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ESTIMATING THE DIMENSION OF THE THURSTON SPINE

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ABSTRACT. For $g \geq 2$, the Thurston spine $\mathcal{P}_g$ is the subspace of Teichmüller space $\mathcal{T}_g$, consisting of the marked surfaces for which the set of shortest curves, the systoles, cuts the surface into polygons. Our main result is the existence of an infinite set A of integers $g \geq 2$ such that

$$\operatorname{codim} \mathcal{P}_g \in o(g/\sqrt{\log g}),$$

when $g \in A$ goes to ∞. This proves the recent conjecture of M. Fortier Bourque.

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INTRODUCTION

0.1 General Introduction

Let $g \geq 2$. The Teichmüller space $\mathcal{T}_g$ is the space of all marked closed hyperbolic surfaces of genus g. (Precise definitions used in the introduction can be found in Section 1.) It is a smooth variety homeomorphic to $\mathbb{R}^{6g-6}$, see e.g. [5][6], on which the mapping class group Γ_g acts properly. By Harer's Theorem [7], Γ_g has virtual cohomological dimension $4g-5$. This leads to the question, raised in [4]-can we find an equivariant deformation retraction of $\mathcal{T}_g$ onto a subcomplex of dimension $4g-5$, or equivalently, of codimension $2g-1$?

In a remarkable note [17], Thurston considered the subspace $\mathcal{P}_g \subset \mathcal{T}_g$ consisting of marked surfaces for which the systoles *fill* the surface, i.e. the systoles cut the surface into polygons. In *loc. cit.*, he proved[1] that $\mathcal{P}_g$ is an equivariant deformation retract of $\mathcal{T}_g$. Since, $\mathcal{P}_g$ is called the Thurston spine. It follows that $\operatorname{codim}\mathcal{P}_g \leq 2g-1$.

Therefore, one could have expected that $\mathcal{P}_g$ has codimension $2g-1$ for all g. In that direction, P. Schmutz Schaller provided examples of surfaces of genus g which are cut by a minimal set of $2g$ systoles. (We could expect that $\mathcal{P}_g$ has locally codimension $2g-1$, as it will

[1] In [11], some doubts have been raised about Thurston's proof. The results stated below were clearly motivated by his note [17], but their proofs are independent of *loc. cit.*. So, we will not discuss here if the main result of [17] is proved or not.

be explained in Subsection 0.2.) Also it was verified by I. Irmer using a Sage computation that $\dim \mathcal{P}_2 = 3$ [8].

However, a year ago, the breakthrough paper [3] showed that $\mathrm{codim}\mathcal{P}_g < 2g-1$, for infinitely many g. More precisely, M. Fortier Bourque proved in [3] that

$$\liminf_{g\to\infty} \mathrm{codim}\, P_g/g \leq 1.$$

Moreover he had conjectured earlier [2] that

$$\liminf_{g\to\infty} \mathrm{codim}\mathcal{P}_g/g = 0.$$

Our paper provides a proof of his conjecture with, in addition, some explicit bound.

Theorem 1. *We have*

$$\mathrm{codim}\, \mathcal{P}_g < \frac{38}{\sqrt{\ln\ln\ln g}}\ \frac{g}{\sqrt{\ln g}},$$

for infinitely many $g \geq 2$.

This leads to the concrete question -which is the smallest g for which $\mathrm{codim}\, \mathcal{P}_g < 2g-1$? In the last section, we will see that, for $g = 17$, we have $\mathrm{codim}\, \mathcal{P}_{17} < 32$. However, we do not know if $g = 17$ is the smallest g answering the question.

0.2 The main idea of the proof

For $g \geq 2$, let us fix, one and for all, an oriented closed topological surface $\mathcal{S}_g$ of genus g. A *marked* hyperbolic surface S of genus g is a hyperbolic surface endowed with an homeomorphism $f : S \to \mathcal{S}_g$, up to homotopy. Recall that a *curve* is free homotopy class of an embedding $S^1 \hookrightarrow \mathcal{S}_g$. Any curve C of $\mathcal{S}$ can be uniquely represented by a closed geodesic of S. Thus its length $L(C)$ can be viewed as a function on the Teichmüller space $\mathcal{T}_g$.

Denote by $\mathrm{Syst}(S)$ the set of systoles of S, which is viewed as a finite set of curves of $\mathcal{S}$. For a finite set $\mathcal{C} = \{C_1, C_2 \dots\}$ of curves, let $\mathrm{Sys}\,(\mathcal{C})$ be the set of marked hyperbolic surfaces S of genus g such that $\mathrm{Syst}(S) = \mathcal{C}$.

An obvious corollary of the submersion theorem is the following

Lemma. *Let $S \in \mathcal{P}_g$ and let $\mathcal{C}$ be a filling subset of* $\mathrm{Syst}(S)$ *of minimal cardinality. If*

($\mathcal{H}$) *the set of differentials $\{\mathrm{d}L(C) \mid C \in \mathrm{Sys}\,(\mathcal{C})\}$ is linearly independent at S,*

then S is adherent to $\mathrm{Sys}\,(\mathcal{C})$ *and we have*

$$\mathrm{codim}_S\, \mathcal{P}_g = \mathrm{Card}\ \mathcal{C} - 1.$$

Thus the proof is based on the following two ingredients

Step 1 finding hyperbolic surfaces S with $\mathrm{Sys}\,(\mathcal{C})$ or $\mathcal{C}$ small
Step 2 checking the hypothesis ($\mathcal{H}$)

It would be more natural to start with surfaces S whose set of systoles is minimal, but it seems difficult to find such surfaces.

The constructions of hyperbolic surfaces with a small number of systoles are based on Penner systems. Recall that a *Penner system* is a finite set $\mathcal{P}$ of curves with a decomposition $\mathcal{P} = \mathcal{B} \cup \mathcal{R}$, whith the following conditions

(1) The curves in $\mathcal{B}$, called the blue curves, are pairwise disjoint,
(2) The curves in $\mathcal{R}$, called the red curves, are pairwise disjoint,
(1) $\mathcal{P}$ fills S, i.e. it cuts S into polygons.

Each polygon cuts by $\mathcal{P}$ have sides of aternating colors red and blue, so the polygons have an even number of sides. The Penner system $\mathcal{P}$ is called *$2p-gonal$* if all polygons cut by $\mathcal{P}$ are $2p$-gons for some integer p. Since $g \geq 2$, the integer p should be ≥ 3.

Let $\epsilon \in]0, \pi[$. There is a unique oriented hyperbolic $2p$-gon $H_p(\epsilon)$ such that

(1) The side are alternatively coloured in blue and in red, and they have all the same length,
(2) The value of directly oriented inner angles from a red side to a blue side is ϵ, although the other angles are equal to $\overline{\epsilon} : \pi - \epsilon$.

Given a $2p$-gonal Penner system $\mathcal{P}$ on $\mathcal{S}_g$, there is a unique hyperbolic metric on $\mathcal{S}_g$ such that each $2p$-gon cut by $\mathcal{P}$ is isometric to $H_p(\epsilon)$. Of course we assume that the isometry preserves the side colors. Let $S_{\mathcal{P}}(\epsilon)$ the corresponding hyperbolic surface. In this way we obtain a path in $\mathcal{T}_g$

$$\sigma :]0, \pi[\to \mathcal{T}_g, \epsilon \mapsto S_{\mathcal{P}}(\epsilon),$$

We will called it the *Sanki path* of the standard Penner system $\mathcal{P}$.

Each curve of $\mathcal{P}$ is cut into edges by the curves of opposite colours. Let us consider the following axiom

(AX) Each blue curve B contains an edge whose extreme points belongs to two different red curves,

Our first result is the following

Theorem 2. *Let $\mathcal{P}$ be a $2p$-gonal Penner system satisfying (AX). Then for all*

We defined five axioms for Penner systems $(AX1-5)$. Our first result is the following

Theorem 3. *Assume that the Penner system satisfies the axioms* $(AX1-5)$*. Then, the set of differentials*

$$\{\mathrm{d}L(C) \mid C \in \mathrm{Syst}(S_g)\}$$

is linearly independent at $\sigma(\epsilon)$, except for finitely many values of ϵ.

For the proof, we will finding surfaces S satisfying the criterion the hypothesis $(\mathcal{H})$ and with a small filling subset of systoles.

The starting point of the proof is based on the main result of [10], that we now recall.

A regular right-angled hexagon H of the Poincaré half plane $\mathbb{H}$ is called *decorated* if it is oriented and its sides are cyclically indexed by $\mathbb{Z}/6\mathbb{Z}$. Up to direct isometries, there are two such hexagons $\mathcal{H}$ and $\overline{\mathcal{H}}$, with opposide orientations.

A tesselation of a closed oriented hyperbolic surface S is called a *standard tesselation* if each tile is isometric to $\mathcal{H}$ or $\overline{\mathcal{H}}$. Of course, it is presumed that the tiles are glued along edges with the same index, therefore a tile isometric to $\mathcal{H}$ is surrounded by six tiles isometric to $\overline{\mathcal{H}}$ and conversely. A vertex of a standard tesselation is an intersection point of two perpendicular geodesics. Therefore, the 1-skeleton of a standard tesselation consists of a finite family of closed geodesics, called the *curves* of the tesselation.

For a hyperbolic surface S, denote by $\mathrm{Syst}(S)$ the set of systoles of S.

Theorem (Theorem 25 of [10]). *There exists an infinite set A of integers $g \geq 2$, and, for any $g \in A$ a closed oriented hyperbolic surface S_g of genus g endowed with a standard tesselation τ_g such that*

(1) the systoles of S_g are exactly the curves of τ_g, and
(2) we have

$$\text{Card Syst}(S_g) \leq \frac{57}{\sqrt{\ln\ln\ln g}} \frac{g}{\sqrt{\ln g}}.$$

The *index* of a curve of the tesselation τ_g is the common index of its edges. It is clear that the subset $\mathcal{C}$ of curves of index $\neq 1$ or 2 fills the surface and Card $\mathcal{C} = 2/3\,\text{Card}\,\text{Syst}(S_g)$.

Let Sys $(\mathcal{C})$ be the set of marked hyperbolic surfaces S of genus g such that $\text{Syst}(S) = \mathcal{C}$. For any curve, or free homotopy class, C of S_g, let $L(C)$ be its length, viewed as a function on the Teichmüller space $\mathcal{T}_g$. Set $\mathcal{C} = \{C_1, C_2 \dots\}$. The subspace Sys $(\mathcal{C})$ is defined by the Card $\mathcal{C} - 1$ equations

$$L(C_1) = L(C_2) = \dots$$

together with some inequalities. Intuitively, our result should follow from the following two facts

(1) In $\mathcal{T}_g$, the point S_g is adjacent to Sys $(\mathcal{C})$, and
(2) for any point $x \in S(\mathcal{C})$ closed to S_g, we have $\text{codim}_x\, S(\mathcal{C}) = \text{Card}\,\mathcal{C} - 1$

If we assume that the differentials $\{\mathrm{d}L(C) \mid c \in \text{Syst}(S_g)\}$, are linearly independent at the point S_g, the previous two facts would follow from the submersion theorem. However, an argument in Theorem 36 of [16] shows that these differentials are often linearly dependent at S_g. For this reason, the cardinality of $\mathcal{C}$ does not determine the local codimension of Sys $(\mathcal{C})$.

