# Positivity of some intersections in $K_0(G/B)$ *
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*This paper is respectfully dedicated to David Buchsbaum.*

*0. Introduction:* Start with a few notations. Let $G$ be a simply connected connected simple algebraic group over $\mathbf{C}$, let $B$ be a Borel subgroup, let $H$ be a Cartan subgroup of $B$, let $P$ be the weight lattice, let $P^+$ be the set of dominant weights and let $W$ be the Weyl group of $G$. For $\lambda \in P$, we denote by $\mathbf{C}(\lambda)$ the corresponding one-dimensional $B$-module and by $\mathcal{L}(\lambda)$ the sheaf of sections of the line bundle $\mathbf{C}(\lambda) \times_B G \to G/B$. With these definitions, we recall that $\Gamma(G/B, \mathcal{L}(\lambda)) \neq 0$ if and only if $-\lambda \in P^+$. For any $w \in W$, let $S_w = \overline{BwB}/B$ be the Schubert variety, let $\mathcal{O}_w$ be its structural sheaf, and set $\mathcal{L}_w(\lambda) = \mathcal{L}(\lambda) \otimes \mathcal{O}_w$. For any coherent sheaf $\mathcal{M}$ on $G/B$ we denote by $[\mathcal{M}]$ its image in the Grothendieck ring $K(G/B) := K_0(G/B)$ of the category of coherent sheaves on $G/B$.

It follows from the cell decomposition of $G/B$ that $([\mathcal{O}_w])_{w \in W}$ is a $\mathbf{Z}$-basis of $K(G/B)$. Therefore, for any $v, w \in W$, $\mu \in P$, we can define the integer $s_w^v(\mu)$ by the following identity in $K(G/B)$:
$$[\mathcal{L}_w(-\mu)] = \sum_{v \in W} s_w^v(\mu)[\mathcal{O}_v].$$

**Theorem 0.1:** *For any $w, v \in W$ and $\lambda \in P^+$, the integer $s_w^v(\lambda)$ is $\geq 0$.*

To explain the title of the paper, note that $[\mathcal{L}_w(-\mu)]$ is the intersection product of $[\mathcal{L}(-\mu)]$ with $S_w$ in $K(G/B)$. This result has been proved by Fulton and Lascoux [FL94] for the group $G = SL(n)$, and *their paper was the motivation of the present work.* We will show that Theorem 0.1 is a very simple corollary of the main result of [$\mathbf{M_2}$89]. Indeed our proof is slighty more precise. Let $K_H(G/B)$ be the Grothendieck ring of the category of the $H$-equivariant coherent sheaves on $G/B$ and for any $H$-equivariant coherent sheaf $\mathcal{M}$ on $G/B$ we denote by $[\mathcal{M}]_H$ its image in $K_H(G/B)$. By [KK87], $([\mathcal{O}_w]_H)_{w \in W}$ is a $\mathbf{Z}[P]$-basis of $K_H(G/B)$ (see also [Ar86]), and therefore we can define the characters $\sigma_w^v(\mu)$ by the following identities in $K_H(G/B)$:
$$[\mathcal{L}_w(-\mu)]_H = \sum_{v \in W} \sigma_w^v(\mu)[\mathcal{O}_v]_H.$$
We prove that for any dominant weight $\lambda$, the characters $\sigma_w^v(\lambda)$ are effective, hence their degrees $s_w^v(\lambda)$ are non-negative integers. Also in [$\mathbf{M_3}$89], we proved that for any $\lambda \in P^+$, the sheaf $\mathcal{O}_w \otimes \mathbf{C}(-\lambda)$ has a $B$-equivariant filtration whose any subquotient is some $\mathcal{L}_v(\mu)$. At the end of Section 5, we will explain why this result is closely related with the effectivity of $\sigma_w^v(\lambda)$.

Using the same ideas, it easy to deduce two other posivity results. These results hold in $K_H(G/B)$, but for simplicity we will state them in $K(G/B)$. For $w \in W$, set $\partial S_w = \cup_{v < w} S_v$, let $\mathcal{I}_w \subset \mathcal{O}_w$ be the ideal defining $\partial S_w$ in $S_w$ and set $\mathcal{I}_w(\lambda) = \mathcal{L}(\lambda) \otimes \mathcal{I}_w$. It is clear that $([\mathcal{I}_w])_{w \in W}$ is another basis of $K(G/B)$. Therefore, we can define some integers $k_w^v(\mu)$ for $v, w \in W$, $\mu \in P$ by the following identities in $K(G/B)$:

$$[\mathcal{I}_w(-\mu)] = \sum_{v \in W} k_w^v(\mu)[\mathcal{I}_v].$$

**Theorem 0.2:** *For any* $w, v \in W$ *and* $\lambda \in P^+$, *the integer* $k_w^v(\lambda)$ *is* $\geq 0$.

Similarly we define the mixed numbers $m_w^v(\mu)$ by the following identity in $K(G/B)$:
$$[\mathcal{I}_w(-\mu)] = \sum_{v \in W} m_w^v(\mu)[\mathcal{O}_v].$$
For $\lambda \in P^+$, denote by $W_\lambda$ its stabilizer in $W$. Recall that each coset $w.W_\lambda$ is an ordered set which is isomorphic to $W_\lambda$. Therefore each $W_\lambda$-coset contains a unique maximal and a unique minimal representative.

**Theorem 0.3:** *Let* $w, v \in W$ *and* $\lambda \in P^+$. *If* $w$ *is minimal in* $w.W_\lambda$, *the integer* $m_w^v(\lambda)$ *is* $\geq 0$.

Indeed Theorem 0.3 is a corollary of [$\mathbf{M_2}$**89**] only when $\lambda$ is strictly regular. However it follows from a refinement of a result of [$\mathbf{M_2}$**89**], whose the proof is similar (a detailed proof will be given in a subsequent paper). Also we will also introduce some refined tensor product multiplicities in order to explain the relationship of these positivity results with the PRV conjecture (independently proved by Kumar [**K88**] and the author [$\mathbf{M_1}$**89**]) and the refined PRV conjecture (due to Kumar [**K89**]). At the end of the paper, we show some identities between the numbers $s_w^v(\mu)$, $k_w^v(\mu)$, $m_w^v(\mu)$, from which we deduce a generalization of Pieri's identity.

