Jordan algebras and weight modules

Michael Lau[1*] and Olivier Mathieu[2]

[1]Université Laval
Département de mathématiques et de statistique
Québec, QC, Canada G1V 0A6
Michael.Lau@mat.ulaval.ca

[2]Institut Camille Jordan
UMR 5028 du CNRS, Université Claude Bernard Lyon 1
SUSTech International Center for Mathematics, Shenzhen, China
69622 Villeurbanne Cedex, France
mathieu@math.univ-lyon1.fr

Abstract: We consider bounded weight modules for the universal central extension $\mathfrak{sl}_2(J)$ of the Tits-Kantor-Koecher algebra of a unital Jordan algebra J. Universal objects called Weyl modules are introduced and studied, and a combinatorial dominance criterion is given for analogues of highest weights.

Specializing J to the free Jordan algebra $J(r)$ of rank r, the category $\mathcal{C}^{fin}$ of finite-dimensional $\mathbb{Z}$-graded $\mathfrak{sl}_2(J)$-modules shares many properties with the representation theory of algebraic groups. Using a deep result of Zelmanov, we show that this subcategory admits Weyl modules. By analogy, we conjecture that $\mathcal{C}^{fin}$ is a highest weight category. The resulting homological properties would then imply cohomological vanishing results previously conjectured as a way of determining graded dimensions of free Jordan algebras.

Keywords: weight modules, Weyl modules, Tits-Kantor-Koecher construction, Jordan algebras, highest weight categories

*Corresponding author

MSC2010: primary 17B10; secondary 17B60, 17C05, 17C50

Statements and Declarations: The authors report that they have no competing interests to declare. Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

1 Introduction

Let J be a unital Jordan algebra over an algebraically closed field k of characteristic 0. Tits defined a Lie algebra structure on the space $(\mathfrak{sl}_2(k) \otimes_k J) \oplus \operatorname{Inn} J$, where $\operatorname{Inn} J$ is the Lie algebra of inner derivations [15]. This construction was later generalized by Kantor [9] and Koecher [12] and is now called the *Tits-Kantor-Koecher algebra* and denoted by $TKK(J)$. The Lie algebra $TKK(J)$ is perfect and admits a universal central extension $\mathfrak{sl}_2(J)$ described [2]. See also [1, 13].

With the exception of $r = 1$ and $r = 2$, the structure of the free Jordan algebras $J(r)$ is unknown. However, it was proved in [10] that its structure is determined by the $\mathfrak{sl}_2(k)$-invariants of $H_*(\mathfrak{sl}_2(J(r)))$. The following conjecture was provided.

Conjecture A [10, Conj. 3]. $H_n(\mathfrak{sl}_2(J(r)))^{\mathfrak{sl}_2(k)} = 0$, *for all* $n > 0$.

Verification of this conjecture would give a recursive method to compute the dimensions of the graded components of $J(r)$. In the present paper, we will interpret this conjecture in terms of representation theory.

Let $\{e, f, h\}$ be the standard basis of $\mathfrak{sl}_2(k)$, and let $h(a) = h \otimes a \in \mathfrak{sl}_2(J)$ for all $a \in J$. For any $\mathfrak{sl}_2(J)$-module M and integer j, let M_j be the weight space $M_j = \{m \in M \,:\, h(1).m = j\, m\}$. The module M is said to be *bounded of level* n if

$$M = \bigoplus_{-n \le j \le n} M_j$$

for some nonnegative integer n with $M_n \neq 0$.

A vector space V endowed with a linear map $\rho : J \to \operatorname{End}(V)$ is called a *J-space* if it satisfies

(J1) $[\rho(a), \rho(a^2)] = 0$,

(J2) $[[\rho(a), \rho(b)], \rho(c)] = 4\rho(\partial_{a,b}\, c)$, for all $a, b, c \in J$, where $\partial_{a,b}(c) = a(cb) - (ac)b$. When $\rho(1)$ acts by multiplication by n, V is called a *J-space of level n*. For any bounded $\mathfrak{sl}_2(J)$-module M of level n, the weight space M_n has a structure of J-space of level n, where $\rho(a)$ is the action of $h(a)$. The J-space V is said to be *dominant of level n* if $V = M_n$ for some bounded $\mathfrak{sl}_2(J)$-module M of level n.

Our first main result characterizes dominant J-spaces of level n. For any partition $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_m)$ of $n+1$, we write $|C_\sigma|$ for the cardinality of the corresponding conjugacy class in S_{n+1} and $\mathrm{sgn}(\sigma)$ for its signature. We write $\rho_\sigma(a)$ for the expression $\rho(a^{\sigma_1})\rho(a^{\sigma_2})\cdots\rho(a^{\sigma_m})$ for all $a \in J$. The following result appears as Theorem 2.6.

Theorem B. *Let (V, ρ) be a J-space of level n. Then V is dominant if and only if it satisfies the following condition*

$$\sum_{\sigma \ \vdash \ n+1} sgn(\sigma)\, |C_\sigma|\, \rho_\sigma(a) = 0.$$

For any dominant J-space of level n, the *Weyl module* $\Delta(M)$ is the bounded $\mathfrak{sl}_2(J)$-module of level n defined by the following universal property:

$$\mathrm{Hom}_{\mathfrak{sl}_2(J)}(\Delta(V), M) = \mathrm{Hom}_J(V, M_n),$$

for all bounded $\mathfrak{sl}_2(J)$-modules M of level n, where homomorphisms of J-spaces are defined to be linear maps commuting with the action of J.

Assume now that $J = \oplus_{n \geq 0} J_n$ is a finitely generated $\mathbb{Z}_+$-graded unital Jordan algebra with $J_0 = k1$. The grading on J clearly induces a grading on the Lie algebra $\mathfrak{sl}_2(J)$. We show that the category $\mathcal{C}^{fin}$ of finite-dimensional $\mathbb{Z}$-graded $\mathfrak{sl}_2(J)$-modules admits a Weyl module for each dominant J-space.