Following an idea of Sanki [15], we can deform the angles of the tiles, by alternately replacing the right angles by angles of value ϵ and $\pi - \epsilon$, for any $\epsilon \in]0, \pi[$. In this way we obtain a path $\sigma :]0, \pi[\to \mathcal{T}_g$, such that $\sigma(\pi/2)$ is the hyperbolic surface S_g. We will called it the *Sanki* path of the tesselation τ_g. The main idea of the proof is the following

Theorem 4 (see Section 5)**.** *The set of differentials*

$$\{\mathrm{d}L(C) \mid C \in \text{Syst}(S_g)\}$$

is linearly independent at $\sigma(\epsilon)$, except for finitely many values of ϵ.

It implies that the assertions (1) and (2) are correct for $x = \sigma(\epsilon)$, where $\epsilon \neq \pi/2$ is closed enough to $\pi/2$.

The proof of Theorem 2 is based on a duality, which is expressed in terms of the Poisson product associated with the Weil-Petersson symplectic structure on $\mathcal{T}_g$. For any curve B of the tesselation, we define a dual function $L(B^*)$ which is a linear combination (with coefficients $\pm 1/2$) of lengths of three curves, which are the boundary components of a well-chosen pair of pants.

It results from an asymptotic analysis of Wolpert's formula [19] that

$$\lim_{\epsilon \to 0} \{L(B), L(A^*)\}(\sigma(\epsilon)) = \delta_{A,B} \tag{1}$$

for any two curves A, B of the tesselation, where, as usual, $\delta_{A,B}$ denotes the Kronecker's symbol. Since $\delta := \det(\{L(B), L(A^*)\})$ is an analytic function, it follows that $\delta(\sigma(\epsilon))$ is not zero for any $\epsilon \neq \pi/2$ closed to $\pi/2$.

In fact the proof of equation (1) is based on elementary but lengthy trigonometric computations of Sections 2-4. To present the computations and the figures as simply as possible, we have restricted ourself to hexagonal tesselations. However similar results hold for tesselations by $2p$-gons for any $p \geq 3$.

1. Background and Definitions

1.1 Marking of surfaces and the Teichmüller space
Let $g \geq 2$. By definition, the *Teichmüller* space $\mathcal{T}_g$ is the space of all marked oriented closed hyperbolic surfaces of genus g. It means that $\mathcal{T}_g$ parametrizes the set of those hyperbolic surfaces, where the marking is a datum that distinguishes isometric surfaces corresponding to distinct paramaters.

There are various equivalent definitions of the marking [5]. Here we will adopt the most convenient for our purpose. Let Π_g be the group given by the following presentation

$$\langle a_1, b_1, \ldots, a_g, b_g \mid (a_1, b_1)(a_2, b_2) \ldots (a_g, b_g) = 1 \rangle.$$

Then a point x of the Teichmuller space is a loxodromic (i.e. faithfull with discrete image) representation $\rho_x : \Pi_g \to \mathrm{PSL}(2, \mathbb{R})$ modulo linear equivalence. At the point x, the corresponding hyperbolic surface is $S_x := \mathbb{H}/\rho_x(\Pi_g)$. With this definition, $\mathcal{T}_g$ is a connected component of a real algebraic variety.

Formally, a *curve* c is a nontrivial conjugacy class of Π_g. For a hyperbolic surface, any free homotopy class has a unique geodesic representative. Thus c defines a closed geodesic c_x of S_x, for any $x \in \mathcal{T}_g$. Here closed geodesics are nonoriented, so we will not distinguish the conjugacy classes of c and c^{-1}. Concretely, the marking of the surface S_x means that each geodesic C of S_x is marked by a conjugacy class in Π_g.

In this setting, the *mapping class group* Γ_g is the group of all outer automorphisms of Π_g which act trivially on $H^2(\Pi_g) \simeq \mathbb{Z}$. It acts on $\mathcal{T}_g$ by changing the marking, or, more formally, by twisting the representation of Π_g.

1.2 Length of curves
The length of an arc, or a closed geodesic, e will be denoted $l(e)$. When there is no possibility of confusion, we will use the same letter for an arc e and its length. For example the expression $\cosh H$ in the proof of Lemma 6 stands for $\cosh l(H)$.

Let $x \in \mathcal{T}_g$. Given a curve c, set $L(c)(x) = l(c_x)$ where c_x is the geodesic representative of c at x. The formula $2\,\mathrm{ch}(L(c(x))) = |\mathrm{Tr}(\rho_x(c))|$ shows that the function $L(c) : \mathcal{T}_g \to \mathbb{R}$ is analytic. Let C be a closed geodesic of S_x. We set $L(C) = L(c)$, where c is the curve marking C.

1.3 The Thurston's spine $\mathcal{P}_g$
In riemannian geometry, a *systole* is an essential closed geodesic of minimal length. In fact, for a hyperbolic surface, any closed geodesic is essential. Let $\mathcal{P}_g$ be the set of all points $x \in \mathcal{T}_g$ such that the set of systoles fills S_x, i.e. it cuts S_x into polygons. The subspace $\mathcal{P}_g$ is called the *Thurston spine*, see [17].

By definition, the Thurston spine $\mathcal{P}_g$ is a semi-analytic subset [17], and therefore it admits a triangulation by [12]. In particular, the dimension $\dim_x \mathcal{P}_g$ at any point $x \in \mathcal{P}_g$ is well defined. Set

$$\dim \mathcal{P}_g = \mathrm{Max}_{x \in \mathcal{P}_g} \dim_x \mathcal{P}_g \text{ and } \mathrm{codim}\mathcal{P}_g := \dim \mathcal{T}_g - \dim \mathcal{P}_g.$$

1.4 Orientation of the boundary components
In what follows, all surfaces S are given with an orientation. When S has a boundary ∂S, it is oriented by the rule that, while moving forward along ∂S, the interior of S is on the right. With this convention, when a circle of the plane is viewed as the boundary of its interior, it is oriented in the clockwise direction.

1.5 Angles
Let C, D be two distinct geodesic arcs of a surface and let P be an intersection point. The angle of C and D at P, denoted $\angle_P CD$ is measured anticlockwise from $T_P C$ to $T_P D$, where $T_P C$ and $T_P D$ are the tangent line at P of C and D. By definition $\angle_P CD$ belongs to $]0, \pi[$. When we permutes C and D there is the formula

$$\angle_P DC = \pi - \angle_P CD.$$

In what follows, it will be convenient to set $\overline{\alpha} = \pi - \alpha$ for any $\alpha \in [0, \pi]$. Also it will be convenient to use the notation $\angle DC$ when the point P is unambiguously defined.

The notion of *inner angles* is different. Let S be a surface whose boundary ∂S is piecewise geodesic. Let c and d be two consecutive geodesic arcs of ∂S meeting at a point P. The inner angle is a real number $\alpha \in]0, 2\pi[$. We have $\alpha < \pi$ when S is locally convex around P. In that case, the equality $\angle\, cd = \alpha$ means that the arc c preceeds d when going forward along ∂S.

2. Trigonometry in $\mathbb{H}$

As stated in the Introduction, the analysis of Sanki's paths, defined in Section 5, is based on many trigonometric computations. This section involves trigonometric computations in the Poincaré half-plane $\mathbb{H}$. Subsequent computations in the pairs of pants $\Pi(k, \epsilon)$ will be done in Section 4.

Let $d_{\mathbb{H}}$ be the hyperbolic distance on $\mathbb{H}$. By definition, a *line* is a complete geodesic Δ of $\mathbb{H}$. For any $P, Q \in \Delta$, the closed arc between P and Q is called a *segment* and it will be denoted PQ. When necessary, the segment PQ is oriented from P to Q.

Given three points A, B and $C \in \mathbb{H}$, we denote by ABC the triangle T whose sides are AB, BC and CA. By our convention, ∂T is oriented clockwise, but we do not require a specific orientation of the sides. Given four points A, B, C and $D \in \mathbb{H}$, we define in the same way the quadrilateral $ABCD$, not ruling out the possibility that one pair of opposite sides intersects.

For the whole section, we will fix an angle $\epsilon \in]0, \pi[$. In the pictures, we will assume that $\epsilon < \pi/2$.

2.1 The ϵ-pencil $\mathcal{F}_\epsilon(\Delta)$ in $\mathbb{H}$
Let $\Delta \subset \mathbb{H}$ be a line. For any $P \in \Delta$, let $F(P)$ be the line passing through P with $\angle \Delta F(P) = \epsilon$, see Section 1.5 for our convention concerning angles. Since no triangle has two angles of values ϵ and $\overline{\epsilon}$, any two lines $F(P)$ and $F(P')$ are parallel. The set $\mathcal{F}_\epsilon(\Delta) := \{F(P) \mid P \in \Delta\}$ will be called the *ϵ-pencil* along the line Δ.

Let Δ' be a geodesic arc of $\mathbb{H}$ with $\Delta \cap \Delta' = \emptyset$. When $F(P)$ meets Δ', set $\Omega(P) = F(P) \cap \Delta'$ and

$$\omega(P) := \angle \Delta' F(P).$$

The line Δ is oriented by the convention that, while going forward, the arc Δ' is on the right. Therefore the notion of an increasing function $f : \Delta \to \mathbb{R}$ is well-defined.

In contrast with the Euclidean geometry, the angle $\omega(P)$ is not constant. On the contrary, it can vary from 0 to π, as it will now be shown.

Lemma 5. *We have:*

1. *The set $I := \{P \in \Delta \mid F(P) \cap \Delta' \neq \emptyset\}$ is an interval of Δ. Moreover, the map $\omega : I \to]0, \pi[$ is increasing.*
2. *Furthermore, if Δ' is a line of $\mathbb{H}$ and $d_{\mathbb{H}}(\Delta', \Delta) \neq 0$, then ω is bijective.*

Proof of claim (1). Let PP' be a positively oriented arc of I. Consider the quadrilateral $Q := PP'\Omega(P')\Omega(P)$. With our conventions, its inner angles are $\epsilon, \overline{\epsilon}, \overline{\omega(P')}$ and $\omega(P)$. Since the area of Q is

$$2\pi - [\epsilon + \overline{\epsilon} + \overline{\omega(P')} + \omega(P)] = \omega(P') - \omega(P),$$

the function ω is increasing.

Next let P'' in the interior of the segment PP'. Since the line $F(P'')$ enters Q at P'', it should left Q at another point. Since $F(P'')$ is parallel to $F(P)$ and $F(P')$, the exit point lies in the segment $\Omega(P)\Omega(P')$, hence P'' belongs to I. Since I contains a segment whenever it contains its two extremal points, I is an interval.

Proof of Claim (2). We can assume that $\Delta = \mathbb{R}i$. By the assumption that $d_{\mathbb{H}}(\Delta, \Delta') > 0$, the lines Δ and Δ' do not intersect and their endpoints in $\partial\mathbb{H}$ are distinct. Therefore the endpoints a, b of Δ' in $\partial\mathbb{H}$ are real numbers with same signs. Without loss of generality, we can also assume that $0 < a < b$, as shown in Figure 1.

There is a line F^+ (resp. F^-) in $\mathcal{F}_\epsilon(\Delta)$ whose the endpoint in $\mathbb{R}_{>0}$ is b (resp. a). Set

$$P^\pm = \Delta \cap F^\pm.$$

Let $\mathbf{B}$ be the open band delimited by F^+ and F^-. When P belongs to the interior of P^-P^+, the line $F(P)$ lies in the interior of the band $\mathbf{B}$ and $F(t)$ meets Δ'. It is clear from the definition of $F^\pm$ that

$$\lim_{P \to P_-} \omega(P) = 0 \text{ and } \lim_{P \to P_+} \omega(P) = \pi.$$

Hence by Assertion (1), ω is bijective.