*Main notations: Throughout the whole paper, we will use the notations of the introduction, together with the following ones.* We will denote by $\Delta^+$ the set of positive roots, by $\Pi$ the subset of simple positive roots. We set $\rho = 1/2 \sum_{\alpha \in \Delta^+} \alpha$ and for $\alpha \in \Pi$ we denote by $s_\alpha$ the corresponding simple reflection. We will use that $W$ is an ordered set relative to the Bruhat ordering and $w_0$ denotes its maximal element. For $w \in W$, let $\epsilon(w)$ be its signature.

## 1. Demazure operators.

Denote by $\mathcal{H}$ the algebra $End_{\mathbf{C}(H)^W}(\mathbf{C}(H))$ where $\mathbf{C}(H)$ denotes the field of rational functions on $H$. By Galois theory, any $X \in \mathcal{H}$ can be uniquely written as $X = \sum_{w \in W} \phi_w w$ and also as $X = \sum_{w \in W} w \, \psi_w$, where $\phi_w, \psi_w \in \mathbf{C}(H)$. For $X \in \mathcal{H}$ and $\phi \in \mathbf{C}(H)$, it is important to distinguish the operator $X \phi \in \mathcal{H}$ from the character $X(\phi) \in \mathbf{C}(H)$: the first expression is the composition of $X$ by the multiplication by $\phi$, and the second one is the value of $X$ at $\phi$.

**Lemma 1.1:** *There is a unique anti-involution $X \mapsto X^*$ of the algebra $\mathcal{H}$ such that*
*(i) $\phi^* = \phi$, for any $\phi \in \mathbf{C}(H)$.*
*(ii) $w^* = \epsilon(w)e^{\rho - w^{-1}\rho} \, w^{-1}$.*

*Proof:* For any $\phi$, $\psi \in \mathbf{C}(H)$ and any $v$, $w \in W$, we have:
$$((\phi w) \circ (\psi v))^* = (\phi \psi^w w v)^*$$
$$= \epsilon(wv) \, e^{\rho - (wv)^{-1}\rho} \, (wv)^{-1}\phi\psi^w$$
and we also have:
$$(\psi v)^* \circ (\phi w)^* = (\epsilon(v) \, e^{\rho - v^{-1}\rho} \, v^{-1}\psi)(\epsilon(w) \, e^{\rho - w^{-1}\rho} \, w^{-1}\phi)$$
$$= \epsilon(vw) \, e^{\rho - v^{-1}\rho} \, e^{v^{-1}\rho - v^{-1}w^{-1}\rho} \, v^{-1}w^{-1}\psi^w\phi$$
$$= \epsilon(vw) \, e^{\rho - v^{-1}w^{-1}\rho} \, v^{-1}w^{-1}\psi^w\phi.$$
Hence $(XY)^* = Y^*X^*$ for any $X, Y \in \mathcal{H}$, and $*$ is anti-morphism of algebras. Moreover $w^{**} = (\epsilon(w)e^{\rho - w^{-1}\rho}w^{-1})^* = e^{\rho - w\rho} \, w \, e^{\rho - w^{-1}\rho} = w$, therefore $*$ is involutive. Q.E.D.

For any $\alpha \in \Pi$, set $\nabla_{s_\alpha} = \frac{1}{1-e^\alpha}(1-e^\alpha s)$. For any reduced decomposition $w = \sigma_1 \ldots \sigma_n$ of an element $w \in W$, set $\nabla_w = \nabla_{\sigma_1} \ldots \nabla_{\sigma_n}$. As it proved by Demazure [**D74**], the element $\nabla_w \in \mathcal{H}$ is independent on a choice of a reduced decomposition for $w$. It is called the *Demazure operator*. Moreover we have $(\nabla_{s_\alpha})^2 = \nabla_{s_\alpha}$. Therefore, for any $v$, $w \in W$, we have $\nabla_v \nabla_w = \nabla_{v*w}$, where the element $v * w \in W$ is defined by $\overline{BvB}.\overline{BwB} = \overline{B(v*w)B}$.

**Lemma 1.2:** *For any $\phi \in \mathbf{Z}[P]$ and for any $w \in W$, we have*
$$\phi\nabla_w = \sum_{v \in W} \nabla_v \, S_w^v(\phi),$$
*for some $S_w^v(\phi) \in \mathbf{Z}[P]$.*

*Proof:* It is clear that any $X \in \mathcal{H}$ can be uniquely written as $X = \sum_{v \in W} \nabla_v \phi_v$, with $\phi_v \in \mathbf{C}(H)$. Therefore we have $\phi\nabla_w = \sum_{v \in W} \nabla_v \, S_w^v(\phi)$, for some $S_w^v(\phi) \in \mathbf{C}(X)$. We only have to prove that $S_w^v(\phi)$ belongs to $\mathbf{Z}[P]$. Let $\alpha \in \Pi$. We have $\phi\nabla_{s_\alpha} = \frac{\phi}{1-e^\alpha}(1 - e^\alpha \, s_\alpha)$, therefore we get:
$$\phi\nabla_{s_\alpha} = \frac{\phi - \phi^{s_\alpha}}{1-e^\alpha} + \nabla_{s_\alpha}\phi^{s_\alpha}.$$
It is clear that $\frac{\phi - \phi^{s_\alpha}}{1-e^\alpha}$ belongs to $\mathbf{Z}[P]$. Then one proves the lemma by induction over $w$. For $w = 1$, the lemma is obvious. For $w \neq 1$, we set $w = s_\alpha u$, where $\alpha \in \Pi$ and $u < w$. It follows that $\phi\nabla_w = \frac{\phi - \phi^{s_\alpha}}{1-e^\alpha}\nabla_u + \nabla_{s_\alpha}\phi^{s_\alpha}\nabla_u$, therefore we get:
$$\phi\nabla_w = \sum_{v \in W} \nabla_v S_u^v(\frac{\phi - \phi^{s_\alpha}}{1-e^\alpha}) + \sum_{v \in W} \nabla_{s_\alpha * v} S_u^v(\phi^{s_\alpha}),$$
what proves the lemma. Q.E.D.

