On the Free Jordan Algebras

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Abstract

A conjecture for the dimension and the character of the homogenous components of the free Jordan algebras is proposed. As a support of the conjecture, some numerical evidences are generated by a computer and some new theoretical results are proved. One of them is the cyclicity of the Jordan operad.

Introduction. Let K be a field of characteristic zero and let $J(D)$ be the free Jordan algebra with D generators $x_1, \dots x_D$ over K. Then

$$J(D) = \oplus_{n\geq 1} J_n(D)$$

where $J_n(D)$ consists of all degree n homogenous Jordan polynomials in the variables $x_1, \dots x_D$. The aim of this paper is a conjecture about the character, as a $GL(D)$-module, of each homogenous component $J_n(D)$ of $J(D)$. In the introduction, only the conjecture for dim $J_n(D)$ will be described, see Section 1.10 for the whole Conjecture 1.

Conjecture 1 (weakest version). *Set* $a_n = \dim J_n(D)$. *The sequence* a_n *is the unique solution of the following equation:*

$$(\mathcal{E}) \qquad Res_{t=0}\, \psi \prod_n^\infty (1 - z^n(t + t^{-1}) + z^{2n})^{a_n} dt = 0,$$

where $\psi = Dzt^{-1} + (1 - Dz) - t$.

It is easy to see that equation $\mathcal{E}$ provides a recurrence relation to uniquely determine the integers a_n, but we do not know a closed formula.

Some computer calculations show that the predicted dimensions are correct for some interesting cases. E.g., for $D = 3$ and $n = 8$ the conjecture predicts that the space of special identities has dimension 3, which is correct: those are the famous Glennie's Identities [6]. Similarly for $D = 4$ the conjecture agrees that some tetrads are missing in $J(4)$, as it has been observed by Cohn [3]. Other interesting numerical evidences are given in Section 2. Since our input is the quite simple polynomial ψ, these numerical verifications provide a good support for the conjecture.

Conjecture 1 is elementary, but quite mysterious. Indeed it follows from two natural, but more sophisticated, conjectures about Lie algebras cohomology. Conjecture 3 will be now stated, see Section 3 for Conjecture 2.

Let $\mathbf{Lie_T}$ be the category of Lie algebras $\mathfrak{g}$ on which $\mathfrak{sl}_2$ acts by derivation such that $\mathfrak{g} = \mathfrak{g}^{\mathfrak{sl}_2} \oplus \mathfrak{g}^{ad}$ as an $\mathfrak{sl}_2$-module. For any Jordan algebra J, Tits has defined a Lie algebra structure on the space $\mathfrak{sl}_2 \otimes J \oplus \mathrm{Inner}\, J$ [24]. It has been later generalized by Kantor [10] and Koecher [11] and it is now called the TKK-construction and denoted by $TTK(J)$. Here we use another refinement of Tits construction, due to Allison and Gao [1]. The corresponding Lie algebra will be denoted by $TAG(J)$, or, more simply, by $\mathfrak{sl}_2\, J$. The Lie algebra $\mathfrak{sl}_2\, J$ belongs to the category $\mathbf{Lie_T}$

Since the TAG-construction is functorial, it is obvious that $\mathfrak{sl}_2\, J(D)$ is a free Lie algebra in the category $\mathbf{Lie_T}$. Therefore it is very natural to expect some cohomology vanishing, as the following

Conjecture 3. *We have*

$$H_k(\mathfrak{sl}_2\, J(D))^{\mathfrak{sl}_2} = 0, \text{ for any } k > 0.$$

The conjecture, obvious for $k = 1$, follows from Allison-Gao paper [1] for $k = 2$. It is also proved for $k = 3$ in Section 4. To connect the two conjectures, we first prove in Section 5

Theorem 1 (imprecise version). *The Jordan operad is cyclic.*

This Theorem has some striking consequences. E.g the space $\mathcal{SI}(D)$ of multilinear special identities of degree D, which is is obviously a $\mathfrak{S}_D$-module, is indeed a $\mathfrak{S}_{D+1}$-module for any $D \geq 1$. Also it allows to easily compute dim $J_n(D)$ for any $n \leq 7$, for any D.

In Section 6, we use a more technical version of Theorem 1 to prove that:

Theorem 2. *Conjecture 3 implies Conjecture 1.*

As a conclusion, the reader could find Conjecture 3 too optimistic. However, it is clear from the paper that the groups $H_*(\mathfrak{sl}_2\, J(D))^{\mathfrak{sl}_2}$ are strongly connected with the structure of the free Jordan algebras and they provide an interesting approach for these questions.

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1. Statement of Conjecture 1

The introduction describes the *weakest version of Conjecture 1*, which determines the dimensions of the homogenous components of $J(D)$. In this section, Conjecture 1 will be stated, as well as a weak version of it.

Let Inner $J(D)$ be the Lie algebra of inner derivations of $J(D)$. Conjecture 1, stated in Section 1.9, provides the character, as $GL(D)$-modules, of the homogenous components of $J(D)$ and of Inner $J(D)$. The *weak version of Conjecture 1* is a formula only for the dimensions of those homogenous components, see Section 1.10. The Subsections 1.1 to 1.8 are devoted to define the main notations of the paper, and to introduce the combinatorial notions which are required to state Conjecture 1.

1.1 Main notations and conventions

Throughout this paper, the ground field K has characteristic zero, and all algebras and vector spaces are defined over K.

Recall that a commutative algebra J is called a *Jordan algebra* if its product satisfies the following *Jordan identity*

$$x^2(yx) = (x^2y)x$$

for any $x, y \in J$. For x, $y \in J$, let $\partial_{x,y} : J \to J$ be the map $z \mapsto x(zy)-(xz)y$. It follows from the Jordan identity that $\partial_{x,y}$ is a derivation. A derivation ∂ of J is called an *inner derivation* if it is a linear combination of some $\partial_{x,y}$.

The space, denoted Inner J, of all inner derivations is a subalgebra of the Lie algebra Der J of all derivations of J.

In what follows, the positive integer D will be given once for all. Let $J(D)$ be the free Jordan algebra on D generators. This algebra, and some variants, has been investigated in many papers by the Novosibirsk school of algebra, e.g. [20], [21], [22], [26],[27].

1.2 The ring R(G)

Let G be an algebraic reductive group and let $Z \subset G$ be a central subgroup isomorphic to K^*. In what follows a rational G-module will be called a *G-module* or a *representation of* G.