□

2.2 The ϵ-edge

Let $\epsilon \in]0, \pi[$ and let Δ, Δ' be two lines with $d_{\mathbb{H}}(\Delta', \Delta) > 0$. Let H be the the common perpendicular arc to Δ and Δ' and let $S \in \Delta$ and $S' \in \Delta'$ be its endpoints.

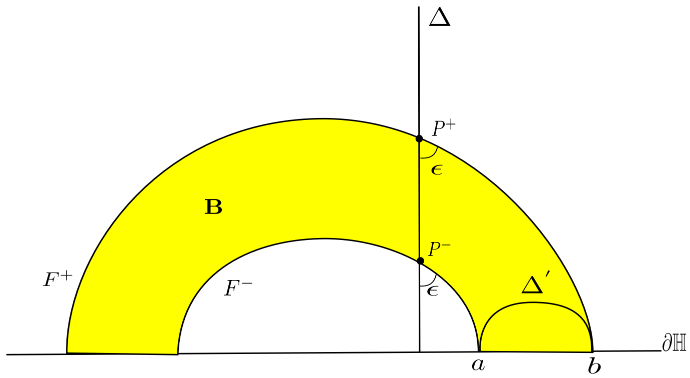

FIGURE 1. The band **B**

By Lemma 5 there is a unique $P \in \Delta$ such that $\omega(P) = \epsilon$. The edge $e = P\Omega(P)$ will be called the ϵ-*edge* of Δ and Δ'. For $\epsilon = \pi/2$, the ϵ-edge is the perpendicular arc H.

Lemma 6. *Set $L = d_{\mathbb{H}}(P, P')$ where $P' = \Omega(P)$. If $\epsilon \neq \pi/2$, then*

(1) H and e intersect at their midpoints.

(2) $d_{\mathbb{H}}(P, S) = d_{\mathbb{H}}(P', S') < L/2$.

Moreover the segment SP is positively oriented whenever $\epsilon < \pi/2$.

Proof. We have $\angle \Delta e = \epsilon$ and $\angle e\Delta' = \overline{\epsilon}$. The sum of the four angles of the quadrilateral $Q := SP\Omega(P)S'$ is 2π. It follows that a pair of opposite edges must intersect, so e meets H at some point M. The two triangles SPM and $M\Omega(P)S'$ have the same three angles, and are therefore isometric. In particular $d_{\mathbb{H}}(P, S) = d_{\mathbb{H}}(P', S')$.

It follows that e and the arc $H = SS'$ intersect at their midpoint M, thus we have $d_{\mathbb{H}}(P, M) = L/2$. Since PM is the hypothenuse of the right-angled triangle PSM, we have $d_{\mathbb{H}}(P, S) < L/2$. The second claim follows. ☐

2.3 $2p$-gons trigonometry and edge colouring

Let $p \geq 3$ be an integer.

Lemma 7. *Up to isometry, there exists a unique hyperbolic $2p$-gons $H(\epsilon)$ whose sides all have the same length $L = L(\epsilon)$ and whose inner angles are alternately ϵ and $\overline{\epsilon}$.*

Moreover we have

$$\cosh L = 1 + \frac{2\cos(\pi/p)}{\sin\epsilon}.$$

Proof of the existence of $H(\epsilon)$ and of the formula for $\cosh L$.

Let T be an oriented triangle whose angles are $\epsilon/2$, $\overline{\epsilon}/2$ and $\pi/3$ and let X be the vertex at the $\pi/3$-angle. Let $\overline{T}$ be the triangle isometric to T with opposite orientation. Let $H(\epsilon)$ be the hexagon obtained by alternately gluing three copies of T and three copies of $\overline{T}$ around

X. This hexagon satisfies the required conditions, so the existence is proved.

By the law of cosines for the triangle T, we have

$$\cosh L = \frac{\cos\epsilon/2\cos\overline{\epsilon}/2 + \cos\pi/3}{\sin\epsilon/2\sin\overline{\epsilon}/2}$$

$$= 1 + \frac{1}{2\sin\epsilon/2\cos\epsilon/2}$$

$$= 1 + \frac{1}{\sin\epsilon}.$$

Proof of the uniqueness.
Let H be a hexagon whose sides all have the same length l, and whose angles are alternately ϵ and $\overline{\epsilon}$. Let $(A_i)_{i\in\mathbb{Z}/6\mathbb{Z}}$ be the six vertices, arranged in a cyclic order. Moreover, we will assume that the angles at A_2, A_4 and A_6 are $\overline{\epsilon}$.

The triangles $A_1A_2A_3$, $A_3A_4A_5$ and $A_5A_6A_1$ have the same angles at the point A_2, A_4 and A_6, and the two sides originating from these points have the same length l. Hence they are isometric. It follows that the triangle $A_1A_3A_5$ is equilateral.

Let X be the center of $A_1A_3A_5$, and let $T' = A_1XA_2$ and $\overline{T}' = A_2A_3X$. Since T' and $\overline{T}'$ have same side lengths they are isometric, with opposite orientations. Hence H is obtained by alternately gluing three copies of T' and three copies of $\overline{T}'$ around X. It follows that the angles of T' are $\epsilon/2$, $\overline{\epsilon}/2$ and $\pi/3$. Therefore T' is isometric the the triangle T of the existence proof. Hence H is isometric to $H(\epsilon)$, proving uniqueness. □

We can alternately assign the colours blue and red to the sides of $H(\epsilon)$, as follows: If S, S' are consecutive sides with $\angle \mathrm{S\,S'} = \epsilon$, then S is red and S' is blue, or, quickly speaking, $\angle\text{red blue} = \epsilon$. For $\epsilon \neq \pi/2$ this colouring is unique.

2.4 The Saccheri quadrilateral in $H(\pi/2)$
In the hexagon $H(\pi/2)$, let S_1, S_2 and S_3 be three consecutive sides and let D be the arc joining the vertex of $S_1 \diagdown S_2$ and the vertex of $S_3 \diagdown S_2$. Thus S_1, S_2, S_3 and D are the four sides of a Saccheri quadrilateral. Set $L' = l(D)$.

Lemma 8. *We have*

$$L' < 2L(\pi/2).$$

Proof. Set $L = L(\pi/2)$. The perpendicular line at the middle of S_2 cuts the Saccheri quadrilateral into two isometric Lambert quadrilaterals. It follows that

$$\sinh L'/2 = \cosh L/2\,\sinh L/2,$$

or equivalently $\sinh^2 L'/2 = 4\sinh^2 L/2$, which implies that $\cosh L' = 5$. On another hand $\cosh 2L = 2\cosh^2 2L - 1 = 7$. It follows that $L' < 2L$. □

3. The three-holed sphere $\Sigma(k,\epsilon)$ and the pair of pants $\Pi(k,\epsilon)$

From now on, let $k \geq 3$ be an integer and let $\epsilon \in]0,\pi[$. We will often think of ϵ as being an acute angle, as in the figures of this section. A topological three-holed sphere endowed with an hyperbolic structure such that the three boundary components are arcwise geodesics will be called a *three-holed sphere*. Moreover a three-holed sphere whose three boundary components are geodesics will be called *a pair of pants*.

In this section, we will define a pair of pants $\Pi(k,\epsilon)$ endowed with a certain tesselation. We will first consider a three-holed sphere $\Sigma(k,\epsilon)$ which has two geodesic boundary components C and C' and one piecewise geodesic boundary component $\mathcal{D}$. Since the inner angles of $\mathcal{D}$ are $< \pi$, it is homotopic to a unique geodesic D. Then $\Pi(k,\epsilon)$ is the pair of pants lying in $\Sigma(k,\epsilon)$, whose boundary components are C, C' and D.

3.1 The tesselated three-holed sphere $\Sigma(k,\epsilon)$
Let us start with the planar graph Γ represented in Figure 2.

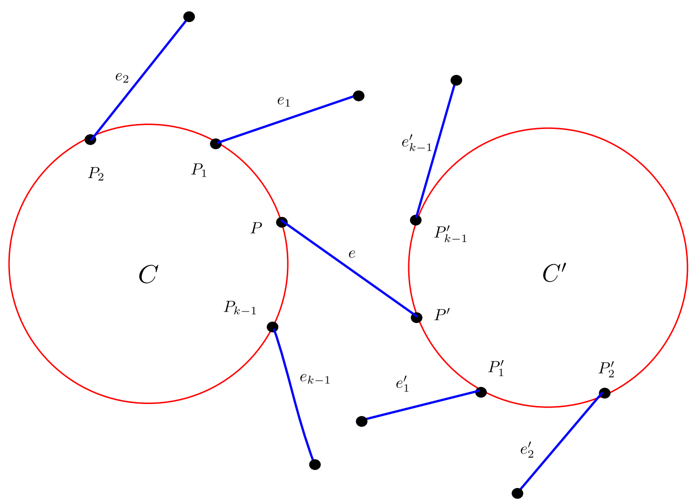

Figure 2. The graph Γ

It consists of two cycles C and C' of length k, which are connected by an edge e with endpoints $P \in C$ and $P' \in C'$. Starting from P in an anticlockwise direction, the other points of C are denoted by $P_1, \dots, P_{k-1}$. For each $1 \leq i \leq k-1$ there is an additional edge e_i, pointing outwards from C. One endpoint of e_i is P_i and the other endpoint has valency one. The vertices $P'_1, \dots, P'_{k-1}$ of C' and the edges e'_i are defined similarly.

The edges of the cycles C and C' are coloured in red, the other edges are coloured in blue. Now we will attach $2k-2$-hexagons $H(\epsilon)$ along the edges of Γ to obtain a hyperbolic surface $\Sigma(\epsilon,k)$. It will be convenient to define a metric on Γ by requiring that each edge, red or blue, has length $L = 1 + 1/\sin\epsilon$.

First we attach two copies of $H(\epsilon)$ along five consecutive edges of Γ. The first copy, denoted $H_1(\epsilon)$, is glued along the edges e_{k-1}, $P_{k-1}P$, e, $P'P'_1$ and e'_1. The second copy, denoted $H_2(\epsilon)$, is glued along the edges e'_{k-1}, $P'_{k-1}P'$, e, PP_1 and e_1, see Figure 3.

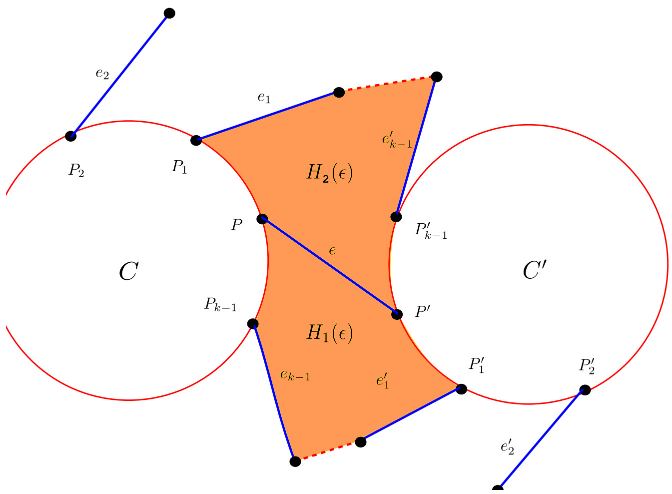

FIGURE 3. Gluing two the first two copies $H_1(\epsilon)$ and $H_2(\epsilon)$ to Γ

Next we attach the remaining $2k-4$ hexagons along three edges. Indeed, for each integer i with $1 \leq i \leq k-2$, we glue one copy of $H(\epsilon)$ along the edges e_i, P_iP_{i+1} and e_{i+1} and, symmetrically, we glue another copy along e'_i, P''_iP_{i+1} and e'_{i+1}, see Figure 4.