Denotes by $\mathcal{H}_{int}$ the subring of $\mathcal{H}$ generated by $\mathbf{Z}[P]$ and the Demazure operators $(\nabla_w)_{w \in W}$. The following statement follows from Lemma 1.2:

**Corollary 1.3:** *Any $X \in \mathcal{H}_{int}$ can be uniquely written as $X = \sum_{w \in W} \phi_w \nabla_w$ and as $X = \sum_{w \in W} \nabla_w \psi_w$, where the elements $\phi_w$, $\psi_w$ belong to $\mathbf{Z}[P]$.*

For any $\alpha \in \Pi$, set $\nabla'_{s_\alpha} = \frac{e^\alpha}{1-e^\alpha}(1 - s_\alpha)$. For any reduced decomposition $w = \sigma_1 \ldots \sigma_n$ of an element $w \in W$, set $\nabla'_w = \nabla'_{\sigma_1} \ldots \nabla'_{\sigma_n}$. The element $\nabla'_w \in \mathcal{H}$ is also well-defined. It is called the *modified Demazure operator*. Moreover we have $(\nabla'_{s_\alpha})^2 = -\nabla'_{s_\alpha}$. Therefore, for any $v$, $w \in W$, we have $\nabla'_v \nabla'_w = \epsilon(vw)\epsilon(v * w)\nabla'_{v*w}$.

**Lemma 1.4:** *For any $\phi \in \mathbf{Z}[P]$ and any $w \in W$, we have:*
$$\phi \nabla'_w = \sum_{v \in W} \nabla'_v K^v_w(\phi), \text{ and}$$
$$\phi \nabla'_w = \sum_{v \in W} \nabla_v M^v_w(\phi),$$
*for some $K^v_w(\phi), M^v_w(\phi) \in \mathbf{Z}[P]$.*

*Proof:* We have $\nabla_{s_\alpha} = 1 + \nabla'_{s_\alpha}$. By induction we get:
$$\nabla_w = \sum_{v \leq w} \nabla'_v, \text{ and } \nabla'_w = \epsilon(w) \sum_{v \leq w} \epsilon(v) \nabla_v.$$
Therefore $(\nabla'_w)_{w \in W}$ is another left and right basis of the $\mathbf{Z}[P]$-module $\mathcal{H}_{int}$ and the lemma follows. Q.E.D.

**Lemma 1.5:** *We have $(\nabla_w)^* = \nabla_{w^{-1}}$ and $(\nabla'_w)^* = \nabla'_{w^{-1}}$ for any $w \in W$.*

*Proof:* For any $\alpha \in \Pi$, we have $s^*_\alpha = -e^\alpha s_\alpha$ and
$$(\nabla_{s_\alpha})^* = \frac{1}{1-e^\alpha} + e^\alpha s_\alpha \frac{e^\alpha}{1-e^\alpha}$$
$$= \frac{1}{1-e^\alpha} + e^\alpha \frac{e^{-\alpha}}{1-e^{-\alpha}} s_\alpha$$
$$= \frac{1}{1-e^\alpha} - \frac{e^\alpha}{1-e^\alpha} s_\alpha.$$
Therefore $(\nabla_{s_\alpha})^* = \nabla_{s_\alpha}$. Choose a reduced decomposition $\sigma_1 \ldots \sigma_n$ of $w$. We get $(\nabla_w)^* = (\nabla_{\sigma_n})^* \ldots (\nabla_{\sigma_1})^* = \nabla_{w^{-1}}$. The proof for $\nabla'_w$ is similar. Q.E.D.

**Corollary 1.6:** *For any $\phi \in \mathbf{Z}[P]$ and any $w \in W$, we have:*
$$\nabla_w \phi = \sum_{v \in W} S^{v^{-1}}_{w^{-1}}(\phi) \nabla_v,$$
$$\nabla'_w \phi = \sum_{v \in W} K^{v^{-1}}_{w^{-1}}(\phi) \nabla'_v,$$
$$\nabla'_w \phi = \sum_{v \in W} M^{v^{-1}}_{w^{-1}}(\phi) \nabla_v.$$

*Proof:* By Lemma 1.2 and 1.5, we have
$$\nabla_w \phi = (\phi \nabla_{w^{-1}})^* = \left(\sum_{v \in W} \nabla_{v^{-1}} S^{v^{-1}}_{w^{-1}}(\phi)\right)^* = \sum_{v \in W} S^{v^{-1}}_{w^{-1}}(\phi) \nabla_v,$$
what proves the first equality. The last two equalities are similar (use Lemma 1.4 instead of Lemma 1.2). Q.E.D.

## 2. Filtrations of $B$-modules.

For any $w \in W$, $\lambda \in P^+$, set $J_w(\lambda) = \Gamma(S_w, \mathcal{L}_w(-\lambda))$ and $K_w(\lambda) = \Gamma(S_w, \mathcal{I}_w(-\lambda))$. It is easy to prove that the $B$-modules $J_w(\lambda)$ are never 0, and $J_w(\lambda) = J_v(\mu)$ if and only if $w\lambda = v\mu$. It is also easy to prove that $K_w(\lambda) \neq 0$ if and only if $w$ is minimal in $W_\lambda$. Moreover two non-zero $B$-modules $K_w(\lambda)$ and $K_v(\mu)$ are isomorphic if and only if $w = v$ and $\lambda = \mu$. For a $B$-equivariant coherent sheaf $\mathcal{M}$ on $G/B$, we set $\chi_B(\mathcal{M}) = \sum_{k \geq 0} (-1)^k \mathrm{ch}\, H^k(G/B, \mathcal{M})$.