Let $n \geq 0$. A G-module on which any $z \in Z$ acts by z^n is called a *G-module of degree n*. Of course this notion is relative to the the subgroup Z and to the isomorphism $Z \simeq K^*$. However we will assume that these data are given once for all.

Let $Rep_n(G)$ be the category of the finite dimensional G-modules of degree n. Set

$$\mathcal{R}(G) = \prod_{n=0}^{\infty} K_0(Rep_n(G))$$
$$\mathcal{M}_{>n}(G) = \prod_{k>n} K_0(Rep_k(G))$$
$$\mathcal{M}(G) = \mathcal{M}_{>0}(G).$$

There are products

$$K_0(Rep_n(G)) \times K_0(Rep_m(G)) \to K_0(Rep_{n+m}(G))$$

induced by the tensor product of the G-modules. Therefore $\mathcal{R}(G)$ is a ring and $\mathcal{M}(G)$ is an ideal.

Moreover $\mathcal{R}(G)$ is complete with respect to the $\mathcal{M}(G)$-adic topology, i.e. the topology for which the sequence $\mathcal{M}_{>n}(G)$ is a basis of neighborhoods of 0. Any element a of $\mathcal{R}(G)$ can be written as a formal series

$$a = \sum_{n\geq 0} a_n$$

where $a_n \in K_0(Rep_n(G))$.

As usual, the class in $K_0(Rep_n(G))$ of a G-module $V \in Rep_n(G))$ is denoted by $[V]$. Also let $Rep(G)$ be the category of the G-modules V, with a decomposition $V = \oplus_{n\geq 0} V_n$, such that $V_n \in Rep_n(G)$ for all $n \geq 0$. For such a module V, its class $[V] \in \mathcal{R}(G)$ is defined by $[V] := \sum_{n\geq 0} [V_n]$.

1.3 Analytic representations of $GL(D)$ and their natural gradings

A finite dimensional rational representation ρ of $GL(D)$ is called *polynomial* if the map $g \mapsto \rho(g)$ is polynomial into the entries $g_{i,j}$ of the matrix g. The center of $GL(D)$ is $Z = K^*\mathrm{id}$, relative to which the degree of a representation has been defined in the previous section. It is easy to show that a polynomial representation ρ has degree n iff $\rho(g)$ is a degree n homogenous polynomial into the entries $g_{i,j}$ of the matrix g. Therefore the notion of a polynomial representation of degree n is unambiguously defined.

By definition an *analytic* $GL(D)$*-module* is a $GL(D)$-module V with a decomposition

$$V = \oplus_{n\geq 0} V_n$$

such that each component V_n is a polynomial representation of degree n. In general V is infinite dimensional, but it is always required that each V_n is finite dimensional. The decomposition $V = \oplus_{n\geq 0} V_n$ of an analytic module V is called its *natural grading*.

The free Jordan algebra $J(D)$ and its associated Lie algebra Inner $J(D)$ are examples of analytic $GL(D)$-modules. The natural grading of $J(D)$ is the previously defined decomposition $J(D) = \oplus_{n\geq 0} J_n(D)$ and the degree n component of Inner $J(D)$ is denoted $\mathrm{Inner}_n J(D)$.

Let $Pol_n(GL(D))$ be the category of polynomial representations of $GL(D)$ of degree n, let $An(GL(D)$ be the category of all analytic $GL(D)$-modules. Set

$$\mathcal{R}_{an}(GL(D)) = \prod_{n\geq 0} K_0(Pol_n(GL(D))), \text{ and}$$
$$\mathcal{M}_{an}(GL(D)) = \prod_{n>0} K_0(Pol_n(GL(D))).$$

The class $[V] \in \mathcal{R}_{an}(GL(D))$ of an analytic module is defined as before.

Similarly a finite dimensional rational representation ρ of $GL(D)\times PSL(2)$ is called *polynomial* if the underlying $GL(D)$-module is polynomial. Also an *analytic* $GL(D) \times PSL(2)$*-module* is a $GL(D) \times PSL(2)$-module V with a decomposition

$$V = \oplus_{n\geq 0} V_n$$

such that each component V_n is a polynomial representation of degree n.

1.4 Weights and Young diagrams

Let $H \subset GL(D)$ be the subgroup of diagonal matrices. The subsection is devoted to the combinarics of the weights and the dominant weights of the polynomial representations.

A D-uple $\mathbf{m} = (m_1, \dots, m_D)$ of non-negative integers is called a *partition*. It is called a *partition of* n if $m_1 + \cdots + m_D = n$. The weight decomposition of an analytic module V is given by

$$V = \oplus_{\mathbf{m}} V_{\mathbf{m}}$$

where $\mathbf{m}$ runs over all the partitions, and where $V_{\mathbf{m}}$ is the subspace of all $v \in V$ such $h.v = h_1^{m_1} h_2^{m_2} \dots h_D^{m_D}.v$ for all $h \in H$ with diagonal entries

$h_1, h_2, \dots h_D$. Relative to the natural grading $V = \oplus_{n \geq 0} V_n$ of V, we have

$$V_n = \oplus_{\mathbf{m}} V_{\mathbf{m}}$$

where $\mathbf{m}$ runs over all the partition of n.

With these notations, there is an isomorphism [18]

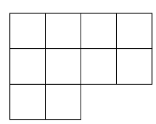
$$\mathcal{R}_{an}(GL(D)) \simeq \mathbb{Z}[[z_1, \dots, z_D]]^{\mathfrak{S}_D}$$

where the symmetric group $\mathfrak{S}_D$ acts by permutation of the variables $z_1, \dots, z_D$. Then the class of an analytic module V in $\mathcal{R}_{an}(GL(D)$ is given by

$$[V] = \sum_{\mathbf{m}} \dim V_{\mathbf{m}}\ z_1^{m1} z_2^{m_2} \dots z_D^{m_D}.$$

For example, let $x_1, \dots,\ x_D$ be the generators of $J(D)$. Then for any partition $\mathbf{m} = (m_1, \dots,\ m_D)$, $J_{\mathbf{m}}(D)$ is the space of Jordan polynomials $p(x_1, \dots,\ x_D)$ which are homogenous of degree m_1 into x_1, homogenous of degree m_2 into x_2 and so on... Thus the class $[J(D)] \in \mathcal{R}_{an}(GL(D))$ encodes the same information as $\dim J_{\mathbf{m}}(D)$ for all $\mathbf{m}$.