Theorem C. *For any $\mathbb{Z}$-graded finite-dimensional dominant J-space V of level n, the Weyl module $\Delta(V)$ is finite-dimensional.*

Theorem C is nontrivial and uses a deep result of Zelmanov on nil Jordan algebras. It appears as Theorem 3.1 in the paper, and shows that the category $\mathcal{C}^{fin}$ shares many properties with categories of representations of reductive algebraic groups in positive characteristic. This leads to the following conjecture.

Conjecture D. *The category $\mathcal{C}^{fin}$ is a highest weight category, in the sense of Cline, Parshall, and Scott.*

A proof of Conjecture D would also settle Conjecture A of [10].

Theorem E. *Let $r \geq 1$. If $\mathcal{C}^{fin}$ is a highest weight category, then Conjecture A holds.*

Theorem E appears below as Theorem 3.2.

2 Bounded weight modules

Let J be a unital Jordan algebra over an algebraically closed field k of characteristic zero. All vector spaces, algebras, and tensor products will be taken over k. For every $a, b \in J$, let $L_a : J \to J$ be the multiplication operator $L_a(b) = ab$. Write $\partial_{a,b} = [L_a, L_b]$, and let κ be one half the Killing form on $\mathfrak{sl}_2(k)$. We write $\operatorname{Inn} J$ for the set $\{\partial_{a,b} : a, b \in J\}$ of *inner derivations* of J. The element $x \otimes a$ in the vector space $\mathfrak{sl}_2(k) \otimes J$ will be denoted by $x(a)$. We fix a standard basis $\{h, e, f\}$ of $\mathfrak{sl}_2(k)$ with $[h, e] = 2e$, $[h, f] = -2f$, and $[e, f] = h$.

2.1 Tits construction

In his 1962 paper, Tits defined a Lie algebra structure on the space

$$TKK(J) := \mathfrak{sl}_2(k) \otimes J \oplus \operatorname{Inn} J,$$

with Lie bracket

1. (T1) $[x(a), y(b)] = [x, y](ab) + \kappa(x, y)\partial_{a,b}$
2. (T2) $[\partial, x(a)] = x(\partial\, a)$,

where $x(a) = x \otimes a$ for any $x, y \in \mathfrak{sl}_2$, $\partial \in \operatorname{Inn} J$, and $a, b \in J$. This construction was later generalized to Jordan pairs and triple systems by Kantor and Koecher, and $TKK(J)$ is known as the *Tits-Kantor-Koecher (TKK) algebra.*

2.2 Tits-Allison-Gao construction

The TKK algebra is perfect, that is, $TKK(J) = [TKK(J), TKK(J)]$, so $TKK(J)$ admits a universal central extension, which we denote by $\mathfrak{sl}_2(J)$. This Lie algebra was nicely described in the 1996 paper of Allison and Gao [2] in the context of universal coverings of the Steinberg unitary Lie algebras $\mathfrak{stu}_n(J)$ for $n \geq 3$. The case where $n = 3$ corresponds to $TKK(J)$. See also [1] for equivalent formulas written in terms of $\mathfrak{sl}_2(k)$.

As a vector space,

$$\mathfrak{sl}_2(J) = (\mathfrak{sl}_2(K) \otimes J) \oplus \{J, J\},$$

where $\{J, J\} = (\bigwedge^2 J)/\mathcal{S}$ and $\mathcal{S} = \mathrm{Span}\{a \wedge a^2 \mid \ a \in J\}$. For any $a, b \in J$, we write $\{a, b\}$ for the image of $a \wedge b$ in $\{J, J\}$. The bracket on $\mathfrak{sl}_2(J)$ is given by

(R1) $[x(a), y(b)] = [x, y](ab) + \kappa(x, y)\{a, b\}$

(R2) $[\{a, b\}, x(c)] = x(\partial_{a,b}\, c)$

(R3) $[\{a, b\}, \{c, d\}] = \{\partial_{a,b}\, c, d\} + \{c, \partial_{a,b}\, d\}$.

for all $a, b, c, d \in J$ and $x, y \in \mathfrak{sl}_2(k)$. It is a bit tricky to show that (R3) is skew-symmetric [2].

There is an obvious Lie algebra epimorphism $\mathfrak{sl}_2(J) \to TKK(J)$ which is the identity on $\mathfrak{sl}_2 \otimes J$ and sends the symbol $\{a, b\}$ to $\partial_{a,b}$. When the Jordan algebra J is associative, we have $\{J, J\} = HC_1(J)$, and the construction specializes to results of Kassel and Loday[11].[1]

2.3 The short grading of $\mathfrak{sl}_2(J)$

The Lie algebra $\mathfrak{G} := \mathfrak{sl}_2(J)$ decomposes with respect to the adjoint action of $\frac{1}{2}h(1)$ as $\mathfrak{G} = \mathfrak{G}_{-1} \oplus \mathfrak{G}_0 \oplus \mathfrak{G}_1$, where

$$\mathfrak{G}_{-1} = f \otimes J,$$
$$\mathfrak{G}_0 = h \otimes J \oplus \{J, J\}, \text{ and}$$
$$\mathfrak{G}_1 = e \otimes J.$$

This decomposition is a root grading in the sense of Berman-Moody [3], and is called the *short grading* of $\mathcal{G}$. In fact, every root-graded Lie algebra

[1] The notation in [11] differs slightly from the modern conventions–Kassel-Loday write HC_2 for what is now denoted as HC_1.

of type A_1 has a universal central extension isomorphic to $\mathfrak{sl}_2(J)$ for some unital Jordan algebra J. See [13] or [1] for details.

