following key estimate:

$$|Q(t)| \leq \frac{\|\delta V\|(\overline{B}_t)}{\|V\|(\overline{B}_t)}. \quad (\text{A.3})$$

In particular, from (A.2) and (A.3) one deduces that any stationary varifold V , i.e. such that $\delta V = 0$, must satisfy the well-known monotonicity inequality

$$\frac{\mu(r)}{r^k} - \frac{\mu(\rho)}{\rho^k} = \int_{(\overline{B}_r \setminus \overline{B}_\rho) \times G_{n,k}} \frac{|x_{S^\perp}|}{|x|^{k+2}} dV(x, S) \geq 0, \quad 0 < \rho < r < \infty. \quad (\text{A.4})$$

Identity (A.2) holds for balls centered at a point $a \in \mathbb{R}^n$ close to the support of the varifold, and will be obtained following the technique sketched here (with the assumption $a = 0$):

- (i) the first variation is calculated on a smooth, radially symmetric vector field $g_\theta(x) = \theta(|x|)x$, where $\theta \in \mathcal{D}(\mathbb{R})$;
- (ii) the term $\delta V(g_\theta)$ is, then, written in two equivalent forms, only using the fact that $|x|^2 = |x_S|^2 + |x_{S^\perp}|^2$, where x_S and x_{S^\perp} denote, respectively, the tangential and the orthogonal component of the vector x with respect to the k -plane S ;
- (iii) the resulting identity is represented in terms of one-dimensional Riemann-Stieltjes integrals, and then interpreted as the nullity of a certain distribution $\Psi(\theta)$;
- (iv) finally, to obtain (A.2) one has to test the null distribution Ψ on a suitably chosen, absolutely continuous function $f : [\rho, r] \rightarrow \mathbb{R}$, with $0 < \rho < r < \infty$.

Then, (A.2) can be used to prove the following general *isoperimetric inequality*

Theorem A.2 (Isoperimetric inequality for varifolds). *There exists a constant $C > 0$ such that, for every k -varifold V with locally bounded first variation and every $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $\varphi \geq 0$,*

$$\int_{\{x: \varphi(x)\Theta^k(\|V\|, x) \geq 1\}} \varphi d\|V\| \leq C \left(\int_{\mathbb{R}^n} \varphi d\|V\| \right)^{\frac{1}{k}} \left(\int_{\mathbb{R}^n} \varphi d\|\delta V\| + \int_{\mathbb{R}^n \times G_{n,k}} |\nabla^S \varphi| dV \right) \quad (\text{A.5})$$

The localization of this inequality yields the relative isoperimetric inequality (3.1) shown in section 3.

A.1 Basic facts on Riemann-Stieltjes integrals and consequences

Before entering the proof of (A.2), we recall some basic facts concerning Riemann-Stieltjes integrals of functions of one real variable (see 2.5.17 and 2.9.24 in [10]) and show how they can be used to represent integrals of certain functions with respect to Radon measures on \mathbb{R}^n or $\mathbb{R}^n \times G_{n,k}$. Suppose that $g : [a, b] \rightarrow \mathbb{R}$ is a function of bounded variation, then for every continuous function $f : [a, b] \rightarrow \mathbb{R}$ one can define the *Riemann-Stieltjes integral*

$$\int_a^b f(t) dg(t) = \sup \sum_{i=1}^N f(t_i)(g(a_{i+1}) - g(a_i)), \quad (\text{A.6})$$

where the supremum is calculated over all subdivisions $a_1 = a < a_2 < \dots < a_{N+1} = b$ and all t_1, \dots, t_N such that $t_i \in [a_i, a_{i+1}]$, for $i = 1, \dots, N$.

Proposition A.3. [10, 2.9.24] *Let $f, \theta : [a, b] \rightarrow \mathbb{R}$ be continuous functions and assume θ is absolutely continuous on $[a, b]$. Then*

$$\int_a^b f(t) d\theta(t) = \int_a^b f\theta' d\mathcal{L}^1. \quad (\text{A.7})$$

Moreover, if g has bounded variation in $[a, b]$ then

$$\int_a^b g\theta' d\mathcal{L}^1 + \int_a^b \theta(t) dg(t) = g(b)\theta(b) - g(a)\theta(a). \quad (\text{A.8})$$

Next, we apply the Riemann-Stieltjes integral to reduce integrals with respect to Radon measures defined on the Grassmann bundle $\mathbb{R}^n \times G_{n,k}$ to one-dimensional integrals, as shown in the following