Relative to the standard Borel subgroup, the dominant weights of polynomial representations are the partitions $\mathbf{m} = (m_1, \dots,\ m_D)$ with $m_1 \geq m_2 \geq \dots \geq m_D$ [18]. Such a partition, which is called a *Young diagram*, is represented by a diagram with m_1 boxes on the first line, m_2 boxes on the second line and so on... When a pictorial notation is not convenient, it will be denoted as $(\mathbf{n_1^{a_1}}, \mathbf{n_2^{a_2}} \dots)$, where the symbol $\mathbf{n^a}$ means that the lign with n boxes is repeated a times. E.g., $(\mathbf{4^2}, \mathbf{2})$ is represented by



For a Young diagram $\mathbf{Y}$, the total number of boxes, namely $m_1 + \dots + m_D$ is called its *size* while its *height* is the number of boxes on the first column. When $\mathbf{Y}$ has heigth $\leq D$, the simple $GL(D)$-module with highest weight $\mathbf{Y}$ will be denoted by $L(\mathbf{Y}; D)$. It is also convenient to set $L(\mathbf{Y}; D) = 0$ if the heigth of $\mathbf{Y}$ is $> D$. For example $L(\mathbf{1^3}; D)$ denotes $\Lambda^3 K^D$, which is zero for $D < 3$.

1.5 Effective elements in $\mathcal{R}(G)$.

The classes $[M]$ of the G-modules M are called the *effective classes* in $\mathcal{R}(G)$. Let $\mathcal{M}(G)^+$ be the set of effective classes in $\mathcal{M}(G)$. Then any $a \in \mathcal{M}(G)$ can be written as $a' - a"$, where a', $a" \in \mathcal{M}(G)^+$.

1.6 λ-structure on the ring $\mathcal{R}(G)$

The ring $\mathcal{R}(G)$ is endowed with a map $\lambda : \mathcal{M}(G) \to \mathcal{R}(G)$.

First $\lambda\, a$ is defined for $a \in \mathcal{M}^+(G)$. Any $a \in \mathcal{M}^+(G)$ is the class of a G-module $V \in Rep(G)$. It is clear that $M := \Lambda\, V$ belongs to $Rep(G)$. Set

$$\lambda\, a = \sum_{k\geq 0} (-1)^k\, [\Lambda^k\, V].$$

Moreover we have $\lambda(a+b) = \lambda\, a\, \lambda\, b$ for any $a,\, b \in \mathcal{M}^+(G)$.

For an arbitrary $a \in \mathcal{M}(G)$, there are $a',\, a" \in \mathcal{M}^+(G)$ such that $a = a' - a"$. Since $\lambda\, a" = 1$ modulo $\mathcal{M}(G)$, it is invertible, and $\lambda\, a$ is defined by

$$\lambda\, a = (\lambda\, a")^{-1} \lambda\, a'.$$

1.7 The decomposition in the ring $\mathcal{R}(G \times PSL(2))$

Let G be a reductive group. For any $k \geq 0$, let $L(2k)$ be the irreducible $PSL(2)$-module of dimension $2k+1$. Since the family $([L(2k)])_{k\geq 0}$ is a basis of $K_0(PSL(2))$, any element $a \in K_0(G \times PSL(2))$ can be writen as a finite sum

$$a = \sum_{k\geq 0} [a : L(2k)]\, [L(2k)]$$

where the multiplicities $[a : L(2k)]$ are elements of $K_0(G)$.

Assume now that G is a subgroup of $GL(D)$ which contains the central subgroup $Z = K^*\mathrm{id}$. We consider Z as a subgroup of $G \times PSL(2)$, and therefore the notion of a $G \times PSL(2)$-module of degree n is well defined. Indeed it means that the underlying G-module has degree n. As before any $a \in \mathcal{R}(G \times PSL(2))$ can be decomposed as

$$a = \sum_{k\geq 0} [a : L(2k)]\, [L(2k)]$$

where $[a : L(2k)] \in \mathcal{R}(G)$. Instead of being a finite sum, it is a series whose convegence comes from the fact that

$$[a : L(2k)] \to 0 \text{ when } k \to \infty.$$

1.8 The elements $A(D)$ and $B(D)$ in the ring $\mathcal{R}_{an}(GL(D))$

Let $G \subset GL(D)$ be a reductive subgroup containing $Z = K^*\mathrm{id}$. Let K^D be the natural representation of $GL(D)$ and let $K^D|_G$ be its restriction to G.

Lemma 1. *1. There are elements $a(G)$ and $b(G)$ in $\mathcal{M}(G)$ which are uniquely defined by the following two equations in $\mathcal{R}(G \times PSL(2))$*

$$\lambda(a(G)[L(2)] + b(G)) : [L(0)] = 1$$
$$\lambda(a(G)[L(2)] + b(G)) : [L(2)] = -[K^D|_G].$$

2. For $G = GL(D)$, set $A(D) = a(GL(D))$ and $B(D) = b(GL(D))$. Then $A(D)$ and $B(D)$ are in $\mathcal{M}_{an}(GL(D))$.

3. Moreover $a(G) = A(D)|_G$ and $b(G) = B(D)|_G$.

Proof. In order to prove Assertion 1, some elements a_n and b_n in $\mathcal{M}(G)$ are defined by induction by the following algorithm. Start with $a_0 = b_0 = 0$. Then assume that a_n and b_n are already defined with the property that

$$\lambda(a_n[L(2)]+b_n):[L(0)]=1 \qquad \text{modulo } \mathcal{M}_{>n}(G)$$
$$\lambda(a_n[L(2)]+b_n):[L(2)]=-[K^D|_G] \quad \text{modulo } \mathcal{M}_{>n}(G).$$
Let α and β be in $K_0(Rep_{n+1}(G))$ defined by
$$\lambda(a_n[L(2)]+b_n):[L(0)]=1-\alpha \qquad \text{modulo } \mathcal{M}_{>n+1}(G)$$
$$\lambda(a_n[L(2)]+b_n):[L(2)]=-[K^D|_G]-\beta \quad \text{modulo } \mathcal{M}_{>n+1}(G).$$
Thus set $a_{n+1}=a_n+\alpha$ and $b_{n+1}=b_n+\beta$. Since we have $\lambda(\alpha[L(2)]+\beta)=1-\alpha.[L(2)]-\beta$ modulo $\mathcal{M}_{>n+1}(G)$, we get
$$\lambda(a_{n+1}[L(2)]+b_{n+1}):[L(0)]=1 \qquad \text{modulo } \mathcal{M}_{>n+1}(G)$$
$$\lambda(a_{n+1}[L(2)]+b_{n+1}):[L(2)]=-[K^D|_G] \quad \text{modulo } \mathcal{M}_{>n+1}(G),$$
and therefore the algorithm can continue.