2.4 Bounded modules

For any $\mathfrak{sl}_2(J)$-module V and any $k \in \mathbb{Z}$, let

$$V_k = \{v \in V \,:\, h(1)v = k\,v\}.$$

An $\mathfrak{sl}_2(J)$-module V is said to be a *bounded weight module of level* ℓ if $V = \oplus_{-\ell \leq m \leq \ell} V_m$, with $V_\ell \neq 0$.

2.5 J-spaces

Recall that a vector space M endowed with a linear map $\rho :\; J \to \mathrm{End}(M)$ is called a J-*space* if ρ satisfies

(J1) $[\rho(a), \rho(a^2)] = 0$,

(J2) $[[\rho(a), \rho(b)], \rho(c)] = 4\rho(\partial_{a,b}\, c)$.

A J-space is said to be of *level* n if $\rho(h(1)) = n\,\mathrm{id}$.

Lemma 2.1. *Let $(M, \tilde{\rho})$ be a representation of the Lie algebra $\mathfrak{G}_0$. Then the map $\rho : a \mapsto \tilde{\rho}(h(a))$ determines a J-space structure on M.*

Conversely, if M is a J-space, then there is a unique $\mathfrak{G}_0$-module structure $(M, \tilde{\rho})$ such that $\tilde{\rho}(h(a)) = \rho(a)$ for any $a \in J$.

Proof. Let $(M, \tilde{\rho})$ be a $\mathfrak{G}_0$-module. For $a \in J$, set $\rho(a) = \tilde{\rho}(a)$. Since $[h(a), h(a^2)] = 4\{a, a^2\} = 0$, it follows that $[\rho(a), \rho(a^2)] = 0$, proving (J1). Let $a, b, c \in J$. We have $[[h(a), h(b)], h(c)] = 4h(\partial_{a,b}\, c)$, and therefore $[[\rho(a), \rho(b)], \rho(c)] = 4\rho(\partial_{a,b}\, c)$, proving (J2).

Conversely, assume that M is a J-space. It is clear that $\mathfrak{G}_0$ is generated by the vector space $h \otimes J$ and defined by the relations

(H1) $[h(a), h(a^2)] = 0$, and

(H2) $[[h(a), h(b)], h(c)] = 4h(\partial_{a,b}\, c)$,

for any $a, b, c \in J$. Therefore there is a unique structure $\tilde{\rho}$ of $\mathfrak{G}_0$-module on M such that $\tilde{\rho}(h(a)) = \rho(a)$ □

A linear map $\sigma : J \to \mathrm{End}_k(M)$ is a *Jordan birepresentation* (and M is a *Jordan bimodule*) if the semidirect product $(a, m)(b, n) = (ab, bm + an)$ gives a Jordan algebra structure to the vector space direct sum $J \oplus M$. This condition is equivalent to the conditions

(M1) $[\sigma(a), \sigma(a^2)] = 0$, and

(M2) $\sigma(a^2 b) + 2\sigma(a)\sigma(b)\sigma(a) = 2\sigma(ab)\sigma(a) + \sigma(a^2)\sigma(b)$,

for all $a, b \in J$. It follows from [8, II.9(47')] that (M1) and (M2) imply that $\rho = 2\sigma$ satisfies (J1) and (J2). The converse is not true, however. If (M, ρ) is a J-space of level n, then setting $a = b = 1 \in J$ in (M2), we see that the only possible values for $n = \rho(1)$ are $0, 1$ or 2 (Peirce decomposition), if we wish to induce a J-bimodule structure on M with action $\sigma = \frac{1}{2}\rho$. But as we will see in Example 2.10, there exist J-spaces of any level.

2.6 Dominant J-spaces

Any J-space M of level n can be induced to a generalized Verma module

$$V(M) = \mathcal{U}(\mathfrak{G}) \otimes_{\mathcal{U}(\mathfrak{G}_0 \oplus \mathfrak{G}_1)} M,$$

where $\mathfrak{G}_1$ acts as zero on M. The main result of this section will be an analysis of which J-spaces M of level n determine $\mathfrak{G}$-modules $V(M)$ with bounded quotients $V(M)/X$, such that the composition of natural maps

$$M \hookrightarrow V(M) \to V(M)/X$$

is an injection of $\mathfrak{G}_0 \oplus \mathfrak{G}_1$-modules. Such a quotient $V(M)/X$ will be called a *bounded M-quotient* of $V(M)$ of level n, and in this case, M is said to be *dominant*. Note that if $V(M)/X$ is a bounded M-quotient of level n, then n is a nonnegative integer and the weight n subspace $(V(M)/X)_n$ is precisely M.

Proposition 2.2. *Let M be a J-space of nonnegative integer level n, with $\mathfrak{G}_0$-action given by $\rho : J \to \mathrm{End}_k(M)$. Then the generalized Verma module $V(M)$ has a bounded M-quotient if and only if*

$$e(1)^{n+1} f(a)^{n+1} m = 0,$$

for all $a \in J$ and $m \in M$.

Proof. If $V(M)$ has a nonzero bounded M-quotient $V(M)/X$, then $M = (V(M)/X)_n = V(M)_n$, so $X_n = 0$. By $\mathfrak{sl}_2$-theory, we see that $(V(M)/X)_m = 0$ for all $m < -n$, so $X_m = V(M)_m$ for all $m < -n$. In particular, $f(a)^{n+1}m \in V(M)_{-n-2} = X_{-n-2} \subseteq X$ for all $m \in M$, so $e(1)^{n+1}f(a)^{n+1}m \in X$ for all $a \in J$ and $m \in M$. But then $e(1)^{n+1}f(a)^{n+1}m \in V(M)_n \cap X = 0$.