Proposition A.4. *Let V be a k -varifold, let $\varphi : \mathbb{R}^n \times G_{n,k} \rightarrow \mathbb{R}$ be non-negative and measurable, and let θ be absolutely continuous on $[\rho, r]$. Then*

$$\int_{(\overline{B}_r \setminus \overline{B}_\rho) \times G_{n,k}} \theta(|x|) \varphi(x, S) dV(x, S) = \int_\rho^r \theta(t) dg(t), \quad (\text{A.9})$$

where we have set

$$g(t) = \int_{(\overline{B}_t \setminus \overline{B}_\rho) \times G_{n,k}} \varphi(x, S) dV(x, S).$$

Proof. Simply write the integral in the left-hand side of (A.9) as a sum of integrals over differences of concentric balls. Then, the proof follows from (A.6). \square

In the following lemma, we introduce some special functions of one real variable that will be used later in the proof of the monotonicity identity (A.2). We first define an opportune test vector field $X_{t,\epsilon}(x)$: given

$$\eta_\epsilon(r) = \begin{cases} 1 & \text{if } r \leq 1 \\ 1 - (r - 1)/\epsilon & \text{if } 1 < r \leq 1 + \epsilon \\ 0 & \text{otherwise,} \end{cases}$$

we set for $t, \epsilon > 0$ and $x \in \mathbb{R}^n$

$$X_{t,\epsilon}(x) = \eta_\epsilon(t^{-1}|x|)x. \quad (\text{A.10})$$

Given a k -plane S , we compute

$$\operatorname{div}_S X_{t,\epsilon}(x) = k \eta_\epsilon(t^{-1}|x|) - \frac{1}{\epsilon t} \frac{|x_S|^2}{|x|} \mathbb{1}_{B_{t(1+\epsilon)} \setminus B_t}(x).$$

Lemma A.5. *Let V be a varifold with locally bounded first variation δV . Given $t \in \mathbb{R}$, we define*

$$\mu(t) = \int_{\overline{B}_t \times G_{n,k}} dV(x, S) \quad (\text{A.11})$$

$$\xi(t) = \int_{\overline{B}_t \times G_{n,k}} \frac{|x_{S^\perp}|^2}{|x|} dV(x, S) \quad (\text{A.12})$$

$$\nu(t) = k\mu(t) - \frac{d}{dt} \int_{\overline{B}_t \times G_{n,k}} \frac{|x_S|^2}{|x|} dV(x, S) \quad (\text{A.13})$$

for $t > 0$, and zero elsewhere, with the convention that the integrands are zero in (A.12) and (A.13) whenever $x = 0$. Then, the functions defined above are right-continuous and of bounded variation on \mathbb{R} . Moreover, the function $Q(t) = \frac{\nu(t)}{t\mu(t)}$, defined when t and $\mu(t)$ are both positive, satisfies $Q(t) \leq \frac{\|\delta V\|(\overline{B}_t)}{\mu(t)}$.

Proof. Clearly, $\mu(t)$ and $\xi(t)$ are right-continuous and non-decreasing, thus of bounded variation. On the other hand, one can easily see that, for almost all $t > 0$,

$$\nu(t) = \lim_{\epsilon \rightarrow 0^+} \delta V(X_{t,\epsilon}).$$

Therefore, by taking $0 < r < t$ one has

$$\begin{aligned} |\nu(t) - \nu(r)| &= \left| \lim_{\epsilon \rightarrow 0^+} \delta V(X_{t,\epsilon} - X_{r,\epsilon}) \right| \\ &\leq \liminf_{\epsilon \rightarrow 0^+} t(1 + \epsilon) \|\delta V\|(B_{t(1+\epsilon)} \setminus \overline{B}_r) \\ &= t[\|\delta V\|(\overline{B}_t) - \|\delta V\|(\overline{B}_r)]. \end{aligned}$$

Since $\|\delta V\|$ is a Radon measure, we conclude that $\nu(t)$ is of bounded variation. Moreover, one has

$$\limsup_{t \rightarrow r^+} |\nu(t) - \nu(r)| \leq \lim_{t \rightarrow r^+} t [\|\delta V\|(\overline{B}_t) - \|\delta V\|(\overline{B}_r)] = 0,$$

hence $\nu(t)$ is right-continuous at almost all $t \in \mathbb{R}$. The last assertion about $Q(t)$ is also an immediate consequence of the previous estimates on $\nu(t)$. \square