Since $a_{n+1}-a_n$ and $b_{n+1}-b_n$ belong to $K_0(Rep_{n+1}(G))$, the sequences a_n and b_n converge. The elements $a(G):=\lim a_n$ and $b(G):=\lim b_n$ satisfies the first assertion. Moreover, it is clear that $a(G)$ and $b(G)$ are uniquely defined.

The second assertion follows from the fact that, for the group $G=GL(D)$, all calculations arise in the ring $\mathcal{R}_{an}(GL(D))$: so the elements $A(D)$ and $B(D)$ are in $\mathcal{M}_{an}(GL(D))$.

For Assertion 3, it is enough to notice that the pair $(a(G),b(G))$ and $(A(D)|_G,B(D)|_G)$ satisfy the same equation, so they are equal. □

1.9 The conjecture 1

After these long preparations, we can now state Conjecture 1.

Conjecture 1. *Let $D\geq 1$ be an integer. In $\mathcal{R}_{an}(GL(D))$ we have*
$$[J(D)]=A(D) \text{ and } [\mathrm{Inner}\, J(D)]=B(D),$$
where the elements $A(D)$ and $B(D)$ are defined in Lemma 1.

1.10 The weak form of Conjecture 1

We will now state the weak version of Conjecture 1 which only involves the dimensions of homogenous components of $J(D)$ and Inner $J(D)$.

Here G is the central subgroup $Z=K^*\mathrm{id}$ of $GL(D)$. As in the subsection 1.4, $\mathcal{R}(Z)$ is identified with $\mathbb{Z}[[z]]$. An Z-module $V\in Rep(G)$ is a graded vector space $V=\oplus_{n\geq 0}V_n$ and its class [V] is
$$[V]=\sum_n \dim V_n\, z^n.$$
Let α be a root of the Lie algebra $\mathfrak{sl}_2$ and set $t=e^\alpha$. Then $K_0(PSL(2))$ is the subring $\mathbb{Z}[t+t^{-1}]$ of $\mathbb{Z}[t,\, t^{-1}]$ consisting of the symmetric polynomials in t and t^{-1}. If follows that

$$\mathcal{R}(G \times PSL(2)) = \mathbb{Z}[t + t^{-1}][[z]].$$

Next let $a \in K_0(PSL(2))$ and set $a = \sum_i c_i\, t^i$. Since $[a : L(0)] = c_0 - c_{-1}$ and $[a : L(2)] = c_{-1} - c_{-2}$ it follows that

$$[a : L(0)] = \mathrm{Res}_{t=0}\, (t^{-1} - 1)a\, \mathrm{d}t \text{ and } [a : L(2)] = \mathrm{Res}_{t=0}\, (1 - t)a\, \mathrm{d}t.$$

Indeed the same formula holds when a and b are in $\mathcal{R}(G \times PSL(2))$. In this setting, Lemma 1 can be expressed as

Lemma 2. *Let $D \geq 1$ be an integer. There are two series $a(z) = \sum_{n \geq n} a_n(D) z^n$ and $b(z) = \sum_{n \geq n} b_n(D) z^n$ in $\mathbb{Z}[[z]]$ which are uniquely defined by the following two equations:*

$$\mathrm{Res}_{t=0}\, (t^{-1} - 1)\Phi\, \mathrm{d}t = 1$$
$$\mathrm{Res}_{t=0}\, (1 - t)\Phi\, \mathrm{d}t = -Dz$$

where $\Phi = \prod_{n \geq 1} (1 - z^n t)^{a_n} (1 - z^n t^{-1})^{a_n} (1 - z^n)^{a_n + b_n}$, $a_n = a_n(D)$ and $b_n = b_n(D)$.

The weak version of Conjecture 1 is

Conjecture 1 (weak version). *Let $D \geq 1$. We have*

$$\dim\, J_n(D) = a_n(D) \textit{ and } \dim\, \mathrm{Inner}_n\, J(D) = b_n(D)$$

where $a_n(D)$ and $b_n(D)$ are defined in Lemma 2.

Indeed, Lemma 2 and the weak version of Conjecture 1 are the specialization of Lemma 1 and Conjecture 1 by the map $\mathcal{R}(GL(D) \times PSL(2)) \to \mathcal{R}(Z \times PSL(2))$.

1.11 About the weakest form of Conjecture 1

It is now shown that the version of Conjecture 1, stated in the introduction, is a consequence of the weak form of Conjecture 1.

It is easy to prove, as in Lemma 1, that the series a_n of the introduction is uniquely defined. It remains to show that the series a_n of Lemma 2 is the same.

Let's consider the series $a_n = a_n(D)$ and $b_n = b_n(D)$ of Lemma 2. We have

$$\mathrm{Res}_{t=0}\, (t^{-1} - 1)\Phi\, \mathrm{d}t = 1, \text{ and}$$
$$\mathrm{Res}_{t=0}\, (1 - t)\Phi\, \mathrm{d}t = -Dz.$$

Using that the residue is $\mathbb{Z}[[z]]$-linear, and combining the two equations we get $\mathrm{Res}_{t=0}\, \psi\, \Phi\, \mathrm{d}t = 0$, or, more explicitly

$$\mathrm{Res}_{t=0}\, \psi \prod_{n \geq 1} (1 - z^n)^{a_n + b_n} (1 - z^n t)^{a_n} (1 - z^n t^{-1})^{a_n}\, \mathrm{d}\, t = 0.$$

By $\mathbb{Z}[[z]]$-linearity we can remove the factor $\prod_{n \geq 1} (1 - z^n)^{a_n + b_n}$ and so we get

$$\mathrm{Res}_{t=0}\, \psi \prod_{n \geq 1} (1 - z^n t)^{a_n} (1 - z^n t^{-1})^{a_n}\, \mathrm{d}\, t = 0$$

which is the equation of the introduction.

2. Numerical Evidences for Conjecture 1

The numbers $\dim J_n(D)$ and $\dim \mathrm{Inner}_n J(D)$ are known in the following cases:

D	$\dim J_n(D)$	Proof in	$\dim \mathrm{Inner}_n J(D)$	Proof in
$D = 1$	any n	folklore	any n	folklore
$D = 2$	any n	Shirshov	any n	Sect. 2.4
$D = 3$	$n \leq 8$	Shirshov & Glennie	$n \leq 8$	Sect. 5
D any	$n \leq 7$	Sect. 5	$n \leq 8$	Sect. 5

The formulas for $\dim J_n(D)$, respectively for $\dim \mathrm{Inner}_n J(D)$, are provided in Section 2.1, respectively in Section 2.2. Then we will describe for which cases Conjecture 1 has been checked.