Conversely, suppose that $e(1)^{n+1}f(a)^{n+1}m = 0$ for all $a \in J$ and $m \in M$. Let $Z \subset V(M)$ be a $\mathfrak{G}$-submodule which is maximal with respect to the property that $Z_n = 0$. Linearisation of the relation $e(1)^{n+1}f(a)^{n+1}m = 0$ gives

$$e(1)^{n+1}f(b_1)\cdots f(b_{n+1})m = 0,$$

for all $b_1, \ldots, b_{n+1} \in J$ and $m \in M$. Then for any $a_1, \ldots, a_{n+1} \in J$, we have

$$h(a_1)\ldots h(a_{n+1})e(1)^{n+1}f(b_1)\cdots f(b_{n+1})m = 0,$$

from which it follows that

$$e(a_1)\cdots e(a_{n+1})f(b_1)\cdots f(b_{n+1})m = 0,$$

for all $a_i, b_j \in J$ and $m \in M$. In particular, this shows that $f(b_1)\cdots f(b_{n+1})m \in Z$, and $V(M)_k \subseteq Z$ for all $k \leq -n-2$. Therefore, $V(M)_k = Z_k$ for all $k < -n$, and $V(M)/Z$ is a bounded M-quotient. □

We now introduce some notation. Let $\sigma = (\sigma_1, \ldots, \sigma_m)$ be a partition of $n+1$, that is,

$$\sigma_1 \geq \cdots \geq \sigma_m \geq 1, \text{ for some } m \geq 1, \text{ where } \sigma_1 + \cdots + \sigma_m = n+1.$$

We write $|C_\sigma|$ for the cardinality of the conjugacy class C_σ of permutations in the symmetric group S_{n+1} with cycle structure σ. The sign of these permutations will be denoted by $\mathrm{sgn}(\sigma)$, and we write $\rho_\sigma(a)$ for the expression $\rho(a^{\sigma_1})\rho(a^{\sigma_2})\cdots\rho(a^{\sigma_m})$ for all $a \in J$ and $\sigma \vdash n+1$. It follows easily from Condition (C1) that $[\rho(a^i), \rho(a^j)] = 0$ for all i, j, so this product is independent of the order of the factors.

The Newton polynomials $N_\ell(x) = x_1^\ell + \cdots + x_n^\ell$ in n indeterminates $x_1, \ldots, x_n$ generate the ring of symmetric polynomials $k[x_1, \ldots, x_n]^{S_n}$. We write

$$N_\sigma(x) = N_{\sigma_1}(x)N_{\sigma_2}(x)\cdots N_{\sigma_m}(x)$$

for each partition $\sigma = (\sigma_1, \ldots, \sigma_m)$ of $n+1$. The space $P_{n+1} \subset k[x_1, \ldots, x_n]^{S_n}$ of symmetric polynomials of total degree $n+1$ is clearly of dimension $p(n+$

1) − 1, where $p(n+1)$ is the number of partitions of $n+1$. The Newton polynomials $N_1(x), \ldots, N_n(x)$ are algebraically independent, so

$$\{N_\sigma(x) \;:\; \sigma \;\vdash\; n+1 \text{ such that } \sigma \neq (1,1,\ldots,1)\}$$

is a basis for P_{n+1}, and the set

$$\{N_\sigma(x) \;:\; \sigma \;\vdash\; n+1\}$$

has exactly one linear dependence relation, up to scalar multiple. We include an amusing representation-theoretic argument below, that we have not seen elsewhere in the literature.

Proposition 2.3. *Up to scalar multiple, the unique linear dependence relation on the set* $\{N_\sigma(x) \;:\; \sigma \;\vdash\; n+1\}$ *is* $\sum_{\sigma \;\vdash\; n+1} sgn(\sigma)|C_\sigma|N_\sigma(x) = 0.$

Proof. By the Frobenius character formula,

$$N_\sigma(x) = \sum_\lambda \chi_\lambda(\sigma) S_\lambda,$$

where the sum is taken over all partitions $\lambda \neq (1,1,\ldots,1)$ of $n+1$. Here $\chi_\lambda(\sigma)$ is the character (evaluated at any permutation of cycle structure σ) of the Specht module associated with λ, and S_λ is the Schur polynomial associated to λ. See [6] for details.

Since sgn is the character of the sign representation, the Specht module associated to $(1,\ldots,1)$, we see that

$$\begin{aligned}
\sum_{\sigma \;\vdash\; n+1} \operatorname{sgn}(\sigma)|C_\sigma|N_\sigma(x) &= \sum_{\sigma \in S_{n+1}} \operatorname{sgn}(\sigma) N_\sigma(x) \\
&= \sum_{\sigma \in S_{n+1}} \chi_{(1,\ldots,1)}(\sigma) \sum_{\lambda \neq (1,\ldots,1)} \chi_\lambda(\sigma) S_\lambda \\
&= \sum_{\lambda \neq (1,\ldots,1)} S_\lambda \sum_{\sigma \in S_{n+1}} \chi_{(1,\ldots,1)}(\sigma)\chi_\lambda(\sigma),
\end{aligned}$$

and the inner product

$$(\chi_{(1,\ldots,1)}, \chi_\lambda) = \sum_{\sigma \in S_{n+1}} \chi_{(1,\ldots,1)}(\sigma)\chi_\lambda(\sigma)$$

is 0 whenever $\lambda \neq (1,\ldots,1)$, by the orthogonality relations. □

Remark 2.4. Proposition 2.3 can also be proved with a more standard combinatorial argument: Let V be a vector space of dimension n, with basis $\{e_1, \dots, e_n\}$. Let $h : V \to V$ be defined by $h(e_i) = x_i e_i$, and let $h^{\otimes(n+1)} : V^{\otimes(n+1)} \to V^{\otimes(n+1)}$ be the induced endomorphism. For $\sigma \in S_{n+1}$, we have

$$\mathrm{Tr}(h^{\otimes(n+1)} \circ \sigma) = N_\sigma(x).$$

Since $\sum_{\sigma \in S_{n+1}} \mathrm{sgn}(\sigma)\sigma$ acts as zero on $V^{\otimes(n+1)}$, the dependence relation follows.