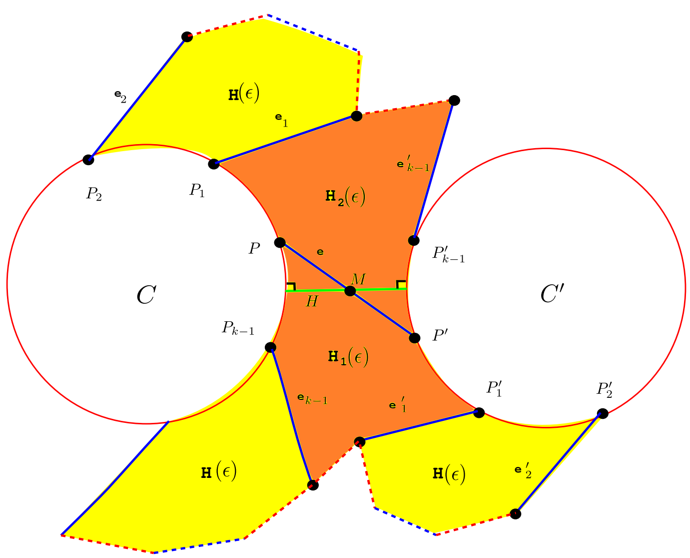

FIGURE 4. Gluing the remaining hexagons $H(\epsilon)$ to Γ

It is tacitly assumed that all gluings respect the metric and the colours of edges.

This defines a hyperbolic surface $\Sigma(k,\epsilon)$ which is homeomorphic to the 3-holed sphere $S_{0,3}$. Two boundary boundary components C and C' of $\Sigma(k,\epsilon)$ are geodesics. The third component, call it $\mathcal{D}$, is piecewise geodesic.

Lemma 9. *The boundary component $\mathcal{D}$ is freely homotopic to a unique geodesic D. Moreover*

(1) D lies in the interior of $\Sigma(k,\epsilon)$,
(2) D meets each arc e_i and e'_i exactly once, and
(3) D does not intersect e.

Proof. The curve $\mathcal{D}$ is piecewise geodesic. It is alternatively composed of $2(k-2)$ blue arcs and $2(k-2)$ red arcs. Since $k \geq 3$, $\mathcal{D}$ is not a geodesic. The inner angles of $\mathcal{D}$ are each less than π, therefore $\mathcal{D}$ is freely homotopic to a unique geodesic D lying in the interior of $\Sigma(k,\epsilon)$.

Let d be a blue edge. There is a 1-parameter family of curves $(\mathcal{D}_t)_{t\in[0,1]}$, realizing a homotopy from $\mathcal{D} = \mathcal{D}_0$ to $D = \mathcal{D}_1$, such that the number of bigons formed by d and $D = \mathcal{D}_t$ does not increase. Since there are no bigons formed by d and $\mathcal{D}_0$, the geometric intersection numbers $i(d,\mathcal{D}_t)$ is constant, which proves the last two claims. □

3.2 The pair of pants $\Pi(k,\epsilon)$ and its central octogon $\mathbf{Q}(\epsilon)$

The geodesic D decomposes $\Sigma(k,\epsilon)$ into two pieces. The component $\Pi(k,\epsilon) \subset \Sigma(k,\epsilon)$ with geodesic curves C, C' and D is a pair of pants.

The blue arcs decompose $\Pi(k,\epsilon)$ into two hexagons adjacent to the central edge e and $2k-4$ quadrilaterals. Let $\mathbf{H}_1(\epsilon) := H_1(\epsilon) \cap \Pi(k,\epsilon)$ and $\mathbf{H}_2(\epsilon) := H_2(\epsilon) \cap \Pi(k,\epsilon)$ be the two hexagons of the decomposition. Their union $\mathbf{Q}(\epsilon) := \mathbf{H}_1(\epsilon) \cup \mathbf{H}_2(\epsilon)$ is a convex octogon.

Let H be the unique perpendicular arc joining C and C' and let $S \in C$ and $S' \in C'$ be its endpoints. For $\epsilon = \pi/2$, we have $H = e$. Otherwise H and e meet as shown in the next lemma.

Lemma 10. *The arc H lies in the octogon $\mathbf{Q}(\epsilon)$. When $\epsilon \neq \pi/2$, H and e intersect at their midpoint.*

Moreover we have

(1) $d(P,S) < L/2$, and
(2) for $\epsilon < \pi/2$, the point P belongs to $\mathbf{H}(\epsilon)$.

Proof. The inner angles of $\mathbf{Q}(\epsilon)$ are less than π, hence there is an isometric embedding

$$\pi : \mathbf{Q}(\epsilon) \to \mathbb{H}.$$

Let Δ and Δ' be the lines of $\mathbb{H}$ containing, respectively, the arcs $\pi\big(C \cap \mathbf{Q}(\epsilon)\big)$ and $\pi\big(C' \cap \mathbf{Q}(\epsilon)\big)$. Let $\overline{H}$ be the common perpendicular arc to Δ and Δ' and let $\overline{S} := \overline{H} \cap \Delta$ and $\overline{S}' := \overline{H} \cap \Delta'$ be its feet. By Lemma 6, we have

$$d_{\mathbb{H}}(\pi(P), \overline{S}) < L/2.$$

Since $\pi\big(C \cap \mathbf{Q}(\epsilon)\big)$ is the geodesic arc of Δ, centered at $\pi(P)$, of length $2L$, it follows that $\overline{S}$ is on the boundary of $\pi\big(C \cap \mathbf{Q}(\epsilon)\big)$. Similarly $\overline{S}'$ is on the boundary of $\pi\big(C \cap \mathbf{Q}(\epsilon)\big)$. By convexity, the arc $\overline{H}$ lies in $\pi\big(\mathbf{Q}(\epsilon)\big)$.

Hence $H = \pi^{-1}(\overline{H})$ belongs to $\mathbf{Q}(\epsilon)$. The other claims follow from Lemma 6 and the fact that π is an isometry. □

4. Trigonometry in the pair of pants $\Pi(k, k', p, \epsilon)$

Let $k, k' \geq 2$ and $p \geq 3$ be integers and let $\epsilon \in]0, \pi[$. The pair of pants $\Pi(k, k', p, \epsilon)$ from the previous section is endowed with an ϵ-edge e joining C and C', whose endpoints are $P \in C$ and $P' \in C'$. By Lemma 7, the length L of e satisfies

$$\cosh(L) = 1 + 2\cos(\pi/p)/\sin\epsilon.$$

Let H (resp. h, resp. h') be the unique common perpendicular arc to C and C' (resp. to C and D, resp. to C' and D). Cutting $\Pi(k, \epsilon)$ along $H \cup h \cup h'$ provides the usual decomposition of the pair of pants into two right-angled hexagons.

4.1 Formula for $\cosh H$
As usual, we will use the same letter for an arc and for its length. As a matter of notation, let $S \in C$ and $S' \in C'$ be the endpoints of H.

Lemma 11. *We have*

$$\cosh H = 1 + 2\cos(\pi/p)\sin\epsilon.$$

Proof. When $\epsilon = \pi/2$, we have $e = H$ and $\cosh H = \cosh L = 2$. Therefore we can assume that $\epsilon \neq \pi/2$.

By Lemma 10, e and H belong to the octogon $\mathbf{Q}(\epsilon)$ and they intersect at their midpoint M. It follows that $SM = H/2$ and $PM = L/2$. By the sine law applied to the triangle PSM, we have

$$\sinh H/2 = \sin\epsilon \sinh L/2.$$

From the identities $2\sinh^2 H/2 = \cosh H - 1$ and $2\sinh^2 L/2 = \cosh L - 1$, it follows that

$$\cosh H - 1 = \sin^2\epsilon\,(\cosh L - 1).$$

By Lemma 7, we have $\cosh L - 1 = 2\cos(\pi/p)/\sin\epsilon$, from which it follows that

$$\cosh H = 1 + 2\cos(\pi/p)\sin\epsilon.$$

□

4.2 Conventions concerning the asymptotics of angle functions
In what follows, we will consider analytic functions $f :]0, \pi[\to \mathbb{R}$. In order to study their asymptotic growth near 0, we will use the following simplified notations. For any pair of functions A, $B :]0, \pi[\to \mathbb{R}$, the expression

$$A \sim B$$

means that

$$\lim_{\epsilon \to 0} \frac{A}{B} = 1.$$

Moreover the expression

$$A \sim *B$$

means that $A \sim aB$ for some positive real number a. Similarly, the expression $A << B$ means that

$$\lim_{\epsilon \to 0} \frac{A}{B} = 0.$$

4.3 Length estimates

Lemma 12. *We have*

(1) $\cosh rL \sim \sinh rL \sim (e^{rL})/2 \sim *\epsilon^{-r}$, *for any* $r > 0$,
(2) $\cosh H - 1 \sim *\epsilon$ *and* $H \sim *\epsilon^{1/2}$,
(3) $d(S, P) = L/2 + o(1)$, *and*
(4) $h \sim *\epsilon^{\frac{k-1}{2}}$ *and* $h' \sim *\epsilon^{\frac{k'-1}{2}}$.

Proof. By Lemma 7, we have $\cosh L \sim *\epsilon^{-1}$. Therefore we have $e^L \sim *\epsilon^{-1}$ and

$$\cosh rL \sim \sinh rL \sim (e^{rL})/2 \sim *\epsilon^{-r}, \text{ for any } r > 0,$$

which prove Assertion 12.1. Lemma 11, implies Assertion 12.2. By Lemma 10, H and e intersect at their midpoint M. Since

$$|d(P, S) - d(P, M)| \leq d(S, M) = H/2$$

we also have $|d(P, S) - L/2| \in o(1)$, which proves Assertion 12.3.

We turn now to the proof of Assertion 12.4. The perpendicular arcs H, h and h' decompose the pair of pants $\Pi(k, \epsilon)$ into two isometric right-angled hexagons (see [5], Proposition 3.1.5) and let $\mathbf{A}$ be one of them.

In an circular order, the hexagon $\mathbf{A}$ has sides $h', k'L/2, H, kL/2, h$ and $D/2$. In order to prove the final statement, we first estimate $\cosh D/2$. By the law of cosines, we have

$$\cosh H = \frac{\cosh(kL/2)\cosh(k'L/2) + \cosh D/2}{\sinh(kL/2)\sinh(k'L/2)}$$

It follows that

$$\begin{aligned}
&\sinh(kL/2)\sinh(k'L/2)(\cosh H - 1) \\
&\quad = \cosh(kL/2)\cosh(k'L/2) - \sinh(kL/2)\sinh(k'L/2) + \cosh D/2 \\
&\quad = \cosh((k - k')L/2) + \cosh D/2.
\end{aligned}$$

Thus we have

$$\cosh D/2 = \sinh(kL/2)\sinh(k'L/2)(\cosh H - 1) - \cosh((k - k')L/2).$$

By Assertions 12.1 and 12.2, we have

$$\cosh((k - k')L/2) \sim *\epsilon^{-|k-k'|/2}, \text{ and}$$
$$\sinh(kL/2)\sinh(k'L/2)(\cosh H - 1) \sim *\epsilon^{1-(k+k')/2}.$$

Since k, k' are ≥ 2, we have $(k + k')/2 - 1 > |k - k'|/2$, therefore we have

$$\sinh D/2 \sim \cosh D/2 \sim *\epsilon^{1-(k+k')/2}.$$

By the law of sines, we have

$$\sinh h = \frac{\sinh H \sinh k'L/2}{\sinh d}$$

and therefore $h \sim \sinh h \sim *\epsilon^{(k-1)/2}$. The assertion concerning h' is similar. □

4.4 The angles ω_i and ω'_i

By Lemma 9, the geodesic D meets the arcs e_i and e'_i exactly once each, so we can define

$$\omega_i := \angle De_i \text{ and } \omega'_j := \angle De'_i$$

for $1 \leq i \leq k-1$ and $1 \leq j \leq k'-1$.