2.1 General results about free Jordan algebras

Let $D \geq 1$ be an integer, let $T(D)$ be the non-unital tensor algebra on D generators $x_1, x_2, \ldots, x_D$. Let σ be the involution on $T(D)$ defined by $\sigma(x_i) = x_i$.

Given an associative algebra A, a subspace J is called a *Jordan subalgebra* if J is stable by the Jordan product $x \circ y = 1/2(xy + yx)$. The Jordan subalgebra $CJ(D) = T(D)^\sigma$ will be called the *Cohn's Jordan algebra*. The Jordan subalgebra $SJ(D)$ generated by $x_1, x_2, \ldots, x_D$ is called the *free special Jordan algebra*. The kernel of the map $J(D) \to CJ(D)$, which is denoted $SI(D)$, is called the *space of special identities*. Its cokernel $M(D)$ will be called the *space of missing tetrads*.

The spaces $J(D)$, $T(D)$, $CJ(D)$, $SJ(D)$, $SI(D)$, $M(D)$ are all analytic $GL(D)$-modules. Relative to the natural grading, the homogenous component of degree n is respectively denoted by $J_n(D)$, $T_n(D)$, $CJ_n(D)$, $SJ_n(D)$, $SI_n(D)$, and $M_n(D)$. There is an exact sequence

$$0 \to SI_n(D) \to J_n(D) \to CJ_n(D) \to M_n(D) \to 0.$$

Set $s_n(D) = \dim CJ_n(D)$.

Lemma 3. *We have*

$$\dim J_n(D) = s_n(D) + \dim SI_n(D) - \dim M_n(D), \textit{ where}$$
$$s_{2n}(D) = \tfrac{1}{2}(D^{2n} + D^n), \textit{and}$$
$$s_{2n+1}(D) = \tfrac{1}{2}(D^{2n+1} + D^{n+1})$$

for any integer n.

Proof. The first assertion comes from the previous exact sequence. The (obvious) computation of $\dim CJ_n(D)$ will be explained in Lemma 6. $\square$

The previous elementary lemma shows that dim $J_n(D)$ is determined whenever dim $SI_n(D)$ and dim $M_n(D)$ are known, as in the following three results:

Glennie's Theorem. *[6] We have $SI_n(D) = 0$ for $n \leq 7$.*

Let $t_4 \in T(4)$ be the element

$$t_4 = \sum_{\sigma \in \mathfrak{S}_4} \epsilon(\sigma)\, x_{\sigma_1} \dots x_{\sigma_4}.$$

Observe that t_4 belongs to $CJ(4)$. Since $SJ(4)$ is commutative, it is clear that $t_4(x_i, x_j, x_k, x_l) \notin SJ(D)$.

Cohn's Reversible Theorem. *The Jordan algebra $CJ(D)$ is generated by the elements $x_1, x_2 \dots x_D$ and $t_4(x_i, x_j, x_k, x_l)$ for all $1 \leq i < j < k < l$. In particular $M_n(D) = 0$ for $D \leq 3$.*

The next lemma, together with Glennie Theorem, provides an explicit formula for dim $J_n(D)$ for any $n \leq 7$ and any $D \geq 1$.

Lemma 4. *For any $D \geq 1$, we have have*
$\dim M_4(D) = \binom{D}{4}$,
$\dim M_5(D) = D\binom{D}{4}$,
$\dim M_6(D) = 2\binom{D+1}{2}\binom{D}{4}$,
$\dim M_7(D) = 2D\binom{D+1}{2}\binom{D}{4} - \dim\, L(\mathbf{3}, \mathbf{2}, \mathbf{1}^{\mathbf{2}}; D)$.

Proof. The case $D = 4$ follows from the fact that $M_4(D) \simeq \Lambda^4 K^D$.

By Corollary 4, we have $M_5(D) = L(\mathbf{1}^{\mathbf{5}}, D) \oplus L((\mathbf{2}, \mathbf{1}^{\mathbf{3}}), D)$ which is isomorphic to $K^D \otimes \Lambda^4 K^D$ and the formula follows.

It follows also from Corollary 4 that $M_6(D) \simeq L(\mathbf{2}, \mathbf{1}^{\mathbf{4}}; D)^2 \oplus L(\mathbf{3}, \mathbf{1}^{\mathbf{3}}; D)^2$, which is isomorphic to $(S^2 K^D \otimes \Lambda^4 K^D)^2$. It follows also from the proof of Corollary 4 that, as a virtual module, we have $[M_7(D)] = [K^D][M_6(D)] - [L(\mathbf{3}, \mathbf{2}, \mathbf{1}^{\mathbf{2}}; D)]$, what proves the formula.

$\square$

2.2 General results about inner derivations

Given an associative algebra A, a subspace L is called a *Lie subalgebra* if L is stable by the Lie product $[x, y] = (xy - yx)$.

Lemma 5. *Let A be an associative algebra, let $Z(A)$ be its center and let $J \subset A$ be a Jordan subalgebra. Assume J contains a set of generators of A and that $Z(A) \cap [A, A] = 0$. Then we have* Inner $J \simeq [J,\, J]$.

Proof. Set $C(J) = \{a \in A | [a,\ J] = 0\}$. As J contains a set of generators of A, we have $C(J) = Z(A)$. Note that $4\,\partial_{x,y}\,z = [[x, y], z] = \mathrm{ad}([x, y])(z)$ for any $x, y, z \in J$. Since $[J,\ J] \cap C(J) = 0$, we have $\mathrm{Inner}\, J = \mathrm{ad}([J, J]) \simeq [J, J]$ $\square$

For $D \geq 1$, the space $A(D) = T(D)^{-\sigma}$ is a Lie subalgebra of $T(D)$. By the previous lemma we have

$$\mathrm{Inner}\, SJ(D) = [SJ(D), SJ(D)] \subset \mathrm{Inner}\, CJ(D) = [CJ(D), CJ(D)].$$

Therefore $\mathrm{Inner}\, SJ(D)$ and $\mathrm{Inner}\, CJ(D)$ are Lie subalgebras of $A(D)$. There is a Lie algebra morphism

$$\mathrm{Inner}\, J(D) \to \mathrm{Inner}\, CJ(D).$$

Its kernel $SD(D)$ will be called the *space of special derivations* and its cokernel $MD(D)$ will be called the space of *missing derivations*.