The following well-known formula, originally due to Garland [7] and reinterpreted by Chari and Pressley [4], will be used to prove boundedness conditions.

Lemma 2.5. *Let $p : \mathcal{U}(\mathfrak{G}) \to \mathcal{U}(\mathfrak{G}_{-1})\mathcal{U}(\mathfrak{G}_0)$ be the projection relative to the (vector space) decomposition $\mathcal{U}(\mathfrak{G}) = \mathcal{U}(\mathfrak{G})\mathfrak{G}_1 \oplus \mathcal{U}(\mathfrak{G}_{-1})\mathcal{U}(\mathfrak{G}_0)$. Then $p(e(1)^r f(a)^{n+1})$ is the coefficient of u^{n+1} in the generating function*

$$\frac{(-1)^r r!(n+1)!}{(n+1-r)!} \left(\sum_{s=1}^{\infty} f(a^s) u^s \right)^{n+1-r} \exp\left(-\sum_{t=1}^{\infty} \frac{h(a^t)}{t} u^t \right).$$

□

Theorem 2.6. *Let (M, ρ) be a J-space of level n. Then the following conditions are equivalent:*

(1) *M is dominant.*

(2) *$e(1)^{n+1} f(a)^{n+1} m = 0$ for all $a \in J$ and $m \in M$.*

(3) $\displaystyle\sum_{\sigma \,\vdash\, n+1} sgn(\sigma) |C_\sigma| \rho_\sigma(a) = 0.$

Proof. By Proposition 2.2, Conditions (1) and (2) are equivalent, so we need only prove that (2) and (3) are equivalent.

Since $e(1)^{n+1} f(a)^{n+1}$ is homogeneous of degree 0 with respect to the grading induced by $\mathrm{ad}\,(h \otimes 1)$, we see that its action on any highest weight vector m is given by the action of its projection p on the subspace $\mathcal{U}(\mathfrak{G}_0)$ with respect to the decomposition $\mathcal{U}(\mathfrak{G})_0 = \mathcal{U}(\mathfrak{G}_0) \oplus (\mathcal{U}(\mathfrak{G})\mathfrak{G}_1 \cap \mathcal{U}(\mathfrak{G})_0)$ of the space $\mathcal{U}(\mathfrak{G})_0$ of degree 0 elements of $\mathcal{U}(\mathfrak{G})$.

By Lemma 2.5, $p(e(1)^{n+1}f(a)^{n+1})$ is the coefficient of u^{n+1} in the generating series

$$(-1)^{n+1}(n+1)!(n+1)!\exp\left(-\sum_{k=1}^{\infty}\frac{h(a^k)}{k}u^k\right).$$

Computing directly, the coefficient of u^{n+1} in $\exp\left(-\sum_{k=1}^{\infty}\frac{h(a^k)}{k}u^k\right)$ is

$$\sum_{\sigma\vdash n+1}(-1)^{r_\sigma}\frac{h_\sigma(a)}{\left(\prod_{i=1}^{r_\sigma}\sigma_i\right)\left(\prod_{j=1}^{m_\sigma}a_j!\right)},$$

where σ_i is the length of the ith row of the Young frame T_σ associated to the partition σ, r_σ is the number of rows of T_σ, m_σ is the number of columns of T_σ, a_j is the number of rows of length j in T_σ, and $h_\sigma(a) = h(a^{\sigma_1})h(a^{\sigma_2})\cdots h(a^{\sigma_{r_\sigma}})$. If $\mathrm{odd}(r_\sigma)$ (respectively, $\mathrm{even}(r_\sigma)$ is the number of odd-length (respectively, even-length) rows of T_σ, we see that $(-1)^{n+1} = (-1)^{\mathrm{odd}(r_\sigma)}$, so

$$(-1)^{n+1}(-1)^{r_\sigma} = (-1)^{n+1}(-1)^{\mathrm{odd}(r_\sigma)}(-1)^{\mathrm{even}(r_\sigma)} = (-1)^{\mathrm{even}(r_\sigma)} = \mathrm{sgn}(\sigma).$$

By elementary counting arguments,

$$|C_\sigma| = \frac{(n+1)!}{\left(\prod_{i=1}^{r_\sigma}\sigma_i\right)\left(\prod_{j=1}^{m_\sigma}a_j!\right)}.$$

See [14, Proposition 1.1.1], for instance. The projection of $e(1)^{n+1}f(a)^{n+1}$ on $\mathcal{U}(\mathfrak{G}_0)$ is thus $(n+1)!\sum_{\sigma\ \vdash\ n+1}\mathrm{sgn}(\sigma)|C_\sigma|h_\sigma(a)$, so Conditions (2) and (3) are equivalent. $\square$

Example 2.7. By Theorem 2.6, any dominant J-space M of level 0 is trivial, in the sense that $\rho:\ J\to\mathrm{End}_k(M)$ is the zero map and M, equipped with the trivial $\mathfrak{G}$-action, is the unique bounded M-quotient of $V(M)$.

Example 2.8. Dominant J-spaces (M,ρ) of level 1 satisfy $\rho(a^2)=\rho(a)^2$ for all $a\in J$, so

$$\rho(ab) = \frac{1}{2}(\rho(a)\rho(b)+\rho(b)\rho(a)), \tag{2.9}$$

for all $a, b \in J$ by linearization. Dominant J-spaces (M, ρ) of level 1 are thus precisely *associative specializations*, Jordan algebra homomorphisms ρ from J to special Jordan algebras of linear operators on a vector space M.