**Theorem 2.1** (Demazure character formulas) *Let $\lambda \in P$ and $w \in W$. We have:*
*(i) $\chi_B(\mathcal{L}_w(\lambda)) = \nabla_w(e^\lambda)$. Moreover if $\lambda$ is dominant, we have $\mathrm{ch}\, J_w(\lambda) = \nabla_w(e^{-\lambda})$.*
*(ii) $\chi_B(\mathcal{I}_w(\lambda)) = \nabla'_w(e^\lambda)$. Moreover if $\lambda$ is dominant, we have $\mathrm{ch}\, K_w(\lambda) = \nabla'_w(e^{-\lambda})$.*

The reference for the theorem is [**MR85**][**An85**]. For any dominant weight $\lambda$, Demazure formulas follow from the vanishing of $H^k(G/B, \mathcal{L}_w(\lambda))$ and $H^k(G/B, \mathcal{I}_w(\lambda))$ for

any $k > 0$. Formula 2.1(i) first appears in [**D74**]. Unfortunately, V. Kac found a serious gap in the Demazure's beautiful paper [**D74**]. This gap has been filled by the introduction of Frobenius splittings in [**MR85**] and by slighty different method in [**A85**].

We say that a $B$-module $M$ has a *Joseph filtration* (respectively: a *van der Kallen filtration*) if it admits a filtration whose any subquotient is some $J_w(\lambda)$ (respectively: some $K_w(\lambda)$).

**Theorem 2.2** *Let $\lambda$, $\mu \in P^+$ and $w \in W$. Then:*
*(i) $J_w(\mu) \otimes \mathbf{C}(-\lambda)$ has a Joseph filtration and*
*(ii) $K_w(\mu) \otimes \mathbf{C}(-\lambda)$ has a van der Kallen filtration.*

The reference for this theorem is [**M$_2$89**]. The Joseph filtrations has been first considered in [**J85**] and Joseph conjectured Theorem 2.2 (i) for $w = w_0$. The first step of a Joseph filtration of $J_w(\mu) \otimes \mathbf{C}(-\lambda)$ is $J_u(\nu)$, where $\lambda + w\mu = v\nu$: this fact, which is a crucial ingredient in the proof of PRV conjecture independently proved by S. Kumar and the author in [**K88**][**M$_2$89**], is an easy consequence of Joseph's paper. Then for $G = SL(n)$, Theorem 2.1 (i) has been first proved by Polo [**P89**] by an had hoc method (a refinement of Wang's trick [**W82**]) based on the fact that any fundamental weight is minuscule. The van der Kallen filtrations has been introduced in [**vdK89**] and van der Kallen proved the remarkable cohomological criterion 7.1 for the existence of a Joseph filtration or a van der Kallen filtration. It follows that Assertions (i) and (ii) are equivalent. For an arbitrary group $G$, Theorem 2.1 was proved by the author [**M$_2$89**] and the proof is strongly based on van der Kallen criterion [**vdK89**] and the Frobenius splittings invented by Metha, Ramanan and Ramanathan (it was stated as an open question in [**P89**]). See also later works [**M$_3$89**][**M90**] (extension to finite characteristics), [**P93**] (a more elementary proof for some cases) and [**vdK93**] (a very nice introduction to the subject).

**Theorem 2.3:** *Let $w \in W$, $\lambda$, $\mu \in P^+$. Assume that $w^{-1}$ is minimal in $w^{-1}.W_\lambda$. Then $K_w(\mu) \otimes \mathbf{C}(-\lambda)$ has a Joseph filtration.*

The references for this theorem is as follows: it is proved in [**M$_2$89**] under the condition $\lambda \in \rho + P^+$. The proof of this refined version is similar, and the details will be given in a subsequent paper.

*Remark:* As a matter of terminology, the Joseph filtrations has been previously called strong or excellent, and the van der Kallen filtration has been called weak or relative Schubert. The terminology "excellent filtrations" conflicts with the well-established terminology "good filtrations" (not defined in the present paper). For any union of Schubert varieties $S$ and any dominant weight $\lambda$, set $J_S(\lambda) = \Gamma(S, \mathcal{L}(-\lambda) \otimes \mathcal{O}_S)$. Following [**P89**], the modules $J_S(\lambda)$ are called the *Schubert modules*, and we call a *Polo filtration* of a $B$-module any filtration whose any subquotient is a Schubert module. Any Joseph filtration is obviously a Polo filtration. Moreover any Polo filtration can be refined to a van der Kallen filtration (what justified the strong/weak terminlogy). For any two unions of Schubert varieties $S_1$, $S_2$, we have $\operatorname{ch} J_{S_1}(\lambda) + \operatorname{ch} J_{S_2}(\lambda) = \operatorname{ch} J_{S_1 \cup S_2}(\lambda) + \operatorname{ch} J_{S_1 \cap S_2}(\lambda)$, therefore the multiplicity of a Schubert module in a Polo filtration depends on the filtration. However

in [**L98**] there is a combinatorial description of the multiplicities of Schubert modules in a Polo filtration of $J_w(\mu) \otimes C(-\lambda)$.

## 3. The characters $S$, $K$ and $M$ are effective.

Let $\mu \in P$, $w$, $v \in W$. Recall from Section 1 that we define the characters $S_w^v(e^\mu)$, $K_w^v(e^\mu)$, $M_w^v(e^\mu) \in \mathbf{Z}[P]$ by the following equalities in $\mathcal{H}$:

$$e^\mu \nabla_w = \sum_{v \in W} \nabla_v S_w^v(e^\mu),$$
$$e^\mu \nabla'_w = \sum_{v \in W} \nabla'_v K_w^v(e^\mu),$$
$$e^\mu \nabla'_w = \sum_{v \in W} \nabla_v M_w^v(e^\mu).$$

Denote by $\mathbf{N}$ the set of non-negative integers.