Lemma 13. *We have*

(1) $\omega_1 < \omega_2 < \cdots < \omega_{k-1}$, and
(2) $\omega'_1 < \omega'_2 < \cdots < \omega'_{k'-1}$.

Proof. The pair of arcs e_1 and e_{k-1} decompose $\Pi(k,\epsilon)$ into two connected components. Let $\mathbf{Q}_0$ be the contractible component, which is a quadrilateral whose sides are e_1, e_{k-1}, an arc of C and an arc of D. It is larger than the quadrilateral $\mathbf{Q}$ of Figure 5, because the top edge is e_1 instead of h.

Let $\pi : \mathbf{Q}_0 \to \mathbb{H}$ be a isometric embedding. Set $\Delta' = \pi(D \cap \mathbf{Q})$ and let Δ be the line of $\mathbb{H}$ containing the arc $\pi(C \cap \mathbf{Q}_0)$. The arcs $\pi(e_1), \ldots, \pi(e_{k-1})$ belongs to the ϵ-pencil $\mathcal{F}_\epsilon(\Delta)$ of the line Δ. The first claim therefore follows from Lemma 5 (1). The second claim is similar. □

4.5 Angle estimates

Lemma 14. *For any integers $1 \leq i \leq k-1$ and $1 \leq i \leq k'-1$, we have*

$$\lim_{\epsilon\to 0} \omega_i = 0 \text{ and } \lim_{\epsilon\to 0} \omega'_j = 0.$$

Proof. By Lemma 13, it is enough to show that $\lim_{\epsilon\to 0} \omega_{k-1} = 0$ and $\lim_{\epsilon\to 0} \omega'_{k'-1} = 0$.

The pair of arcs h and e_{k-1} cut $\Pi(k,\epsilon)$ into two connected components, where $\mathbf{Q}$ is the contractible component, as shown in Figure 5. Then $\mathbf{Q}$ is a convex quadrilateral whose vertices are the endpoints of h, namely $N := h \cap C$ and $\Omega := h \cap D$ and the endpoints of e_{k-1}, namely $\Omega_{k-1} := e_{k-1} \cap D$ and P_{k-1}.

Let v be the diagonal of $\mathbf{Q}$ joining P_{k-1} and Ω. Set

$$\epsilon^- = \angle Cv,\ \epsilon^+ = \angle ve_{k-1},\ \gamma^- = \angle hv \text{ and } \gamma^+ = \angle vD.$$

By definition we have

$$\epsilon^- + \epsilon^+ = \epsilon, \text{ and } \gamma^- + \gamma^+ = \pi/2.$$

First, we look at the trigonometry of the triangle $P_{k-1}N\Omega$. Set $a = d(N, P_{k-1})$. We have $a = (k/2-1)L + d(S,P)$. By Lemma 11, we have $d(S,P) = L/2 + o(1)$, therefore $a = uL + o(1)$, where $u = (k-1)/2$. It follows that

$$\cosh a \sim *\epsilon^{-u}.$$

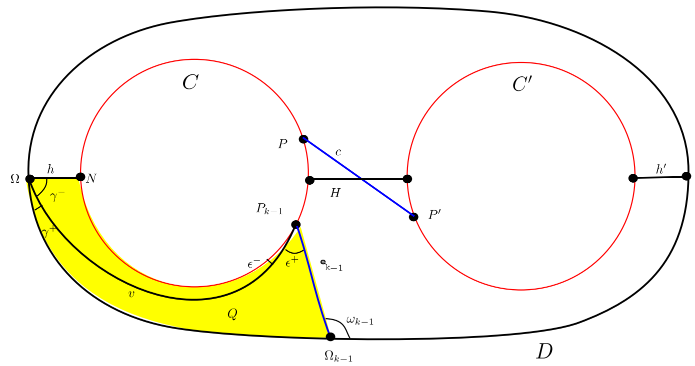

FIGURE 5. This figure illustrates the notation from the proofs of Lemmas 14 and 13.

Since $\cosh v = \cosh a \cosh h$, it follows from Lemma 12 that

$$\cosh v \sim \cosh a \sim *\epsilon^{-u}.$$

By the sine law, we have $\sin \epsilon^- = \sinh h / \sinh v$. It follows from Lemma 12 and the previous estimate that $\epsilon^- \sim *\epsilon^{2u}$. Since $k \geq 3$, we have $\epsilon^- << \epsilon$ and therefore

$$\epsilon^+ \sim \epsilon.$$

By combining the cosine and sine laws, we have $\cos \gamma^- = \sin \epsilon^- \cosh a = \sinh h \cosh a / \sinh v$, and therefore

$$\cos \gamma^- \sim *\epsilon^u.$$

Next, we will look at the trigonometry of the triangle $P_{k-1}\Omega\Omega_{k-1}$. Since $\sin \gamma^+ = \cos \gamma^-$, we have

$$\gamma^+ \sim \sin \gamma^+ \sim *\epsilon^u.$$

Using the cosine law, we have $\cos \overline{\omega}_{k-1} = \sin \epsilon^+ \sin \gamma^+ \cosh v - \cos \epsilon^+ \cos \gamma^+)$. Adding one on each side and using that $\cos \overline{\omega}_{k-1} = -\cos \omega_{k-1}$, we obtain

$$1 - \cos \omega_{k-1} = \sin \epsilon^+ \sin \gamma^+ \cosh v + (1 - \cos \epsilon^+ \cos \gamma^+).$$

We will now estimate the right term of the previous identity. Using a Taylor expansion, it is clear that

$$1 - \cos \epsilon^+ \cos \gamma^+ \in O(\epsilon^2) + O(\epsilon^{2u}) = O(\epsilon^2).$$

On the other hand, we have

$$\sin \epsilon^+ \sin \gamma^+ \cosh v \sim *\epsilon.$$

Hence we have $1 - \cos \omega_{k-1} \sim *\epsilon$, i.e.

$$\omega_{k-1} \sim *\epsilon^{1/2},$$

and therefore we have proved that $\lim_{\epsilon\to 0}\omega_{k-1}=0$.

□

5. Sanki's paths and curve duality

For the whole section, we assume given an oriented topological closed manifold $\mathcal{S}$ of genus g endowed with an isomorphism $\rho:\Pi_g\to\pi_1(\mathcal{S})$, defined modulo the inner conjugations. It will be called the *marking* of the topological surface $\mathcal{S}$.

We will consider a set $\mathrm{Tess}(\mathcal{S})$ of hexagonal tesselations of $\mathcal{S}$, which are defined by a set $\mathcal{C}_R$ of red curves and a set $\mathcal{C}_B$ of blue curves. Following an idea of [15][2], we define, for each $\tau\in\mathrm{Tess}(\mathcal{S})$, a path, called a Sanki's path, $\sigma:]0,2\pi[\to\mathcal{T}_g$. Intuitively, Sanki's path are infinitesimal analogs of Penner's construction [13] of quasi-Anosov homeomorphisms.

When τ satisfies some additional properties, we define, for each blue curve B a dual object B^*, which is a linear combination of three curves with coefficients $\pm 1/2$. Of course, B^* is not a multicurve, but its length function $L(B^*)$ is well defined. The first result of the paper is Theorem 17, showing a kind of duality between B and B^*. It is expressed in terms of the Poisson bracket $\{L(A),L(B^*)\}$ relative to the Weil-Petersson symplectic form [19].

5.1 The set of hexagonal tesselations $\mathrm{Tess}(\mathcal{S})$
Let $\mathcal{H}$ be an oriented topological hexagon whose six sides are alternatively coloured in red and blue. Strictly speaking $\mathcal{H}$ is a closed disc whose boundary is divided into six components, but the terminology hexagon is more suggestive.

Let $\mathrm{Tess}(\mathcal{S})$ be the set of all tesselations τ of $\mathcal{S}$ satisfying the following two axioms:

(AX1) The tiles are homeomorphic to $\mathcal{H}$ and they are glued pairwise along edges of the same colour.

(AX2) Each vertex of the tesselation has valence four.

The last axiom implies that each vertex is the endpoint of four edges, which are alternately red and blue. The graph consisting of red edges is a disjoint union of cycles. Those cycles are called the *red curves of the tesselation* and the set of red curves is denoted $\mathcal{C}_{red}$. Similarly, we define the *blue curves of the tesselation* and the set $\mathcal{C}_{blue}$ of blue curves. The set $\mathrm{Curv}(\tau):=\mathcal{C}_{red}\cup\mathcal{C}_{blue}$ is called the set of *curves of the tesselation*.

5.2 Sanki paths
We will now define the Sanki's path of a tessalation $\tau\in\mathrm{Tess}(\mathcal{S})$. Let $\epsilon\in]0,\pi[$. Define a metric on the 1-skeleton τ_1 of τ by requiring that all edges have length L. Recall that $L=\operatorname{arcosh}(1+1/\sin\epsilon)$ is the side lengths of the hexagon $H(\epsilon)$ defined in Subsection 2.3.

For each closed face f of the tesselation, let $\phi_f:H(\epsilon)\to f$ be a homeomorphism such that its restriction to the boundary $\delta f:\partial H(\epsilon)\to\partial f$ preserves the metric and the colour of the edges.

A tesselation of $\mathcal{S}$ is obtained where each tile is endowed with a hyperbolic structure. Along each edge of τ_1, two geodesic arcs have been glued isometrically. Around each vertex of τ_1, the four angles are alternatively ϵ and $\overline{\epsilon}$, hence their sum is 2π. By Theorem 1.3.5 of

[5], there is a hyperbolic metric on $\mathcal{S}$ extending the metric of the tiles. Together with the marking ρ of $\mathcal{S}$, we obtain a well defined marked hyperbolic surface $S_\tau(\epsilon)$.

The idea of deforming right-angled regular polygons by polygons with angles of value alternatively ϵ and $\pi-\epsilon$ first appeared in [15] and it was used in [2]. Therefore the corresponding path $\sigma_\tau :]0,\pi[\to \mathcal{T}_g$, $\epsilon \mapsto S_\tau(\epsilon)$ will be called the *Sanki's path* of the tessalation τ. Since the function $\cosh L = 1 + 1/\sin\epsilon$ is analytic, the path σ_τ is analytic.

It should be noted that, around each vertex the colours, blue or red, and the angles, ϵ or $\overline{\epsilon}$, of the edges alternate. Therefore the blue curves and the red curves are geodesics with respect to the hyperbolic metric on $S_\tau(\epsilon)$.

5.3 k-regular tesselations

For a closed oriented surface $\mathcal{S}$, $(AX1)$ and $(AX2)$ is the minimal set of axioms required to define the Sanki's path. We will now define more axioms. The axiom (AX3) will ensure that the curves have the same length, while the axioms (AX4) and (AX5) are connected with the duality construction.