The spaces $\mathrm{Inner}\, J(D)$, $\mathrm{Inner}\, CJ(D)$, $\mathrm{Inner}\, SJ(D)$, $A(D)$, $SD(D)$ and $MD(D)$ are all analytic $GL(D)$-modules. Relative to the natural grading, the homogenous component of degree n is respectively denoted by $\mathrm{Inner}_n\, J(D)$, $\mathrm{Inner}_n\, CJ(D)$, $\mathrm{Inner}_n\, SJ(D)$, $A_n(D)$, $SD_n(D)$ and $MD_n(D)$.

There is an exact sequence

$$0 \to SD_n(D) \to \mathrm{Inner}_n\, J(D) \to \mathrm{Inner}_n\, CJ(D) \to MD_n(D) \to 0.$$

Set $r_n(D) = \dim \mathrm{Inner}_n\, CJ(D)$.

Lemma 6. *We have* $\mathrm{Inner}\, CJ(D) = A(D) \cap [T(D), T(D)]$, *and*

$$\dim \mathrm{Inner}\, J_n(D) = r_n(D) + \dim\, SD_n(D) - \dim\, MD_n(D),$$ *where*

$$r_{2n}(D) = \tfrac{1}{2}D^{2n} + \tfrac{1}{4}(D-1)D^n - \tfrac{1}{4n}\textstyle\sum_{i|2n} \phi(i) D^{\frac{2n}{i}},$$

$$r_{2n+1}(D) = \tfrac{1}{2}D^{2n+1} - \tfrac{1}{4n+2}\textstyle\sum_{i|2n+1} \phi(i) D^{\frac{2n+1}{i}}.$$

for any $n \geq 1$, where ϕ is the Euler's totient function.

Proof. 1) We have $[T(D), T(D)] = \sum_i [x_i, T(D)]$, so we get

$$A(D) \cap [T(D), T(D)] = \textstyle\sum_i [x_i, CJ(D)] \subset [CJ(D), CJ(D)].$$

Therefore we have $[CJ(D), CJ(D)] = A(D) \cap [T(D), T(D)]$, and it follows from Lemma 5 that $\mathrm{Inner}\, CJ(D) = A(D) \cap [T(D), T(D)]$.

2) Let σ be an involution preserving a basis B of some vector space V. An element $b \in B$ is called *oriented* if $b \neq \sigma(b)$. Thus B is union of B^σ and of its *oriented pairs* $\{b, b^\sigma\}$. The following formulas will be used repeatively

$\dim V^\sigma = \frac{1}{2}\,(\mathrm{Card}\, B + \mathrm{Card}\, B^\sigma)$,

$\dim V^{-\sigma}$ is the number of oriented pairs.

3) The set of words in $x_1, \ldots, x_D$ is a σ-invariant basis of $T(D)$, thus the formula for $s_n(D)$ (which was stated in Lemma 3) and for $\dim A_n(D)$ follows from the previous formulas.

4) A *cyclic word* is a word modulo cyclic permutation: for example $x_1x_2x_3$ and $x_2x_3x_1$ define the same cyclic word. For n, $D \geq 1$, let $c_n(D)$ be the number of pairs of oriented cyclic words of length n on a alphabet with D letters. E.g. $c_6(2) = 1$ since x^2y^2xy and yxy^2x^2 is the only pair of oriented words of length 6 in two letters.

In the literature of Combinatorics, a cyclic word is often called a necklace while a non-oriented word is called a bracelet, and their enumeration is quite standard. There are closed formulas for both, the webpage [25] is nice. From this it follows that

$c_{2n}(D) = \frac{1}{4n}\sum_{i|2n}\phi(i)D^{\frac{2n}{i}} - \frac{1}{4}(D+1)D^n$, and

$c_{2n+1}(D) = \frac{1}{4n}\sum_{i|2n+1}\phi(i)D^{\frac{2n+1}{i}} - \frac{1}{2}D^{n+1}$

for any $n \geq 1$, where ϕ denotes the Euler's totient function.

Since the set of cyclic words is a basis of $T(D)/[T(D),T(D)]$, we have

$$\dim A_n(D)/[T(D),T(D)] \cap A_n(D) = c_n(D).$$

Using the short exact sequence

$$0 \to \mathrm{Inner}(CJ(D)) \to A(D) \to A(D)/[T(D),T(D)] \cap A(D) \to 0$$

we get that $\dim \mathrm{Inner}_n CJ(D) = \dim A_n(D) - c_n(D)$ from which the explicit formula for $r_n(D)$ follows. $\square$

Since $\dim \mathrm{Inner}_n J(D) = r_n(D) + \dim SD_n(D) - \dim MD_n(D)$, the next two lemmas compute $\dim \mathrm{Inner}_n J(D)$ for any $n \leq 8$ and any $D \geq 1$.

Lemma 7. *We have $SD_n(D) = 0$ for any $n \leq 8$ and any D.*

Proof. The lemma follows from Corollary 7 proved in Section 5. $\square$

Lemma 8. *We have $MD_n(D) = 0$ for $n \leq 4$, and*

$\dim MD_5(D) = D\binom{D}{4} - \binom{D}{5}$,

$\dim MD_6(D) = \binom{D}{6} + D^2\binom{D}{4} - D\binom{D}{5}$,

$\dim MD_7(D) = 2[D \dim L(\mathbf{3}, \mathbf{1^3}; D) + \binom{D}{2}\binom{D}{5} - \binom{D}{7}]$.

Moreover Corollary 5 provides a (very long) formula for $\dim MD_8(D)$.

Proof. We have $[L(\mathbf{2}, \mathbf{1^3}; D)] = [K^D \otimes \Lambda^4 K^D] - [\Lambda^5 K^D]$. Using Corollary 5, we have $\dim MD_5(D) = \dim L(\mathbf{2}, \mathbf{1^3}; D) = D\binom{D}{4} - \binom{D}{5}$.

By Corollary 5 we have $MD_6(D) \simeq L(\mathbf{1^6}; D) \oplus L(\mathbf{2}, \mathbf{1^4})(D) \oplus L(\mathbf{2^2}, \mathbf{1^2}; D) \oplus L(\mathbf{3}, \mathbf{1^3}; D)$ which is isomorphic to $\Lambda^6 K^D \oplus K^D \otimes MD_5(D)$, from which the formula follows.