**Theorem 3.1:** *Let $\lambda \in P^+$, $w$, $v \in W$.*
*(i) $S_w^v(e^{-\lambda})$ and $K_w^v(e^{-\lambda})$ belongs to $\mathbf{N}[P]$.*
*(ii) If $w^{-1}$ is minimal in $w^{-1}.W_\lambda$, then $M_w^v(e^{-\lambda})$ belongs to $\mathbf{N}[P]$.*

*Proof:* Fix $w \in W$ and $\lambda \in P^+$. Set $S_w^v(e^{-\lambda}) = \sum_{\xi \in P} m_v^\xi e^{-\xi}$ and $\Omega = \{\xi \in P | m_v^\xi \neq 0 \text{ for some } v\}$. Let $\mu \in P^+$ and assume that $\mu$ is far away from the walls. Then $\mu - \xi \in P^{++}$ for any weight $\xi \in \Omega$, where $P^{++} = \rho + P^+$. By Demazure formula 2.1, we get:

$$e^{-\lambda} \mathrm{ch}\, J_w(\mu) = \sum_{v \in W} \sum_{\xi \in \Omega} m_v^\xi \mathrm{ch}\, J_v(\mu + \xi).$$

Up to repetitions, the family of characters $(\mathrm{ch}\, J_x(\nu))_{x \in W \nu \in P^+}$ form a basis of $\mathbf{Z}[P]$. Therefore the characters $(\mathrm{ch}\, J_x(\nu))_{x \in W \nu \in P^{++}}$ are linearly independent. By Theorem 2.2(i), $J_w(\lambda) \otimes \mathbf{C}(-\mu)$ has a Joseph filtration. Hence $m_v^\xi$ is the multiplicity of $J_v(\mu + \xi)$ in a Joseph filtration of $J_w(\mu) \otimes \mathbf{C}(-\lambda)$, for any $\xi \in \Omega$, $v \in W$, when $\mu \in P^+$ is far away from the walls. Therefore $m_v^\xi \geq 0$ and $S_w^v(e^{-\lambda})$ belongs to $\mathbf{N}[P]$.

The proof for $K_w^v(e^{-\mu})$ and $M_w^v(e^{-\mu})$ is similar: use Theorem 2.2(ii) and Theorem 2.3 instead of Theorem 2.2(i).

## 4. Harmonic functions on $\mathfrak{h}^*$.

Let $\mathfrak{h}$ be the Lie algebra of $H$. Identify $S\mathfrak{h}^*$ with the space of translation-invariant differential operators on $\mathfrak{h}^*$, and denote by $S^+ \mathfrak{h}^*$ its maximal homogenous ideal (therefore the elements of $S^+ \mathfrak{h}^*$ are differential operators without constant term). A polynomial function $F$ on $\mathfrak{h}^*$ is called *harmonic* if and only if $\theta F = 0$ for any $\theta \in (S^+ \mathfrak{h}^*)^W$. We say that a complex valued function $f$ defined on $P$ or on $P^+$ is *harmonic* if it is the restriction of a harmonic polynomial function $F$ on $\mathfrak{h}^*$ (as $P$ and $P^+$ are Zariski dense in $\mathfrak{h}^*$, $F$ is uniquely determined by $f$). Denote by $Har$ the space of harmonic functions on $P$. As $S\mathfrak{h}$ is a free $(S\mathfrak{h})^W$-module, $Har$ is a regular representation of $W$ and its dimension is $\mathrm{Card}\, W$. For any coherent sheaf $\mathcal{M}$ on $G/B$, denote by $\chi(\mathcal{M}) = \sum_{k \geq 0} (-1)^k \dim H^k(G/B, \mathcal{M})$ its Euler characteristic.

**Theorem: 4.1**
*(i) For any coherent sheaf $\mathcal{M}$ on $G/B$, the map $\lambda \in P \mapsto \chi(\mathcal{M} \otimes \mathcal{L}(-\lambda))$ is harmonic.*
*(ii) The map $\mathbf{C} \otimes K_0(G/B) \to Har$, $[\mathcal{M}] \mapsto \chi(\mathcal{M} \otimes \mathcal{L}(-\lambda))$ is an isomorphism.*

The reference for this statement is [**J85**].

For any $w \in W$ and $\lambda \in P$, set $j_w(\lambda) = \chi(\mathcal{L}_w(-\lambda))$ and $k_w(\lambda) = \chi(\mathcal{I}_w(-\lambda))$. It follows from Theorem 4.1 that $(j_w)_{w \in W}$ and $(k_w)_{w \in W}$ are two bases of $Har$. Therefore we have:

**Corollary 4.2:** *Let $w \in W$ and $\lambda, \mu \in P$. We have*
*(i) $j_w(\lambda + \mu) = \sum_{v \in W} s_w^v(\lambda) j_v(\mu)$,*
*(ii) $k_w(\lambda + \mu) = \sum_{v \in W} k_w^v(\lambda) k_v(\mu)$,*
*(ii) $k_w(\lambda + \mu) = \sum_{v \in W} m_w^v(\lambda) j_v(\mu)$.*

*Proof:* The corollary follows from the previous theorem, Demazure character formulas 2.1 and the definitions of the numbers $s_w^v(\lambda)$, $k_w^v(\lambda)$ and $m_w^v(\lambda)$. Q.E.D.

**5. Proofs of the Theorems 0.1, 0.2, and 0.3 and a filtration of $\mathcal{O}_w \otimes \mathbf{C}(-\lambda)$.**
The *degree* of a character $\phi = \sum_{\xi \in P} n^\xi e^\xi \in \mathbf{Z}[P]$ is the integer $\deg \phi = \sum_{\xi \in P} n^\xi$.