Example 2.10. For levels higher than 2, dominant J-spaces are never Jordan bimodules. See the discussion after Lemma 2.1 for details. Many such J-spaces exist. For example, it follows immediately from Proposition 2.3 and Theorem 2.6, that the map

$$\begin{aligned} \rho: \ k[t] &\to \operatorname{End}_k\left(k[x_1, \ldots, x_n]^{S_n}\right) \\ t^\ell &\mapsto N_\ell(x) \end{aligned} \tag{2.11}$$

defines a dominant J-space of level n for the (associative) Jordan algebra $J = k[t]$, an example we will consider in more detail in Section 3.

3 Weyl modules and highest weight categories

Let n be a nonnegative integer. The categories $\mathcal{C}^b(M)$ of bounded weight modules attached to bounded J-spaces M of level n admit universal objects $\Delta(M)$, called *Weyl modules*. Every bounded M-quotient of level n is a homomorphic image of $\Delta(M)$, and it is clear that

$$\Delta(M) = V(M) \ / \ \mathcal{U}(\mathfrak{G}) \sum_{\ell < -n} V(M)_\ell \ = \ \bigoplus_{\ell=0}^{n} \Delta(M)_{n-2\ell},$$

where $\Delta(M)_{n-2\ell}$ is the vector subspace of weight $n - 2\ell$. Identifying

$$\mathfrak{G}_{-1} = \{f \otimes a \ : \ a \in J\} \subset \mathfrak{G} = (\mathfrak{sl}_2(k) \otimes J) \oplus \{J, J\}$$

with J, the weight space $V(M)_{n-2\ell}$ identifies with the vector space $S^\ell J \otimes M$ for $\ell = 0, \ldots, n$.

3.1 Weyl modules for finite dimensional dominant J-spaces

Let $J = \bigoplus_{\ell=0}^{\infty} J_\ell$ be a finitely generated $\mathbb{Z}_+$-graded unital Jordan algebra with $J_0 = k1$. Let $M = \bigoplus_{\ell=0}^{\infty} M_\ell$ be a $\mathbb{Z}$-graded dominant J-space of level n. We now prove one of our main results, that the category $\mathcal{C}^{fin}$ of finite-dimensional $\mathbb{Z}$-graded $\mathfrak{sl}_2(J)$-modules contains its Weyl modules.

Theorem 3.1. *Let M be a $\mathbb{Z}$-graded dominant J-space of level n, for a Z_+-graded and finitely generated Jordan algebra J with $J_0 = k1$. Then the Weyl module $\Delta(M)$ is finite dimensional if and only if M is finite dimensional.*

Proof. If $\Delta(M)$ is finite dimensional, then $M \subseteq \Delta(M)$ is clearly also finite dimensional. Conversely, assume that M is finite dimensional. Up to a possible shift in grading, we may assume that M is $\mathbb{Z}_+$-graded. Let N be the largest nonnegative integer for which the graded component M_N is nonzero. Let $a \in J$ be a homogeneous element with $\deg a > N$, and let $v \in M$. By Lemma 2.5, $e(1)^n f(a)^{n+1} v$ is the coefficient of u^{n+1} in the formal series

$$(-1)^n n!(n+1)! \sum_{s=1}^{\infty} f(a^s) u^s \exp\left(-\sum_{t=1}^{\infty} \frac{h(a^t)}{t} u^t\right) v.$$

By degree considerations, $h(a^t)v = 0$ for all $t \geq 1$, so

$$e(1)^n f(a)^{n+1} v = (-1)^n n!(n+1)! f(a^{n+1}) v,$$

and $f(a^{n+1})v = 0$ as an element of $\Delta(M)$.

In particular, $f\left(b^{(N+1)(n+1)}\right) M = 0$ for all b in the (non-unital) Jordan subalgebra $J^+ = \bigoplus_{\ell=1}^{\infty} J_\ell \subset J$. Let $I = \{x \in J^+ : f(x)M = 0\}$. For all $x \in J^+$, $y \in I$, and $m \in M$,

$$\begin{aligned} 0 &= h(x)f(y)m \\ &= f(y)h(x)m - 2f(xy)m \\ &= -2f(xy)m \end{aligned}$$

since $h(x)M \subseteq M$ and $f(y)M = 0$. Therefore, $xy \in I$ and I is an ideal of J^+.

Since $b^{(N+1)(n+1)} \in I$ for all $b \in J^+$, the Jordan algebra J^+/I is nil of bounded index, hence locally nilpotent by a result of Zelmanov [16]. But J, and thus J^+/I, is finitely generated, so J^+/I is nilpotent and there exists $N' > 0$ such that every product (in any association) of N' elements of J^+ is in I. The (finitely many) generators of J may be chosen to be homogeneous and of positive degree at most r for some $r > 0$. In particular, $J_s \subseteq I$ for all $s \geq rN'$. That is, $f(a)M = 0$ for all $a \in J$ with $\deg a \geq rN'$.

The weight space $\Delta(M)_{n-2\ell}$ is spanned by monomials of the form

$$f(a_1) \cdots f(a_\ell) w$$

with $a_1, \ldots, a_\ell \in J$ and $w \in M$. Since the $f(a_i)$ commute with each other, the set

$$\{f(a_1) \cdots f(a_\ell) w \,:\, w \in M \text{ and } a_1, \ldots, a_\ell \in J \text{ with } \deg a_i < rN' \text{ for all } i\}$$

already spans $\Delta(M)_{n-2\ell}$. As M and J_i are finite dimensional for all i, it now follows that $\dim \Delta(M)_{n-2\ell} < \infty$ for all ℓ, and the Weyl module $\Delta(M) = \bigoplus_{\ell=0}^{n} \Delta(M)_{n-2\ell}$ is also finite dimensional. □