A.2 Proof of the monotonicity identity (A.2)

We test the first variation of V on a radial vector field Y of the form $Y(x) = \theta(|x|)x$, where $\theta \in \mathcal{D}(\mathbb{R})$. A simple approximation argument shows that the support of θ' may even contain 0 for the proof below to be valid, thus all functions $\theta \in \mathcal{D}(\mathbb{R})$ are allowed for testing. Hence, setting $t = |x|$ we have

$$\delta V(Y) = \int_{G_k(\mathbb{R}^n)} \operatorname{div}_S Y(x) dV(x, S) = \int_{G_k(\mathbb{R}^n)} [\theta'(t) \frac{|x_S|^2}{t} + k\theta(t)] dV(x, S). \quad (\text{A.14})$$

Thanks to the identity $|x|^2 = |x_S|^2 + |x_{S^\perp}|^2$ we rewrite the right-hand side of (A.14) as follows

$$\int_{G_k(\mathbb{R}^n)} [\theta'(t) \frac{|x_S|^2}{t} + k\theta(t)] dV(x, S) = \int_{G_k(\mathbb{R}^n)} [t\theta'(t) + k\theta(t)] dV(x, S) - \int_{G_k(\mathbb{R}^n)} \theta'(t) \frac{|x_{S^\perp}|^2}{t} dV(x, S). \quad (\text{A.15})$$

Defining $\mu(t)$, $\xi(t)$, $\nu(t)$ as in Lemma A.5, and owing to Propositions A.3 and A.4, we can write (A.15) as

$$- \int \nu(t)\theta'(t) dt = \int t\theta'(t) d\mu(t) - k \int \mu(t)\theta'(t) dt - \int \theta'(t) d\xi(t). \quad (\text{A.16})$$

Integrating by parts (see formula (A.8)) we obtain

$$\int [\nu(t) - k\mu(t)]\theta'(t) dt + \int \theta'(t) t d\mu(t) - \int \theta'(t) d\xi(t) = 0. \quad (\text{A.17})$$

In other words, let $\Psi \in \mathcal{D}'(\mathbb{R})$ be the distribution defined by

$$\Psi(\theta) = \int [\nu(t) - k\mu(t)]\theta(t) dt + \int \theta(t) t d\mu(t) - \int \theta(t) d\xi(t).$$

Clearly, (A.17) says that the distributional derivative of Ψ is zero, hence Ψ must be equal to a constant $c \in \mathbb{R}$. On the other hand, choosing $\theta(t) = 0$ for all $t \geq 0$ one concludes that $c = 0$, that is,

$$\Psi(\theta) = \int [\nu(t) - k\mu(t)]\theta(t) dt + \int \theta(t) t d\mu(t) - \int \theta(t) d\xi(t) = 0 \quad (\text{A.18})$$

for all $\theta \in \mathcal{D}(\mathbb{R})$. By approximation, one gets (A.18) valid for all absolutely continuous θ with compact support in \mathbb{R} . Another integration by part as in (A.8) lets us write (A.18) in the form

$$\int_{\rho}^r \{[\nu(t) - k\mu(t)]f(t) - (t f(t))' \mu(t)\} dt + r f(r) \mu(r) - \rho f(\rho) \mu(\rho) = \int_{\rho}^r f(t) d\xi(t), \quad (\text{A.19})$$

which is true for any absolutely continuous function $f : [\rho, r] \rightarrow \mathbb{R}$.

To conclude, we only need to choose $f(t)$ in order that the first integral in (A.19) becomes zero. Another requirement is the term $\rho f(\rho)$ to be equal to ρ^{-k} . In conclusion, we simply take $f(t)$ as the solution to the following Cauchy problem:

$$\begin{cases} \mu(t)(t f(t))' = [\nu(t) - k\mu(t)]f(t) & t \in [\rho, r], \\ f(\rho) = \rho^{-k-1}, \end{cases}$$

that is,

$$f(t) = t^{-k-1} \exp \int_{\rho}^t \frac{\nu(\tau)}{\tau \mu(\tau)} d\tau. \quad (\text{A.20})$$

Defining $Q(t) = \frac{\nu(t)}{t\mu(t)}$ for $t > 0$ and plugging (A.20) into (A.19), one obtains (A.2) as wanted.