**Lemma 5.1:** *For any $w, v \in W$ and $\lambda \in P$, we have $s_w^v(\lambda) = \deg S_{w^{-1}}^{v^{-1}}(e^{-\lambda})$, $k_w^v(\lambda) = \deg K_{w^{-1}}^{v^{-1}}(e^{-\lambda})$, and $m_w^v(\lambda) = \deg M_{w^{-1}}^{v^{-1}}(e^{-\lambda})$.*

*Proof:* By Corollary 1.6, we have
$\nabla_w e^{-\lambda} = \sum_{v \in W} S_{w^{-1}}^{v^{-1}}(e^{-\lambda}) \nabla_v$. Therefore we get for any $\mu \in P$
$$\begin{aligned} j_w(\lambda + \mu) &= \chi(\mathcal{L}_w(-\lambda - \mu)) \\ &= \deg \nabla_w(e^{-\lambda} e^{-\mu}) \\ &= \deg[\sum_{v \in W} S_{w^{-1}}^{v^{-1}}(e^{-\lambda}) \nabla_v(e^{-\mu})] \\ &= \sum_{v \in W} \deg S_{w^{-1}}^{v^{-1}}(e^{-\lambda}) j_v(\mu). \end{aligned}$$
Therefore the equality $s_w^v(\lambda) = \deg S_{w^{-1}}^{v^{-1}}(e^{-\lambda})$ follows from Corollary 4.2 and the fact that $(j_w)_{w \in W}$ is a basis of $Har$. The proof of the other two equalities is similar. Q.E.D.

*Proofs of Theorems 0.1, 0.2, and 0.3.* Under the hypotheses of the theorems (i.e. $\lambda \in P^+$ for the first two, and $\lambda \in P^+$ and $w$ is minimal in $w.W_\lambda$ for the third one) the characters $S_{w^{-1}}^{v^{-1}}(e^{-\lambda})$, $K_{w^{-1}}^{v^{-1}}(e^{-\lambda})$, and $M_{w^{-1}}^{v^{-1}}(e^{-\lambda})$ are effective by Theorem 3.1. Therefore their degrees are $\geq 0$, and the theorems follows from Lemma 5.1. Q.E.D.

*Relation with [$\mathbf{M_3}$89]:* By [KK87], $K_H(G/B)$ is a free $\mathbf{Z}[P]$-module with basis $([\mathcal{O}_w]_H)_{w \in W}$. Theorem 0.1 can be reinforced as follows: in $K_H(G/B)$, we have, $[\mathcal{L}_w(-\lambda)]_H = \sum_{v \in W} S_{w^{-1}}^{v^{-1}}(e^{-\lambda}) [\mathcal{O}_v]_H$ and for any $\lambda \in P^+$ the character $S_{w^{-1}}^{v^{-1}}(e^{-\lambda})$ is effective. Note that $([\mathcal{O}_w \otimes \mathbf{C}(\mu)]_H)_{w \in W, \mu \in P}$ is a $\mathbf{Z}$-basis of $K_H(G/B)$. However there is another natural $\mathbf{Z}$-basis, namely $([\mathcal{L}_w(\mu)])_{w \in W, \mu \in P}$. Define the integers $n_w^{v,\xi}(\mu)$ by the following identity in $K_H(G/B)$:
$$[\mathcal{O}_w \otimes \mathbf{C}(\mu)]_H = \sum_{v \in W, \xi \in P} n_w^{v,\xi}(\mu) [\mathcal{L}_v(\xi)]_H.$$
In [$\mathbf{M_3}$89], we proved that for any $\lambda \in P^+$, $\mathcal{O}_w \otimes \mathbf{C}(-\lambda)$ has a $B$-equivariant filtration whose any subquotient is some $\mathcal{L}_v(\xi)$. Therefore the integers $n_w^{v,\xi}(\lambda)$ are non-negative for any $\lambda \in P^+$. Indeed it is easy to see that $S_w^v(e^{-\lambda}) = \sum_{\xi \in P} n_w^{v,\xi}(\lambda) e^\xi$.

## 6. Some refined tensor product multiplicities.

For any $\lambda \in P^+$, denote by $L(\lambda)$ the simple $G$-module with lowest weight $-\lambda$. For $\lambda, \mu, \nu$, define the tensor product multiplicities $K^\nu_{\lambda\mu}$ by the identity

$$L(\lambda) \otimes L(\mu) = \oplus_{\nu \in P^+} L(\nu)^{K^\nu_{\lambda\mu}}.$$

The tensor product multiplicities $K^\nu_{\lambda\mu}$ have been explicitly computed by Steinberg [**S61**]. We will now define some refined tensor multiplicities $K^\nu_{\lambda\mu}(w, v)$ indexed by the additional indices $w, v \in W$ by the following requirements:

(i) Assume that $w^{-1}$ is minimal in $w^{-1}W_\mu$ and $v$ is maximal in $v.W_\nu$. Then by Theorem 2.3, $K_w(\lambda) \otimes \mathbf{C}(-\mu)$ has a Joseph filtration and by definition $K^\nu_{\lambda\mu}(w, v)$ is the multiplicity of $J_v(\nu)$ in a Joseph filtration of $K_w(\lambda) \otimes \mathbf{C}(-\mu)$.

(ii) Otherwise, set $K^\nu_{\lambda\mu}(w, v) = 0$.

By definition the integers $K^\nu_{\lambda\mu}(w, v)$ are non-negative.

**Theorem 6.1:** *For any $\lambda, \mu, \nu \in P^+$, we have*
$$K^\nu_{\lambda\mu} = \sum_{v, w \in W} K^\nu_{\lambda\mu}(w, v).$$