Let $k \geq 2$ be an integer and let $\mathcal{S}$ be a closed surface. A tesselation $\tau \in \mathrm{Tess}(\mathcal{S})$ is called a *k-regular tesselation* iff it satisfies the following axiom

(AX3) Each curve of τ, blue or red, consists of exactly k edges.

Denote by $\mathrm{Tess}(\mathcal{S}, k)$ the set of all k-regular tesselations. For any k-regular tesselation τ, we will consider two additional axioms. The first axiom is

(AX4) A blue edge and a red curve meet at most once.

Assume now that $\tau \in \mathrm{Tess}(\mathcal{S}, k)$ satisfies (AX4). Let R be a red curve, let b, b' be two blue edges adjacent to R and let N be a small regular neigborhood of R. Since $\mathcal{S}$ is oriented, $N \setminus R$ consists of two open annuli, $N^\pm$. By axiom (AX4), b has only one endpoint in R, therefore b intersect either N^+ or N^-. Similarly, b' intersect either N^+ or N^-. We say that b, b' are *adjacent on the same side* of C if they both intersect N^+ or if they both intersect N^-. Our last axiom is

(AX5) Two distinct blue edges are adjacent on the same side of at most one red curve.

Denote by $\mathrm{Tess}_{45}(\mathcal{S}, k)$ the set of k-regular tesselations satisfying the axioms (AX4) and (AX5).

4.4 The isometric embedding $\pi_b : \Pi(k,\epsilon) \to S_\tau(\epsilon)$

From now on, assume that $k \geq 3$. Let $\tau \in \mathrm{Tess}_{45}(\mathcal{S}, k)$. In order to define the duality, we first associate to each blue edge a pair of pants $\Pi(k,\epsilon) \subset S_\tau(\epsilon)$.

Let b be a blue edge with endpoints Q and Q', and let R and R' be the red curves passing through Q and Q'. By axiom (AX4), the two curves R and R' are distinct, so the graph $\Gamma_0 := R \cup R' \cup b$ is a union of two circles connected by an edge. Since $\mathcal{S}$ is oriented, a small normal open neighbourhood N of Γ_0 is a thickened eight. Then $R \cup R'$ cuts N into three components, two of them are homeomorphic to annuli and the third one, call it Ω, contains b.

In a planar representation of Γ_0, Ω is the exterior of $R \cup R'$. Since R and R' are boundary components of Ω, these curves inherit an orientation. By axiom (AX3), R contains k vertices of the tesselation. Starting from Q in the anticlockwise direction, the other $k-1$ points of R are denoted by $Q_1, \dots, Q_{k-1}$. For each $1 \leq i \leq k-1$ let b_i be the blue edge starting at Q_i on the same side as b. The points $Q'_1, \dots, Q'_{k-1}$ of R' and the edges b'_i are defined similarly. Adding the edges b_i, b'_i to the graph Γ_0, we obtain a graph $\overline{\Gamma}$.

Let $\epsilon \in]0, \pi[$ and recall that $S_\tau(\epsilon)$ is the tesselated surface S_g representing a point in $\mathcal{T}_g$.

Let Γ be the graph defined in Section 4.1. There is a local isometry $\pi : \Gamma \to \overline{\Gamma}$ such that $\pi(P) = Q$, $\pi(P_i) = Q_i$, $\pi(e_i) = b_i$, $\pi(C) = R$, $\pi(P') = Q'$, $\pi(P'_i) = Q'_i$, $\pi(e_i) = b_i$ and $\pi(C') = R'$. Clearly, it can be extended uniquely to a local isometry $\pi : \Pi(k, \epsilon) \to S_\tau(\epsilon)$. Let π_b be its restriction to $\Pi(k, \epsilon)$.

Lemma 15. *The map $\pi_b : \Pi(k, \epsilon) \to S_\tau(\epsilon)$ is an isometric embedding.*

Proof. Since b joins R and R', it follows from Axiom (Ax4) that R and R' are distinct. By Axioms (Ax4) and (Ax5), the point Q_i, respectively Q'_j is the unique endpoint of $b_i \cap \Omega$, respectively of $b'_j \cap \Omega$. Hence the blue edges b, b_i and b'_j are all distinct. For any two edges $e \neq e'$ of $\Gamma \cap \Pi(k, \epsilon)$, we therefore have $\pi(e) \neq \pi(e')$.

Let F, F' be two faces of $\Pi(k, \epsilon)$ such that $\pi(F) = \pi(F')$. Since each face F or F' has at least two blue edges in Γ, it follows that F and F' contain a common blue edge $e \subset \Gamma$. It follows easily that $F = F'$.

Consequently, the restriction of π to $\Pi(k, \epsilon) \setminus \mathcal{D}$ is injective. By Lemma 9, $\Pi(k, \epsilon)$ lies in $\Sigma(k, \epsilon) \setminus \mathcal{D}$. Therefore π induces an isometric embedding $\pi_b : \Pi(k, \epsilon) \to S_\tau(\epsilon)$. ☐

5.5 The dual functions $L(B^*)$
Let $\tau \in \mathrm{Tess}_{45}(\mathcal{S}, k)$ for some integer $k \geq 3$.

We are now going to define the dual function $L(B^*)$, for any blue curve of B of the tesselation. Choose anf fix one edge b of B. Let R, R' be the two red curves containing the endpoints of b and set $D_b = \pi_b(D)$. Set

$$L(B^*) = \frac{1}{2}(L(R) + L(R') - L(D_b)).$$

Informally speaking, $L(B^*)$ is the length function associated with the "dual curve" $B^* = 1/2(R + R' - D_b)$. Strictly speaking, the function $L(B^*)$ depends of the choice of the edge b.

For any $F, G \in C^\infty(\mathcal{T}_g)$, let $\{F, G\}$ be their Poisson bracket induced by the the Weil-Petersson symplectic form on $\mathcal{T}_g$, see e.g. [19]. The duality between B and B^* is demonstrated in the next lemma.

Lemma 16. *Let $\tau \in \mathrm{Tess}_{45}(\mathcal{S}, k)$ and let $\sigma_\tau :]0, \pi[\to \mathcal{T}_g$ be the associated Sanki path.*

For any A, $B \in \mathcal{C}_{blue}$, we have

$$\lim_{\epsilon\to 0}\{L(A), L(B^*)\}(\sigma_\tau(\epsilon)) = \delta_{A,B},$$

where $\delta_{A,B}$ *is the Kronecker delta.*

Proof. Let $B \in \mathcal{C}_{blue}$. By definition there is an edge b of B such that $2L(B^*) = L(R) + L(R') - L(D_b)$ where R and R' are the two red curves containing the endpoints of b.

Set $\overline{\Pi} = \pi_b(\Pi(k,\epsilon))$ and $\overline{D} = \pi_b(D_b)$. For each $i \in \{1, 2, \dots, k-1\}$, set

$$\beta_i = c_i \cap \overline{\Pi}(k,\epsilon) \text{ and } \beta_i' = c_i' \cap \overline{\Pi}(k,\epsilon).$$

By Lemma 15, β_i is an arc, with one endpoint P_i in R and the other endpoint Ω_i on $\overline{D}$. Similarly, β_i' is an arc, with one endpoint Q_i' in R' and the other endpoint, say Ω_i', belongs to $\overline{D}$. We have

(1) β_i does not intersect R',
(2) $\beta_i \cap R = Q_i$ and $\angle R\beta_i = \epsilon$
(3) $\beta_i \cap \overline{D} = \Omega_i$ $\angle \overline{D}\beta_i = \omega_i$,

where the angles ω_i are defined in Section 4.4. Similarly, we have

(1) β_i' does not intersect R,
(2) $\beta_i' \cap R' = P_i'$ and $\angle R\beta_i = \epsilon$
(3) $\beta_i' \cap \overline{D} = \Omega_i$ $\angle \overline{D}\beta_i' = \omega_i'$.

Let $A \in \mathcal{C}_{blue}$ be another blue curve. Set

$$I = \{i \mid 1 \le i \le k-1 \,\text{and}\, \beta_i \subset A\}, \text{ and}$$
$$I' = \{i \mid 1 \le i \le k-1 \text{ and}\beta_i' \subset A\}.$$

When $A \neq B$, the curve A meets $R \cup R' \cup D_b$ exactly at the points P_i, Ω_i for $i \in I$ and P_i', Ω_i' for $i \in I$. Therefore by Wolpert's formula [18], we have

$$\{L(A), L(R) + L(R') - L(D_b)\}(\sigma_\tau(\epsilon))$$
$$= [\textstyle\sum_{i\in I} \cos\epsilon - \cos\omega_i] + [\sum_{i\in I'} \cos\epsilon - \cos\omega_i'].$$

By Lemma 14, we have

$$\lim_{\epsilon\to 0} \omega_i = 0,$$

and therefore

$$\lim_{\epsilon\to 0}\{L(A), L(B^*)\}(\sigma_\tau(\epsilon)) = 0$$

When $A = B$, the computation is similar except that, in addition to the arcs β_i for $i \in I$ and β_i' for $i \in I'$, the geodesic A contains b. Therefore, one obtains

$$\{L(A), L(R) + L(R') - L(D_b)\}(\sigma_\tau(\epsilon))$$
$$= 2\cos\epsilon + [\textstyle\sum_{i\in I} \cos\epsilon - \cos\omega_i] + [\sum_{i\in I} \cos\epsilon - \cos\omega_i'],$$

and therefore

$$\lim_{\epsilon\to 0}\{L(A), L(A^*)\}(\sigma_\tau(\epsilon)) = 1.$$

$\square$

5.6 The duality theorem
Suppose $k \geq 3$ and choose $\tau \in \mathrm{Tess}(\mathcal{S}, k)$. Recall that $\sigma_\tau :]0, \pi[\to \mathcal{T}_g$ is the Sanki path.

Theorem 17. *Assume that τ satisfies the axioms (AX4) and AX(5). Then for any $\epsilon \in]0, \pi[$ outside some finite set F, the set*

$$\{\mathrm{d}L(C) \mid C \in \mathrm{Curv}(\tau)\}$$

is linearly independent at the point $\sigma_\tau(\epsilon)$.

Proof. For $\epsilon \in]0, \pi[$, let $\delta(\epsilon)$ be the determinant of the square matrix

$$(\{L(A), L(B^*)\}(\sigma_\tau(\epsilon)))_{A,B \in \mathcal{C}_{blue}},$$

and set $F = \{\epsilon \in]0, \pi[\mid \delta(\epsilon) = 0\}$.

By Lemma 16, we have $\lim_{\epsilon \to 0} \delta(\epsilon) = 1$. Moreover, changing the orientation of $\mathcal{S}$ amounts to replacing ϵ by $\overline{\epsilon}$, so we also have $\lim_{\epsilon \to \pi} \delta(\epsilon) = \pm 1$. Since δ is an analytic function on $]0, \pi[$, it follows that F is finite.

It remains to show that, for $\epsilon \notin F$, the differentials at $\sigma_\tau(\epsilon)$ of the set of length functions $\{L(C) \mid C \in \mathrm{Curv}(\tau)\}$ are linearly independent.