A.3 Proof of the isoperimetric inequality (A.5)

Define the varifold $V_\varphi = \varphi V$, such that

$$V_\varphi(\alpha) = \int \alpha(x, S) \varphi(x) dV(x, S)$$

for all $\alpha \in C_c^0(\mathbb{R}^n \times G_{n,k})$, and assume that $\|\delta V_\varphi\|$ is a Radon measure (otherwise the result holds trivially). Fix $\lambda \in (1, +\infty)$ and define a suitable radius

$$s = \left(\frac{\lambda \|V_\varphi\|(\mathbb{R}^n)}{\omega_k} \right)^{\frac{1}{k}}.$$

Take $a \in \mathbb{R}^n$ and suppose $\theta(a)\varphi(a) \geq 1$. The monotonicity identity (A.2) thus implies

$$\exp \int_r^s Q_\varphi(t) dt \geq \frac{s^k \|V_\varphi\|(\overline{B(a, r)})}{r^k \|V_\varphi\|(\overline{B(a, s)})}, \quad (\text{A.21})$$

where $Q_\varphi(t)$ is defined as in (A.1), with V_φ replacing V . From (A.21) we infer that

$$\begin{aligned} \liminf_{r \rightarrow 0^+} \exp \int_r^s Q_\varphi(t) dt &\geq \omega_k \theta(a) \varphi(a) \frac{s^k}{\|V_\varphi\|(\overline{B(a, s)})} \\ &\geq \omega_k \theta(a) \varphi(a) \frac{s^k}{\|V_\varphi\|(\mathbb{R}^n)} \\ &= \theta(a) \varphi(a) \lambda \\ &\geq \lambda, \end{aligned}$$

that is,

$$\liminf_{r \rightarrow 0^+} \int_r^s Q_\varphi(t) dt \geq \log \lambda > 0.$$

From Lemma A.5 and the previous inequality, we get

$$\liminf_{r \rightarrow 0^+} \int_r^s \frac{\|\delta V_\varphi\|(\overline{B(a, t)})}{\|V_\varphi\|(\overline{B(a, t)})} dt \geq \log \lambda,$$

thus for any $0 < \epsilon < \log \lambda$ there exists $\hat{r} = \hat{r}(a, \epsilon)$ such that, for all $0 < r < \hat{r}$,

$$\int_r^s \frac{\|\delta V_\varphi\|(\overline{B(a, t)})}{\|V_\varphi\|(\overline{B(a, t)})} dt \geq \log \lambda - \epsilon,$$

whence the existence of $\hat{t} \in (0, s)$ for which

$$\frac{s \|\delta V_\varphi\|(\overline{B(a, \hat{t})})}{\|V_\varphi\|(\overline{B(a, \hat{t})})} \geq \log \lambda - \epsilon \quad (\text{A.22})$$

holds true. By the Besicovich Covering Theorem we deduce

$$\|V_\varphi\|(\{a : \theta(a)\varphi(a) \geq 1\}) \leq C_B \frac{s}{\log \lambda - \epsilon} \|\delta V_\varphi\|(\mathbb{R}^n) = C_B \frac{\|V_\varphi\|(\mathbb{R}^n)^{\frac{1}{k}}}{\omega_k^{\frac{1}{k}}} \frac{\lambda^{\frac{1}{k}}}{\log \lambda - \epsilon} \|\delta V_\varphi\|(\mathbb{R}^n). \quad (\text{A.23})$$

Therefore, the minimization of the function $\lambda \rightarrow \frac{\lambda^{\frac{1}{k}}}{\log \lambda - \epsilon}$ on the interval $(\exp(\epsilon), +\infty)$ leads to the optimal choice $\tilde{\lambda} = \exp(k + \epsilon)$, for which

$$\frac{\tilde{\lambda}^{\frac{1}{k}}}{\log \tilde{\lambda} - \epsilon} = \frac{\exp(1 + \epsilon/k)}{k}.$$

Then, passing to the limit in (A.23) as $\epsilon \rightarrow 0^+$, we obtain

$$\int_{\{a: \varphi(a)\theta(a) \geq 1\}} \varphi d\|V\| \leq C \left(\int \varphi d\|V\| \right)^{\frac{1}{k}} \|\delta V_\varphi\|(\mathbb{R}^n).$$

Combining with

$$\|\delta V_\varphi\|(\mathbb{R}^n) \leq \int \varphi d\|\delta V\| + \int |\nabla^S \varphi(x)| dV(x, S),$$

we obtain (A.5).

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