*Proof:* We have $\operatorname{ch} L(\mu) = \nabla(e^{-\lambda})$, where $\nabla = \nabla_{w_0}$. Therefore $\operatorname{ch} L(\lambda) \otimes L(\mu) = \nabla(e^{-\mu} \operatorname{ch} L(\lambda))$. Moreover $ch\, L(\lambda) = \sum_{w \in W} \operatorname{ch} K_w(\lambda)$. Hence we get:
$$\operatorname{ch} L(\lambda) \otimes L(\mu) = \sum_{w \in W} \nabla(e^{-\mu} \operatorname{ch} K_w(\lambda)).$$
If $w$ in not minimal in $w.W_\lambda$, then $K_w(\lambda) = 0$. Therefore the in previous sum, we let $w$ runs over the minimal representatives of $W/W_\lambda$. Fix such a $w$. If $w^{-1}$ is not minimal in $w^{-1}.W_\mu$, there is a root $\alpha \in \Pi$ such that $\mu(h_\alpha) = 0$ and $s_\alpha w \leq w$, therefore $\nabla_{s_\alpha}(e^{-\mu} \operatorname{ch} K_w(\lambda)) = e^{-\mu} \nabla_{s_\alpha}(\operatorname{ch} K_w(\lambda)) = e^{-\mu} \nabla_{s_\alpha} \nabla'_{s_\alpha}(\operatorname{ch} K_{s_\alpha w}(\lambda)) = 0$, and therefore $\nabla(e^{-\mu} \operatorname{ch} K_w(\lambda)) = 0$. Otherwise $w^{-1}$ is minimal in $w^{-1}.W_\mu$, and by definition we have:
$$e^{-\mu} \operatorname{ch} K_w(\lambda) = \sum_{v \in W} K^\nu_{\lambda\mu}(w, v) \operatorname{ch} J_v(\nu).$$
Therefore, using the identity $\nabla(\operatorname{ch} J_v(\nu)) = ch\, L(\nu)$, we get:
$$\operatorname{ch} L(\lambda) \otimes L(\mu) = \sum_{\nu \in P^+ v, w \in W} K^\nu_{\lambda\mu}(w, v) \operatorname{ch} L(\nu),$$
from which we get the required equality. Q.E.D.

**Lemma 6.2:** *Let $\lambda, \mu, \nu \in P^+$. Let $w, v \in W$ such that $w$ is minimal in $w.W_\lambda$, $w^{-1}$ is minimal in $w^{-1}.W_\mu$, $v$ is maximal in $v.W_\nu$, and $\mu + w\lambda = v\nu$. Then $K^\nu_{\lambda\mu}(w, v) = 1$.*

*Proof:* By Theorem 2.3, the $B$-module $K_w(\lambda) \otimes \mathbf{C}(-\mu)$ has a Joseph filtration. However, $H^0(U, K_w(\lambda) \otimes \mathbf{C}(-\mu))$ is a one dimensional $H$-module of weight $-\mu - w\lambda$, where $U$ is the unipotent radical of $B$. Hence the first step of a Joseph filtration of $K_w(\lambda) \otimes \mathbf{C}(-\mu)$ is $J_v(\nu)$, and $J_v(\nu)$ does not occur further. Therefore $K^\nu_{\lambda\mu}(w, v)$ equals one. Q.E.D.

**Corollary 6.3:** (refined PRV conjecture) *Let $\lambda, \mu, \nu \in P^+$. Let $X$ be the set of all $w, \in W$ such that $\mu + w\lambda \in W.\nu$. Then $K^\nu_{\lambda\mu} \geq W_\mu \backslash X / W_\lambda$.*

*Proof:* It is clear that $X$ is a union of double $W_\mu \times W_\lambda$-cosets. Each such double coset contains a minimal representative $w$, and therefore $w$ is minimal in $w.W_\lambda$ and $w^{-1}$ is minimal in $w^{-1}.W_\mu$. For such a $w$ choose $v \in W$ such that $\mu + w\lambda = v\nu$ and $v$ is maximal

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in $v.W_\nu$. By Lemma 6.2, $K^\nu_{\lambda\mu}(w,v) = 1$. Therefore by Theorem 6.1, $K^\nu_{\lambda\mu} \geq W_\mu \backslash X / W_\lambda$. Q.E.D.

## 7. A generalization of Pieri's identity.

For $w \in W$ set $\overline{w} = ww_0$, for $\lambda \in P$ set $\overline{\lambda} = -w_0\lambda$ and denote by $\phi \in \mathbf{Z}[P] \mapsto \overline{\phi} \in \mathbf{Z}[P]$ the involution sending $e^\mu$ to $e^{\overline{\mu}}$. The following two theorems are due to van der Kallen [**vdK89**].

**Theorem 7.1:** *Let $M$ be a $B$-module.*
*(i) $M$ has a Joseph filtration if and only if $H^k(B, M \otimes K_w(\lambda)) = 0$ for any $k > 0$, $\lambda \in P^+$ and $w \in W$,*
*(ii) $M$ has a van der Kallen filtration if and only if $H^k(B, M \otimes J_w(\lambda)) = 0$ for any $k > 0$, $\lambda \in P^+$ and $w \in W$.*

**Theorem 7.2:** *As a $B \times B$-module, $\mathbf{C}[B]$ has a filtration whose the subquotients are $J_w(\lambda) \otimes K_{\overline{w}}(\overline{\lambda})$ and each of them occurs exactly once.*

**Corollary 7.3:** *Let $M$ be a $B$-module, let $\lambda \in P^+$, $v, w \in W$. Assume that $v$ is minimal in $v.W_\lambda$ and $w$ is maximal in $w.W_\lambda$.*
*(i) If $M$ has a Joseph filtration, then the multiplicity of $J_w(\lambda)$ in a Joseph filtration of $M$ is $\dim H^0(B, M \otimes K_{\overline{w}}(\overline{\lambda}))$.*
*(ii) If $M$ has a van der Kallen filtration, then the multiplicity of $K_v(\lambda)$ in a van der Kallen filtration of $M$ is $\dim H^0(B, M \otimes J_{\overline{v}}(\overline{\lambda}))$.*

*Proof:* We have $M \simeq H^0(B, \mathbf{C}[B] \otimes M)$. Therefore the filtration of $\mathbf{C}[B]$ defined in Theorem 7.1 induces a filtration of $M$. Assume that $M$ has a Joseph filtration. By Theorem 7.1, the subquotient of the filtration are $J_w(\lambda) \otimes H^0(B, K_{\overline{w}}(\overline{\lambda}) \otimes M)$. Therefore Assertion (i) is proved. The second assertion is similar. Q.E.D.

Denote by $\tau : \mathbf{Z}[P] \to \mathbf{Z}[P]$ the involution sending $e^\mu$ to $e^{w_0\mu}$.