3.2 Highest weight categories and character formulas for free Jordan algebras

Cline, Parshall, and Scott [5] introduced the notion of *highest weight category* as a unifying theme in representation theory, modelled after highest weight representations of semisimple algebraic groups and their Lie algebras. Their definition requires labelling simple objects by a poset Λ, and the existence of enough injectives, as well as costandard objects labelled by the same index set as the simples and satisfying various axioms. Given the similarities between the category $\mathcal{C}^{fin}$ of finite-dimensional $\mathbb{Z}$-graded $\mathfrak{sl}_2(J(r))$-modules and the representation theory of reductive algebraic groups in positive characteristic, we conjecture that $\mathcal{C}^{fin}$ is a highest weight category, with the Weyl modules and their duals (twisted by the Cartan involution) as the standard and costandard objects, respectively. In a highest weight category, the higher ext-groups $\mathrm{Ext}^i(\Delta(\lambda), \nabla(\mu)) = 0$ for all $i > 0$ and $\lambda, \mu \in \Lambda$. If $\mathcal{C}^{fin}$ is indeed a highest weight category as conjectured above, the vanishing of higher ext-groups would, in fact, settle the main conjecture of [10] and thus describe the graded dimensions of the free Jordan algebras $J(r)$.

Theorem 3.2. *If $\mathcal{C}^{fin}$ is a highest weight category with Weyl modules and their duals as its standard and costandard objects, then $H_i(\mathfrak{sl}_2(J(r)))$ contains no nonzero trivial $\mathfrak{sl}_2(k)$-modules for $i > 0$.*

Proof. Let J be the free unital Jordan algebra $J(r)$ on r generators, and suppose that $\mathcal{C}^{fin}$ is a highest weight category as in the hypotheses of the theorem. As noted above, in a highest weight category, $\mathrm{Ext}^i(\Delta(\lambda), \nabla(\mu))$ vanishes for all $i > 0$ and indices λ, μ of simples, where $\Delta(\lambda)$ and $\nabla(\mu)$ are the corresponding standard and costandard objects. In $\mathcal{C}^{fin}$, the Weyl and dual Weyl modules corresponding to the trivial 1-dimensional $\mathfrak{sl}_2(J)$-module k are themselves 1-dimensional, so

$$\mathrm{Ext}^i_{\mathcal{C}^{fin}}(k,k) = 0 \quad \text{for all } i > 0.$$

But $\mathrm{Ext}^i_{\mathcal{C}^{fin}}(k,k) = H^i(\mathfrak{sl}_2(J))$, and the cohomology ring

$$H^*(\mathfrak{sl}_2(J)) = H^*(\mathfrak{sl}_2(k)) \otimes H^*(\mathfrak{sl}_2(J), \mathfrak{sl}_2(k)).$$

As $H^0(\mathfrak{sl}_2(J)) = H^0(\mathfrak{sl}_2(k)) = k$, we see that $H^*(sl_2(J)) = k$ and the relative cohomology $H^i(\mathfrak{sl}_2(J), \mathfrak{sl}_2(k)) = 0$, for all $i > 0$. The result now follows from the universal coefficient theorem and the interpretation of the relative cohomology as the $\mathfrak{sl}_2(k)$-invariants in $H^i(\mathfrak{sl}_2(J))$. □

3.3 Example: Weyl modules for free Jordan algebras of rank 1

For any Jordan algebra J with unit 1 and $n \in \mathbb{Z}_+$, let

$$T(J) = k1 \oplus J \oplus \left(J^{\otimes 2}\right) \oplus \left(J^{\otimes 3}\right) \oplus \cdots$$

be its tensor algebra, and let $I \subseteq T(J)$ be the two-sided ideal generated by the relations

$$1 - n1, \tag{3.3}$$
$$a \otimes a^2 - a^2 \otimes a, \tag{3.4}$$
$$a \otimes b \otimes c + c \otimes b \otimes a - b \otimes a \otimes c - c \otimes a \otimes b + b(ac) - a(bc), \tag{3.5}$$
$$\sum_{\sigma \vdash n+1} \mathrm{sgn}(\sigma)|C_\sigma| T_\sigma(a), \tag{3.6}$$

for all $a, b, c \in J$, where $T_\sigma(a) = a^{\sigma_1} \otimes a^{\sigma_2} \otimes \cdots \otimes a^{\sigma_m}$ for all partitions $\sigma = (\sigma_1, \ldots, \sigma_m) \vdash n+1$. The associative algebra $\mathcal{U}_n(J) = T(J)/I$ is called the *universal J-space envelope of level n*. There is a unique associative algebra homomorphism $\check{\rho} : T(J) \to \mathrm{End}_k(M)$ extending the action $\rho : J \to \mathrm{End}_k(M)$ of any J-space (M, ρ) of level n, and in light of Lemma 2.1 and Theorem 2.6, the map $\check{\rho}$ descends to the quotient $\mathcal{U}_n(J)$. By construction, dominant J-spaces of level n and left $\mathcal{U}_n(J)$-modules are equivalent notions, and a J-space (M,ρ) of level n is said to be *free of rank* r if $(M,\check{\rho})$ is a free $\mathcal{U}_n(J)$-module.

Let $F=\mathcal{U}_n(J)$ be the universal J-space envelope of level n for a unital Jordan algebra J. If J is finitely generated as a Jordan algebra, then F is finitely generated as an associative algebra, by Relation (3.6). For example, if $J=k[t]$ is the free Jordan algebra of rank 1, then $F=k[x_1,\dots,x_n]^{S_n}$ is the algebra of symmetric polynomials, where t^ℓ corresponds to the Newton polynomial $N_\ell(x)=x_1^\ell+\cdots+x_n^\ell\in F$. If J is free of rank m, then $\mathcal{U}_0(J)=k$ and $\mathcal{U}_1(J)$ is the quotient of the free associative algebra in m generators by the ideal generated by the relation $a\otimes b\otimes c+c\otimes b\otimes a=b\otimes a\otimes c+c\otimes a\otimes b$ for all $a,b,c\in J$.