We have $K^D \otimes L(\mathbf{3}, \mathbf{1^3}; D) = L(\mathbf{4}, \mathbf{1^3}; D) \oplus L(\mathbf{3}, \mathbf{2}, \mathbf{1^2}; D) \oplus L(\mathbf{3}, \mathbf{1^4}; D)$ and $\Lambda^2 K^D \otimes \Lambda^2 K^D = L(\mathbf{2^2}, \mathbf{1^3}; D) \oplus L(\mathbf{2}, \mathbf{1^5}; D) \oplus \Lambda^5 K^D$. By Corollary 5, $MD_7(D)$ is isomorphic to

$[L(\mathbf{4}, \mathbf{1^3}; D) \oplus L(\mathbf{3}, \mathbf{2}, \mathbf{1^2}; D) \oplus L(\mathbf{3}, \mathbf{1^4}; D) \oplus L(\mathbf{2^2}, \mathbf{1^3}; D) \oplus L(\mathbf{2}, \mathbf{1^5}; D)]^2$

follows that, as a virtual module, we have

$[MD_7(D)] = 2([K^D][L(\mathbf{3}, \mathbf{1^3}; D)] + [\Lambda^2 K^D][\Lambda^2 K^D] - [\Lambda^7 K^D])$,

what proves the formula.

□

2.3 The case $D = 1$

Set $\Phi = \prod_{n=1}^{\infty} (1 - z^n t)(1 - z^n t^{-1})(1 - z^n)$ and, for any $n \geq 0$, set $P_n = t^{-n} + t^{-n+1} + \cdots + t^n$.

Observe that $\mathrm{Res}_{t=0}(t^{-1} - 1)P_n dt$ is 1 for $n = 0$, and 0 when $n > 0$. Similarly we have $\mathrm{Res}_{t=0}(1 - t)P_n dt$ is 1 when $n = 1$ and 0 otherwise. Using the classical Jacobi triple identity [7]

$$\Phi = \sum_{n=0}^{\infty} (-1)^n z^{\frac{n(n+1)}{2}} P_n$$

it follows that

$$\mathrm{Res}_{t=0}(t^{-1} - 1)\Phi dt = 1 \text{ and } \mathrm{Res}_{t=0}(1 - t)\Phi dt = -z.$$

therefore we have $a_n(1) = 1$ and $b_n(1) = 0$ for any n. This is in agreement with the fact that $J(1) = xK[x]$ and $\mathrm{Inner}\, J(1) = 0$, and so Conjecture 1 holds for $D = 1$.

2.4 The case $D = 2$

Recall the following

Shirshov's Theorem. *We have J(2)=CJ(2).*

Therefore, it is easy to compute dim $J_n(2)$. Using that $\dim \mathrm{Inner}_n J(2) = \dim A_n(2) - c_n(2)$, the computation of $\dim \mathrm{Inner}_n J(2)$ is easily deduced from the value of $c_n(2)$. These values are computed in [25] for $n \leq 15$ (it is the number $N(n, 2) - N'(n, 2)$ of [25]). From which it has been checked using a computer that $a_n(2) = s_n(2) = \dim J_n(2)$ and $b_n(2) = r_n(2) = \dim \mathrm{Inner}_n J(2)$ for any $n \leq 15$.

2.5 The case $D = 3$

In the case $D = 3$, recall the following

Shirshov-Cohn Theorem. *The map $J(3) \to CJ(3)$ is onto.*

Macdonnald's Theorem. *The space $SI(3)$ contains no Jordan polynomials of degree ≤ 1 into x_3.*

The space $SI_8(3)$ contains the special identity G_8, discovered in [6], which is called the *Glennie's identity*. It is multi-homogenous of degree $(3, 3, 2)$. In addition of the original expression, there are two simpler formulas due to Thedy and Shestakov [16] and [23].

Glennie's Identity Theorem. *1. We have $G_8 \neq 0$.*

2. $SI_8(3)$ is the 3-dimensional $GL(3)$-module generated by G_8.

Assertion 1 is proved in [6], and the fact G_8 generates a 3-dimensional $GL(3)$-module is implicite in [6]. However, we did not find a full proof of Assertion 2 in the litterature, but the experts consider it as true. Note that by Macdonald's Theorem, no partition $\mathbf{m} = (m_1, m_2, m_3)$ with $m_3 \leq 1$ is a weight of $SI(3)$. It seems to be known that $(4, 2, 2)$ is not a weight of $SI_8(3)$ and that the highest weight $(3, 3, 2)$ has multiplicity one.

It follows from Glennie's Theorem and Shirshov-Cohn's Theorem that

$$\dim\, J_n(3) = \dim SJ_n(3) = \dim\, CJ_n(3) = s_n(3) \text{ for } n \leq 7,$$

while the previous argument shows (or suggests, if the reader do consider Assertion 2 as a conjecture) that $\dim\, J_8(3) = s_8(3) + 3$

It follows from Lemma 7 and Shirshov-Cohn's Theorem that

$$\dim \operatorname{Inner}_n J(3) = \operatorname{Inner}_n SJ(3) = \dim \operatorname{Inner}_n CJ(3) = r_n(3) \text{ for } n \leq 8.$$

This correlates with the computer data that $a_n(3) = s_n(3)$ for $n \leq 7$, while $a_8(3) = s_8(3) + 3$ and similarly $b_n(3) = r_n(3)$ for $n \leq 8$.

2.6 The case $D = 4$

By Glennie's Theorem we have $J_n(4) = SJ_n(4)$ for $n \leq 7$, thus we have $\dim\, J_n(4) = s_n(4) - \dim\, M_n(4)$ for $n \leq 7$. Therefore it follows from Lemma 4 that $\dim\, J_n(4) = s_n(4)$, for $n \leq 3$, while $\dim\, J_4(4) = s_4(4) - 1$, $\dim\, J_5(4) = s_5(4) - 4$, $\dim\, J_6(4) = s_6(4) - 20$, and $\dim\, J_7(4) = s_7(4) - 60$.

Similarly, we have $\operatorname{Inner}_n J(4) = \operatorname{Inner}_n SJ_k(4)$ for $n \leq 8$ by Lemma 7 and therefore we have $\dim \operatorname{Inner} J_n(4) = r_n(4) - \dim\, MD_n(4)$ for $n \leq 8$. It follows from Lemma 8 that $\dim \operatorname{Inner}_n J(D) = r_n(4)$, for $n \leq 4$ while $\dim \operatorname{Inner}_5 J(D) = r_5(4) - 4$ $\dim \operatorname{Inner}_6 J(D) = r_6(4) - 16$ and $\dim \operatorname{Inner}_7 J(D) = r_7(4) - 80$.

Some computer computations show that these dimensions agrees with the numbers $a_n(4)$ and $b_n(4)$ of Conjecture 1.