**Lemma 7.4:** *For any $v, w \in W$, $\phi \in \mathbf{Z}[P]$, we have:*
*(i) $S^v_w(\phi) = \tau K^{\overline{w}}_{\overline{v}}(\overline{\phi})$,*
*(ii) $M^v_w(\phi) = \tau M^{\overline{w}}_{\overline{v}}(\overline{\phi})$,*
*(iii) $K^\nu_{\lambda\mu}(w,v) = K^{\overline{\lambda}}_{\nu\mu}(\overline{v}, \overline{w})$.*

*Proof:* Let $\lambda$ be any dominant weight and set $S^v_w(e^{-\lambda}) = \sum_{\xi \in P} n_\xi\, e^{-\xi}$, $K^{\overline{v}}_{\overline{w}}(e^{-\overline{\lambda}}) = \sum_{\xi \in P} n'_\xi\, e^{-\xi}$. Fix $\xi \in P$ and choose a dominant weight $\mu$ which is far away from the walls. It follows from the proof of Theorem 3.1 that $n_\xi$ is the mutiplicity of $J_v(\mu + \xi)$ in a Joseph filtration of $J_w(\mu) \otimes \mathbf{C}(-\lambda)$. By Corollary 7.3, we get $n_\xi = \dim H^0(B, J_w(\mu) \otimes K_{\overline{w}}(\overline{\mu} + \overline{\xi}) \otimes \mathbf{C}(-\lambda))$. As $\mu$ is an arbitrary dominant weight far away from the walls, we also get $n_\xi = \dim H^0(B, J_w(\mu - \overline{\xi}) \otimes K_{\overline{w}}(\overline{\mu}) \otimes \mathbf{C}(-\lambda))$. Using again Corollary 7.3, we get $n_\xi = n'_{-\overline{\xi}}$, what proves the first assertion for $\phi = e^{-\lambda}$. However, the maps $\phi \mapsto S^v_w(\phi)$ and

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$\phi \mapsto \tau\, K_{\overline{v}}^{\overline{w}}(\overline{\phi})$ are element of the algebra $\mathcal{H}$. Therefore they are uniquely determined on by their values on $e^{-\lambda}$ for $\lambda \in P^+$, and Assertion (i) follows. The proof of the other two assertions is similar. Q.E.D.

For $\alpha \in \Pi$, set $\Delta_{s_\alpha} = \frac{1}{1-e^{-\alpha}}(1-e^{-\alpha}\, s_\alpha)$ and $\Delta'_{s_\alpha} = \frac{e^{-\alpha}}{1-e^{-\alpha}}(1-s_\alpha)$. For any reduced decomposition $\sigma_1 \ldots \sigma_n$ of an element $w \in W$, set $\Delta_w = \Delta_{\sigma_1} \ldots \Delta_{\sigma_n}$ and $\Delta'_w = \Delta'_{\sigma_1} \ldots \Delta'_{\sigma_n}$. These operators do not depend on the reduced decomposition of $w$. Recall that for any $\lambda \in P^+$, the highest weight of $L(\lambda)$ is $\overline{\lambda}$. For any $w \in W$, let $J_w^*(\lambda)$ be the $B$-submodule of $L(\lambda)$ generated by a vector $v_w$ of weight $w\overline{\lambda}$, and set $K_w^*(\lambda) = J_w^*(\lambda)/\sum_{v<w} J_v^*(\lambda)$. By Demazure character formula 2.1, we have
$$\mathrm{ch}\, J_w^*(\lambda) = \Delta_w(e^{\overline{\lambda}}),$$
$$\mathrm{ch}\, K_w^*(\lambda) = \Delta'_w(e^{\overline{\lambda}}).$$

**Lemma 7.5:** *For any $w \in W$, we have $S_{w_0}^{w^{-1}}(e^{-\lambda}) = \Delta'_{\overline{w}}(e^{\overline{\lambda}})$.*

*Proof:* By Corollay 1.6, we have $\nabla'_w \phi = \sum_{v \in W} K_{w^{-1}}^{v^{-1}}(\phi)\, \nabla'_v$. However $\nabla'_v(1) = 0$ for any $v \neq 1$. Hence we get $\nabla'_w(\phi) = K_{w^{-1}}^1(\phi)$. Therefore the lemma follows from Lemma 7.4. Q.E.D.

**Corollary 7.6:** *Let $\lambda \in P^+$. In $K(G/B)$, we have*
$$[\mathcal{L}(-\lambda)] = \sum_{w \in W} k_{\overline{w}}(\overline{\lambda})[\mathcal{O}_w].$$

The corollary follows from Corollary 4.2 and the Lemma 7.5. It provides a generalisation for any group of the classical Pieri formula. It should be noted that the Littelmann' path model provides a nice combinatorial formula for $k_w(\lambda)$. More precisely, in **L94**] Littelmann defined the Lakshmibai-Sheshadri path of shape $\lambda$: they are indexed by a pair $(\mathbf{a}, \tau)$, with $\mathbf{a}$ is an increasing sequence of rational numbers $\mathbf{a} : 0 = a_0 < a_1 < \ldots < a_r = 1$ and a decreasing sequence of $W_\lambda$-cosets $\tau : \tau_1 > \ldots > \tau_r = 1$ together with an integrality condition on $\mathbf{a}$ (this condition depends on $\lambda$). He showed that $j_w(\lambda)$ is the number of Lakshmibai-Sheshadri paths $(\mathbf{a}, \tau)$ with $\tau_1 \leq w.W_\lambda$. As $j_w(\lambda) = \sum_{v \leq w} k_v(\lambda)$, it is easy to deduce that $k_w(\lambda)$ is the number of Lakshmibai-Sheshadi paths $(\mathbf{a}, \tau)$ with $\tau_1 = w.W_\lambda$, whenever $w$ is minimal in $w.W_\lambda$ (otherwise $k_w(\lambda) = 0$).

Using Lemma 7.5, we also get the following identity:

**Corollary 7.7:** *Let $\lambda, \mu \in P^+$. We have*
$$\mathrm{ch}\, L(\lambda + \mu) = \sum_{w \in W} \mathrm{ch}\, J_w(\lambda) \otimes K_{\overline{w}}^*(\mu).$$

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