Let L be the two-dimensional simple $\mathfrak{sl}_2(k)$-module. The Jordan algebra $J=k[t]$ is commutative and associative, and it is easy to see that $\{J,J\}=0$ and the TKK algebra $\mathfrak{G}=\mathfrak{sl}_2(J)=\mathfrak{sl}_2(k)\otimes J$ is centrally closed. The space $L[t]=L\otimes k[t]$ is obviously a $\mathfrak{G}$-module, where

$$(x\otimes p(t)).(v\otimes q(t))=xv\otimes p(t)q(t),$$

for all $x\in\mathfrak{sl}_2(k)$, $v\in L$, and $p(t),q(t)\in J$. This gives a $\mathfrak{G}$-module structure on the space $S^n(L[t])\subset T(L[t])$ of homogeneous symmetric tensors of degree n.

Proposition 3.7. *Let $F=k[x_1,\dots,x_n]^{S_n}$ be the rank 1 free $\mathcal{U}_n(k[t])$-module. Then the Weyl module $\Delta_n(F)$ is isomorphic to $S^n(L[t])$.*

Proof. Let $v\in L$ be a nonzero vector of weight 1 with respect to the action of $h\in\mathfrak{sl}_2(k)$. There is a natural injection $\iota:\ F\longrightarrow S^n(L[t])$, with

$$\iota:\ \sum_{\sigma\in S_n} x_{\sigma(1)}^{a_{\sigma(1)}}\cdots x_{\sigma(n)}^{a_{\sigma(n)}}\longmapsto\sum_{\sigma\in S_n}(v\otimes t^{a_{\sigma(1)}})\otimes\cdots\otimes(v\otimes t^{a_{\sigma(n)}}).$$

This maps extends uniquely to a $\mathfrak{G}$-module epimorphism

$$\begin{aligned} V(F)&\to S^n(L[t])\\ u.p&\mapsto u.\iota(p),\qquad \text{for all } u\in U(\mathfrak{G}) \text{ and } p\in F,\end{aligned}$$

with kernel $\sum_{\ell<-n}V(F)_\ell$. □

Remark 3.8. In fact, for every prime Jordan algebra J and every $n\geq 2$, there is a dominant J-space of level n, on which the Lie algebra $\mathfrak{G}_0(J)=$

$(\mathfrak{h} \otimes J) \oplus \{J, J\}$ acts faithfully. If J is special, then there is a faithful associative specialization $\rho : J \to \mathrm{End}_k(M)$, and M is a J-space of level 1. The faithfulness of the extension $\tilde{\rho} : \mathfrak{G}_0 \to \mathrm{End}_k(M)$ on $\{J, J\}$ follows immediately from the assumption that J is prime. We can then take the n-fold tensor product of M to obtain a faithful J-space of level n.

If J is the Albert algebra $\mathbb{A}$, then we can construct a faithful $\mathfrak{G}_0$-module of level n as a tensor product of copies of the level 2 and level 3 representations of the Albert algebra, obtained from representations of the exceptional Lie algebra E_6, viewed as the subalgebra $(k\, h \otimes \mathbb{A}) \oplus \{\mathbb{A}, \mathbb{A}\}$ of the Lie algebra $\mathfrak{sl}_2(\mathbb{A})$.

This observation is clearly not true for arbitrary (non-prime) Jordan algebras. For example, the Lie algebra $\mathfrak{G}_0(k[t, t^{-1}])$ has a nontrivial centre that acts as 0 on all bounded modules.

References

[1] B. Allison, G. Benkart, and Y. Gao, Central extensions of Lie algebras graded by finite root systems, Math. Ann. **316** (2000), 499–527.

[2] B. Allison and Y. Gao, Central quotients and coverings of Steinberg unitary Lie algebras, Canad. J. Math. **48** (1996), 449–482.

[3] S. Berman and R.V. Moody, Lie algebras graded by finite root systems and the intersection matrix algebras of Slodowy, Invent. Math. **108** (1992), 323–347.

[4] V. Chari and A. Pressley, Weyl modules for classical and quantum affine algebras, Represent. Theory **5** (2001), 191–223.

[5] E. Cline, B. Parshall, and L. Scott, Finite dimensional algebras and highest weight categories, J. Reine Angew. Math. **391** (1988), 85–99.

[6] W. Fulton and J. Harris, *Representation Theory: A First Course*, Grad. Texts in Math., Springer, New York, 1991.

[7] H. Garland, The arithmetic theory of loop groups, J. Algebra **53** (1978), 480–551.

[8] N. Jacobson, Structure and Representations of Jordan Algebras, Colloquium Publications, Amer. Math. Soc., Providence, 1968.

[9] I.L. Kantor, Classification of irreducible transitive differential groups, (Russian) Dokl. Akad. Nauk SSSR **158** (1964), 1271–1274.

[10] I. Kashuba and O. Mathieu, On the free Jordan algebras, Adv. Math. **383** (2021), Paper no. 107690, 35 pp.

[11] C. Kassel and J.-L. Loday, Extensions centrales d'algèbres de Lie, Ann. Inst. Fourier (Grenoble) **32** (1982), 119–142.

[12] M. Koecher, *The Minnesota Notes on Jordan Algebras and Their Applications*, Lecture Notes in Math., vol. 1710, Springer-Verlag, Berlin, 1999.

[13] E. Neher, Lie algebras graded by 3-graded root systems, Amer. J. Math. **118** (1996), 439–491.

[14] B. Sagan, The Symmetric Group, Second ed., Grad. Texts in Math., Springer, New York, 2001.

[15] J. Tits, Une classe d'algèbres de Lie en relation avec les algèbres de Jordan, Indag. Math. **24** (1962) 530–534.

[16] E. Zelmanov, Jordan nil-algebras of bounded index, (Russian) Dokl. Akad. Nauk SSSR **249** (1979), 30–33. English translation in Soviet Math. Dokl. **20** (1979), 1188—1192.