Let $\epsilon \notin F$. Let $(a_A)_{A \in \mathrm{Curv}(\tau)}$ be an element of $\mathbb{R}^{|\mathrm{Curv}(\tau)|}$ such that

$$\sum_{A \in \mathrm{Curv}(\tau)} a_A dL(A)|_{\sigma_\tau(\epsilon)} = 0.$$

Let $B \in \mathcal{C}_{blue}$. Recall that B^* is a linear combination of two red curves R, R' and a certain geodesic D_b. Neither the geodesics D_b nor the red curves meet any red curve transversally. Hence we have $\{L(C), L(B^*)\} = 0$ for any red curve C. It follows that

$$\sum_{A \in \mathcal{C}_{blue}} \{a_A L(A), L(B^*)\}$$

is zero at $\sigma_\tau(\epsilon)$. Since $\delta(\epsilon) \neq 0$, we have $a_A = 0$ for any $A \in \mathcal{C}_{blue}$.

Therefore, it follows that

$$\sum_{A \in \mathcal{C}_{red}} a_A dL(A)|_{\sigma_\tau(\epsilon)} = 0.$$

Since it is a subset of some Fenchel-Nielsen coordinates, the set of differentials $\{\mathrm{d}L(A) \mid A \in \mathcal{C}_{red}\}$ is linearly independent. Therefore we also have $a_A = 0$ for any $A \in \mathcal{C}_{red}$.

Hence the differentials $(\mathrm{d}L(A))_{A \in \mathrm{Curv}(\tau)}$ are linearly independent at $\sigma_\tau(\epsilon)$. □

6. Local structure of $\mathcal{P}_g$ along a Sanki's path

Our previous work [10] provides a systematic construction of k-regular tesselations τ. To apply Theorem 17, we first show that the axioms $(AX4)$ and $(AX5)$ are automatically satisfied when the curves of $S_\tau(\pi/2)$ are the systoles.

Then we deduce the local structure of the Thurston's spine along a Sanki's path.

To prove the main result, we will apply Theorem 17 to the tesselations τ_g of the surface S_g obtained in [10]. Obviously they satisfy the axioms $(AX1-3)$ and it remains to prove that the tesselations τ_g also satisfy $(AX4)$ and $(AX5)$.

6.1 Verification of the axioms $(AX4)$ *and* $(AX5)$
As usual, suppose $\mathcal{S}$ is a surface of genus $g \geq 2$, $k \geq 3$ and $\tau \in \mathrm{Tess}(\mathcal{S}, k)$.

Lemma 18. *Assume that the set of systoles of* $S_\tau(\pi/2)$ *is the set of curves of* τ. *Then the tesselation* τ *satisfies the axioms (AX4) and (AX5).*

Proof. By a well-known lemma of riemannian geometry, two distinct systoles intersect in at most one point, therefore (AX4) is satisfied.

The proof of Axiom (AX5) is more delicate. Let $C, C' \in \mathcal{C}$ and let c, d be two distinct blue edges connecting C and C' on the same side. Let $(P, P') \in C \times C'$ be the endpoints of c, and let $(Q, Q') \in C \times C'$ be the endpoints of d, as shown in Figure 6.

Since c and d are adjacent to C on the same side, there is a planar representation of C and C' where c and d are on the exterior of C. This planar representation provides an orientation of C and C', called the *direct* orientation.

Let F_1 and F_2 be the two hexagons containing d. Set $f_1 = F_1 \cap C$, $f_2 = F_1 \cap C$, $f_1' = F_1 \cap C$, $f_2' = F_1 \cap C$. By definition, f_1', d and f_1 are consecutive edges of F_1. We can assume f_1', d and f_1 are ordered relative to the direct orientation. Consequently f_2 follows f_1 relative to the orientation of C. Since $\mathcal{S}$ is oriented, f_1' follows f_2' in the direct orientation of C'. Therefore the relative position of F_1 and F_2 along C and C' is as in Figure 6.

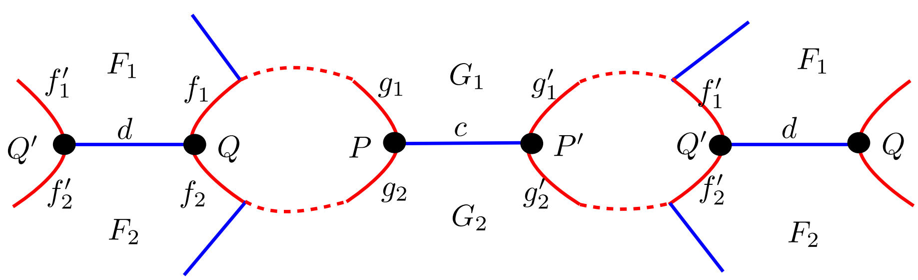

FIGURE 6. Respective positions of F_1 and F_2. They appear on the left side and the right sides of the figure: it should be understood that they lie on a cylinder.

For $i = 1$ or 2, let γ_i be the arc of C from Q to P and containing f_i. Similarly let γ_i' be the arc of C' from P' to Q and containing f_i'. Let us orient c from P to P' and d from Q' to Q.

The arc γ_1, γ_2, γ_1' and γ_2' cover $C \cup C'$, therefore we have

$$L(\gamma_1) + L(\gamma_1') + L(\gamma_2) + L(\gamma_2') = 2kL.$$

It is possible to assume without loss of generality that

$$L(\gamma_1) + L(\gamma_1') \leq kL.$$

The path γ_1 consists of $L(\gamma_1)/L$ edges. Let g_1 be the last edge of γ_1. Similarly, let g_1' be the first edge of γ_1'. By definition, g_1 contains P and g_1' contains P'. Let G_1 be the hexagon with the three consecutive edges g_1, c and g'_1.

Now $f_1 \neq g_1$ and $f'_1 \neq g'_1$, otherwise the edge d would join g_1 and g'_1. Hence we have $F_1 \neq G_1$. Therefore there is a factorization $\gamma_1 = f_1 * \delta_1 * g_1$, where δ_1 is the geodesic arc between f_1 and g_1 and where the notation $*$ stands for the concatenation of paths. Similarly, there is a factorization $\gamma'_1 = f_1 * \delta'_1 * g_1$.

We show now that the loop

$$\gamma := \gamma_1 * c * \gamma'_1 * d,$$

is not null-homotopic. Assume otherwise. Set $S_g = S_\tau(\pi/2)$, let $\pi : \mathbb{H} \to S_g$ be the universal cover of S_g and let $\tilde{\gamma}$ be a lift of γ in $\mathbb{H}$. Since γ is composed of four geodesic arcs, and the angles between them are $\pi/2$, the lift $\tilde{\gamma}$ would bound a quadrilateral whose inner angles are all $\pi/2$ or $3\pi/2$ which is impossible. It follows that γ is not null-homotopic.

The hexagon F_1 contains a Saccheri quadrilateral whose basis is d and with feet given by f_1 and f'_1. Let d' be the fourth side, oriented from f'_1 to f_1. As a path, d' is homotopic to $f'_1 * d * f_1$.

Similarly, let c' be the last side, oriented from g_1 to g'_1, of the Saccheri quadrilateral in G_1 whose basis is c and whose feet are g_1 and g'_1. Similarly, c' is homotopic to $g_1 * d * g'_1$.

Up to a reparametrization, we have

$$\gamma = \delta_1 * g_1 * c * g'_1 * \delta'_1 * f'_1 * d * f_1.$$

Hence γ is homotopic to $\tilde{\gamma} = \delta_1 * c' * \delta'_1 * d'$. Since we have $L(c') = L(d') = L'$, we have

$$L(\tilde{\gamma}) = L(\gamma_1) + L(\gamma'_1) + 2L' - 4L < kL$$

by Lemma 8, which contradicts that $\mathcal{C}$ is the set of systoles. ☐

6.2 Two corollaries

We will now derive two corollaries concerning the structure of $\mathcal{P}_g$ at the neighborhood of a Sanki's path.

Given a finite set of curves $\mathcal{C} = \{C_1, C_2, \dots, C_n\}$, let $E(\mathcal{C})$ be the set of $x \in \mathcal{T}_g$ such that

$$L(C_1)(x) = L(C_2)(x) = \cdots = L(C_n)(x).$$

Also let $\mathrm{Sys}\,(\mathcal{C})$ be the set of points $x \in \mathcal{P}_g$ such that $\mathcal{C}$ is the set of systoles at x.

Let $X \subset \mathcal{P}_g$ be a locally closed subset, and let $x \in \overline{X}$. We say that X is *locally a smooth manifold at x* if $U \cap X$ is a smooth manifold for some open neighborhood of X. When it is the case, the local codimension $\mathrm{codim}_x X$ is well defined. Our previous definition do not require that x belongs to X.

Corollary 19. *Let $\tau \in \mathrm{Tess}(\mathcal{S}, k)$ for some $k \geq 3$ such that $\mathrm{Curv}(\tau)$ is the set of systoles of $S_\tau(\pi/2)$.*

Let $\mathcal{C} \subset \mathrm{Curv}(\tau)$ be any filling subset. Then for any $\epsilon \neq \pi/2$ closed to $\pi/2$, we have

(i) $\sigma_\tau(\epsilon)$ belongs to $\overline{\mathrm{Sys}\,(\mathcal{C})}$,

(ii) $\mathrm{Sys}\,(\mathcal{C})$ is a smooth manifold in the neighborhood of $\sigma_\tau(\epsilon)$, and

(iii) $\mathrm{codim}_{\sigma_\tau(\epsilon)} \mathrm{Sys}\,(\mathcal{C}) = \mathrm{Card}\,\mathcal{C} - 1$.

Proof. By Lemma 18, the tesselation τ belongs to $\mathrm{Tess}_{45}(\mathcal{S}, k)$. Hence by Theorem 17, the map

$$L(\mathcal{C}) : \mathcal{T}(\mathcal{S}) \to \mathbb{R}^{\mathcal{C}}, x \mapsto (L(C)(x))_{C \in \mathrm{Curv}(\tau)}$$

is a submersion at the point $\sigma_\tau(\epsilon)$ for all $\epsilon \neq \pi/2$ closed to $\pi/2$. By the submersion theorem, $E(\mathcal{C})$ is smooth of codimension Card $\mathcal{C} - 1$ around the point $\sigma_\tau(\epsilon)$ and $\sigma_\tau(\epsilon)$ is adherent to the set $E^+(\mathcal{C})$ of all $x \in E(\mathcal{C})$ defined by the inequations

$$L(C)(x) < L(C')(x), \text{ for all } C \in \mathcal{C} \text{ and } C' \in \mathrm{Curv}(\tau) \setminus \mathcal{C}.$$

Thus Assertion (ii) and (iii) follows from the fact that $\mathrm{Sys}\,(\mathcal{C})$ is an open set of $E(\mathcal{C})$, see [16][17]. □

Corollary 20. *Under the hypothesis of Corollary 19, the point $\sigma_\tau(\pi/2)$ is adherent to* $\mathrm{Sys}\,(\mathcal{C})$ *and we have*

$$\mathrm{codim}\, \mathrm{Sys}\,(\mathcal{C}) < \mathrm{Card}\,(\mathcal{C}).$$

6.3 The main result from [10]

A *decoration* of the hexagon $H(\pi/2)$ is a cyclic indexing of it six sides by $\mathbb{Z}/6\mathbb{Z}$. Up to direct isometries, there are exactly two decorated hexagons, say $\mathcal{H}$ and $\overline{\mathcal{H}}$.

Let S be a closed hyperbolic surface. A *standard* hexagonal tesselation τ of S is a tesselation of S, where each tile is isomorphic to $\mathcal{H}$ or $\overline{\mathcal{H}}$. Of course, it is assumed that tiles are glued along edges of the same index.

Theorem (Theorem 25 of [10])**.** *There exists an infinite set A of integers $g \geq 2$, and, for any $g \in A$, a closed oriented hyperbolic surface S_g of genus g endowed with a standard tesselation τ_g satisfying the following assertions*

(1) the systoles of S_g are the curves of τ_g, and
(2) we have

$$\mathrm{Card}\ \mathrm{Syst}(S_g) \leq \frac{57}{\sqrt{\ln \ln \ln g}}\ \frac{g}{\sqrt{\ln g}}.$$

6.4 Proof of Theorem 1

Theorem 1. *There exists an infinite set A of integers $g \geq 2$ such that*

$$\mathrm{codim}\, \mathcal{P}_g < \tfrac{38}{\sqrt{\ln \ln \ln g}}\ \tfrac{g}{\sqrt{\ln g}},$$

for any $g \in A$.

Proof. Let A be the set of the of the theorem of Subsection 6.3. Let $g \in A$ and let τ_g be the corresponding tesselation.

By hypotheses, any curve C of τ consists of edges of the same index. By extension it will be called the index of the curve. Let $\mathcal{C}$ be the set of all curves of index $3, 4, 5$ or 6. We claim that $\mathcal{C}$ fills the surface.

Let P be a vertex at the intersection of two curves of index 1 and 2. Let $\mathbf{Q}$ be the union of the four hexagons surrounding P. It turns out that $\mathbf{Q}$ is a 12-gon whose edges have indices distinct from 1 and 2. It follows that $\mathcal{C}$ cuts the surface into these 12-gons.

It is clear that Card $\mathcal{C} = 2/3\,\text{Card Syst}(S_g)$. To finish the proof, it is enough to show that τ_g satisfies the hypothesis of Corollary 20.

We can assign the red colour to the edges of τ_g, of index $1, 2$ or 3 and the blue colour to other edges. Moreover, since all curves have the same length, the tesselation is k-regular for some k. The case $k = 2$ was excluded from consideration in [10] so we have $k \geq 3$. In fact, the decoration implies that k is even [10], so we have $k \geq 4$. It follows that τ_g belongs to $\text{Tess}(\mathcal{S}, k)$ for some $k \geq 4$.

It follows from Corollary 20 that

$$\text{codim}\,\mathcal{P}_g < \tfrac{38}{\sqrt{\ln\ln\ln g}}\;\tfrac{g}{\sqrt{\ln g}}.$$

□

7. Examples

Before [3], it was a challenging question to know if $\text{codim}\mathcal{P}_g$ was less than $2g - 1$. Since the bounds in [3] are not explicit, it is still interesting to know the smallest g for which $\text{codim}\mathcal{P}_g < 2g - 1$. We will describe our construction for $g = 17$ and show that $\text{codim}\mathcal{P}_{17} < 33$.

We will first briefly explain the case $2k = 2$ which was excluded from consideration in order to avoid some specificity.

7.1 Standard $2k$-regular tesselations

We will briefly explain the construction of all standard $2k$-regular tesselations, following [10]. Let $\mathcal{H}$ be a decorated right-angled regular hexagon of the Poincaré half-plane $\mathbb{H}$. For each $i \in \mathbb{Z}/6\mathbb{Z}$, let s_i be the reflection in the line Δ_i containing the side of index i of $\mathcal{H}$. The group W generated by these reflections is a Coxeter group with presentation

$$\langle s_i \mid (s_i s_{i+1})^2 = 1, \forall i \in \mathbb{Z}/6\mathbb{Z}\rangle.$$

By a theorem of Poincaré, the collection of hexagons $\{w.\mathcal{H}\}$ is the set of tiles of a tesselation of τ of $\mathbb{H}$.

Let W^+ be the subgroup of index two consisting of products of an even number of generators. Let $k \geq 1$. Let H be a subgroup of W satisfying

(1) H is a finite index subgroup of W^+,
(2) $H^w \cap \langle s_i, s_{i+1}\rangle = \{1\}$, and
(3) $H^w \cap \langle s_i s_{i+1}\rangle = \langle (s_i s_{i+1})^k\rangle$,

for any $i \in \mathbb{Z}/6\mathbb{Z}$ and $w \in W$, where H^w stands for wHw^{-1}.
Then $\mathbb{H}/H$ is a closed oriented hyperbolic surface endowed with a $2k$-regular standard tesselation. Conversely, any such tesselated surface is isometric to $\mathbb{H}/H$, where H satisfies the previous conditions, see [10], Theorem 12. This leads to the question, only partially answered by Criterion 18 of [10] - when the curves of the tesselation are the systoles of the surface?

7.2 Schmutz's genus two surface.

The case $2k = 2$ is simple, but it has been excluded because of its particularity. In fact, the three-holed sphere $\Sigma(2, \epsilon)$ is equal to $\Pi(2, \epsilon)$.

There is only one subgroup H of W satisfying the previous three conditions, and we have $W/H \simeq (\mathbb{Z}/2\mathbb{Z})^2$. The corresponding surface $S_2 := \mathbb{H}/H$ is the genus 2 surface tesselated by 4 hexagons, see Figure 7. It has been proved in [16] that the set $\mathcal{C}$ of the curves of the tesselation are the systoles.

The curve S_2 has six points P_i, for $i \in \mathbb{Z}/6\mathbb{Z}$, which are fixed by the hyperelleptic invotion. Denote by $C_{i,i+1}$ the curve of $\mathcal{C}$ containing P_i and P_{i+1}.

Tedious computations show that $\mathcal{C}$ is the set of systoles at the point $\sigma(\epsilon)$ for any $\epsilon \in]\pi/4, \pi/2$. The limit point $\sigma(\pi/4)$ is the Bolza's surface with 12-systoles [1]. Let $\mathcal{C}'$ be the six new systoles of $\sigma(\pi/4)$. Each of these systoles contains the hyperelliptice point P_i and P_{i+2} for some i. For $\epsilon < \pi/4$, $\mathcal{C}'$ is the set of systoles at the point $\sigma(\epsilon)$. Since $\mathcal{C}'$ does not fill, $\sigma(\epsilon)$ is no more in $\mathcal{P}_g$ for $\epsilon < \pi/4$.

A similar analysis can be carried for $\epsilon \geq \pi/2$. The limit points at $\pi/4$ and $3\pi/4$ are Bolza's surface with distinct markings.

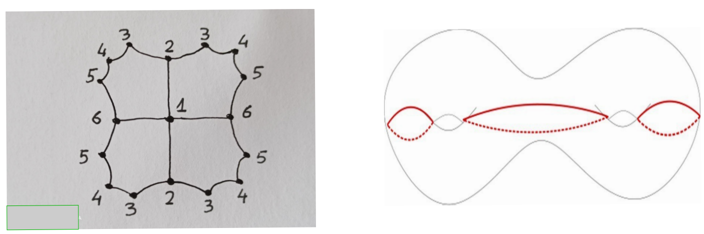

Figure 7. Up to repetition, there are only six vertices on the left side of this figure, which are the points P_i indexed by $1, 2, \ldots, 6$. They are located on the x-axis of the figure on the right. The hyperelliptic involution is a 180-degree rotation around this axis. Three systoles are located on the vertical plane and the other three are on the horizontal plane.

7.3 An exemple of genus 17.
When $2k = 4$, the analysis is more complicated. We will describe a surface of genus 17 endowed with a 4-regular tesselation.

Let $H \subset W$ be the normal subgroup generated by the elements $(s_i s_{i+3})^2$ and set $S_{17} = \mathbb{H}/H$. The quotient $\Gamma := W/H$ is isomorphicto $(\mathbb{Z}/2\mathbb{Z})^6$, hence S_{17} is tesselated by 64 hexagons. It follows that S_{17} has genus 17.

Lemma 21. *The systoles of S_{17} are exactly the curves of the given tesselation.*

Proof. This specific example does not fully satisfies the hypotheses of criterion 18 of [10], so we will briefly explain the proof.

The group Γ is given with 6 generators, and its Caley graph $\operatorname{Cay}\Gamma$ is the one-skeleton of a 6-dimensional cube. There is an embedding of $\operatorname{Cay}\Gamma$ in S_{17}. The vertices are the centers of the hexagons and the edges are the geodeosic arcs connecting two vertices belonging to two adjacent faces and crossing their common edge.

A loop in $\operatorname{Cay}\Gamma$ is a word w on the letters $(s_i)_{i\in\mathbb{Z}/6\mathbb{Z}}$ representing 1 in Γ. The letters s_1, s_3 and s_5 are called the red letters, and the other three are called the blue letters. For any word w, let $l_R(w)$, resp. $l_B(w)$, be the number of occurences of red letters, resp. of blue letters. Also set $l(w) = l_R(w) + l_B(w)$.

As in Lemma 14 of [10], any closed geodesic γ is freely homotopic to a loop $\omega(\gamma)$ in $\operatorname{Cay}\Gamma$. Indeed if γ crosses sucessively some edges of index $i_1, i_2, \dots i_n$ then $\omega(\gamma)$ is the word $s_{i_1}s_{i_2}\dots s_{i_n}$. If at some point γ crosses a vertex at the intersection of two edges of indices i and $i+1$, the previous definition is ambiguous. By convention, we will consider that γ crosses first an edge of index i and then an edge of index $i+1$.

We claim that the systoles of S_{17} are the curves of the tesselation, which have length $4L$, where $L = \operatorname{arcosh} 2$. Let γ be a closed geodesic. Note that $l_R(\omega(\gamma)$ and $l_B(\omega(\gamma)$ are even.

First assume that $l(\omega(\gamma) > 4$. We have $l_R(\omega(\gamma) \geq 4$ or $l_B(\omega(\gamma) \geq 4$ and γ is not a curve of the tesselation. Therefore γ has length bigger that $4L$ by Lemma 17 of [10].

Next assume that $l(\omega(\gamma) \leq 4$ It is obvious that $l(\omega(\gamma)$ is bigger than 2, so we have $l(\omega(\gamma)) = 4$.

Note that $\omega(\gamma)$ cannot contains two identical consecutive letters, so $\omega(\gamma) = s_is_js_is_j$ for some $i \neq j$. Note also that the words $s_is_{i+1}s_is_{i+i}$ are null-homotopic in S_{17}. If $\omega(\gamma) = s_is_{i+2}s_is_{i+2}$, then γ is a curve of the tesselation. If $\omega(\gamma) = s_is_{i+3}s_is_{i+3}$, then γ is a concatenation of four arcs which connects the middles of a side of index i to a side of index $i+3$. If c is one of these arcs, it cut an hexagon into two right-angled pentagons. By the formula of Theorem 3.5.10 of [14], we have $l(c) = \operatorname{arcosh}3$, therefore $l(\gamma) = 4\operatorname{arcosh}3$ is bigger than $4L$. Since γ is defined up to an orientation, we have treated all cases where $l(\omega(\gamma)) = 4$. □

The next lemma shows $\operatorname{codim}\mathcal{P}_g < 2g-1$ for $g = 17$.

Lemma 22. *We have* $\operatorname{codim}\mathcal{P}_{17} < 33$.

Proof. The surface S_{17} has 48 curves. Let $\mathcal{C}$ be the set of all curves of index 3,4,5, or 6. As in the proof of Theorem 1, the set $\mathcal{C}$ fills the surface. Since $\operatorname{Card}\mathcal{C} = 32$, we have $\operatorname{codim}\mathcal{P}_{17} < 32$ by Corollary 20. □

Remark. The set $\mathcal{C}$ of the proof is not a minimal filling subset. Intuitive computations suggest that the minimal filling subsets have cardinality 25, and that $\operatorname{codim}\mathcal{P}_{17} = 24$.

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