2.6 Conclusion

These numerical computations show that the Conjecture 1 takes into account the "erratic" formula for dim Inner $J_n(2)$, also it detects the special identities in $J_8(3)$ and the missing tetrads in $J(4)$. It provides some support for the conjecture, because these facts were not put artificially in Conjecture 1. Later on, we will see that Conjecture 1 is also supported by some theoretical results and by the more theoretical conjectures 2 and 3.

3. The Conjecture 2

Conjecture 1 is an elementary statement, but it looks quite mysterious. In this section, the very natural, but less elementary, Conjecture 2 will be stated. At the end of the section, it will be proved that Conjecture 2 implies Conjecture 1.

3.1 The Tits functor $T : \mathbf{Lie_T} \to \mathbf{Jor}$.
Let $\mathbf{T}$ be the category of $PSL(2)$-modules M such that $M = M^{\mathfrak{sl}_2} \oplus M^{ad}$, where M^{ad} denotes the isotypical component of M of adjoint type. Let $\mathbf{Lie_T}$ be the category of Lie algebras $\mathfrak{g}$ in category $\mathbf{T}$ on which $\mathfrak{sl}_2$ acts by derivation (respectively endowed with an embedding $\mathfrak{sl}_2 \subset \mathfrak{g}$).

Let $\mathbf{Jor}$ be the category of Jordan algebras. Let e, f, h be the usual basis of $\mathfrak{sl}_2$. For $\mathfrak{g} \in \mathbf{Lie_T}$, set

$$T(\mathfrak{g}) = \{x \in \mathfrak{g} \,|\, [h, x] = 2x\}.$$

Then $T(\mathfrak{g})$ has an algebra structure, where the product $x \circ y$ of any two elements x, $y \in T(\mathfrak{g})$ is defined by:

$$x \circ y = \frac{1}{2}\,[x, f \cdot y].$$

It turns out that $T(\mathfrak{g})$ is a Jordan algebra [24]. So the map $\mathfrak{g} \mapsto T(\mathfrak{g})$ is a functor $T : \mathbf{Lie_T} \to \mathbf{Jor}$. It will be called the *Tits functor*.

3.2 The TKK -construction
To each Jordan algebra J is associated a Lie algebra $TKK(J) \in \mathbf{Lie_T}$ which is defined as follows. As a vector space we have

$$TKK(J) = \text{Inner } J \oplus \mathfrak{sl}_2 \otimes J.$$

For $x \in \mathfrak{sl}_2$ and $a \in J$, set $x(a) = x \otimes a$. The bracket $[X, Y]$ of two elements in $TKK(J)$ is defined as follows. When at least one argument lies in Inner J, it is defined by the fact that Inner J is a Lie algebra acting on J. Moreover the bracket of two elements $x(a)$, $y(b)$ in $\mathfrak{sl}_2 \otimes J$ is given by

$$[x(a), y(b)] = [x, y](a \circ b) + \kappa(x, y)\,\partial_{a,b}$$

where κ is the invariant bilinear form on $\mathfrak{sl}_2$ normalized by the condition $\kappa(h,h) = 4$. This construction first appears in Tits paper [24]. Later this definition has been generalized by Koecher [11] and Kantor [10] in the theory of Jordan pairs (which is beyond the scope of this paper) and therefore the Lie algebra $TKK(J)$ is usually called the *TKK-construction*.

However the notion of an inner derivation is not functorial and therefore the map $J \in \mathbf{Jor} \mapsto TKK(J) \in \mathbf{Lie_T}$ is *not* functorial.

3.3 The Lie algebra $TAG(J) = \mathfrak{sl}_2\, J$

More recently, Allison and Gao [1] found another generalization (in the theory of structurable algebras) of Tits construction, see also [2] and [14]. In the context of a Jordan algebra J, this provides a refinement of the TKK-construction. The corresponding Lie algebra will be called the *Tits-Allison-Gao construction* and it will be denoted by $TAG(J)$ or simply by $\mathfrak{sl}_2\, J$.

Let J be any Jordan algebra. First $TAG(J)$ is defined as a vector space. Let $R(J) \subset \Lambda^2 J$ be the linear span of all $a \wedge a^2$ where a runs over J and set $\mathcal{B}J = \Lambda^2 J / R(J)$. Set

$$TAG(J) = \mathcal{B}J \oplus \mathfrak{sl}_2 \otimes J.$$

Next, define the Lie algebra structure on $TAG(J)$. For $\omega = \sum_i a_i \wedge b_i \in \Lambda^2 J$, set $\partial_\omega = \sum_i \partial_{a_i,b_i}$ and let $\{\omega\}$ be its image in $\mathcal{B}J$. By Jordan identity we have $\partial_{a,a^2} = 0$, so there is a natural map

$$\mathcal{B}J \to \mathrm{Inner} J,\ \{\omega\} \mapsto \partial_\omega.$$

Given another element $\omega' = \sum_i a'_i \wedge b'_i$ in $\Lambda^2 J$, set $\delta_\omega.\omega' = \sum_i (\partial_\omega.a'_i) \wedge b'_i + a'_i \wedge \partial_\omega.b'_i$. Since ∂_ω is a derivation, we have $\partial_\omega.R(J) \subset R(J)$ and therefore we can set $\partial_\omega.\{\omega'\} = \{\partial_\omega.\omega'\}$.

The bracket on $TAG(J)$ is defined by the following rules

1. $[x(a), y(b)] = [x,y](a \circ b) + \kappa(x,y)\{a \wedge b\}$,
2. $[\{\omega\}, x(a)] = x(\partial_\omega a)$, and
3. $[\{\omega\}, \{\omega'\}] = \partial_\omega.\{\omega'\}$,

for any $x, y \in \mathfrak{sl}_2$, $a, b \in J$ and $\{\omega\}, \{\omega'\} \in \mathcal{B}J$, where, as before we denote by $x(a)$ the element $x \otimes a$ and where $\kappa(x,y) = \frac{1}{2}\operatorname{Tr}\operatorname{ad}(x) \circ \operatorname{ad}(y)$.

It is proved in [1] that $TAG(J)$ is a Lie algebra (indeed the tricky part is the proof that $[\{\omega\}, \{\omega'\}]$ is skew-symmetric). In general $TKK(J)$ and $TAG(J)$ are different. For $J = K[t, t^{-1}]$, we have $\mathrm{Inner}(J) = 0$, while $\mathcal{B}J$ is a one-dimensional Lie algebra. Therefore $TKK(J) = \mathfrak{sl}_2(K[t, t^{-1}])$ while $TAG(J)$ is the famous affine Kac-Moody Lie algebra $\widehat{\mathfrak{sl}_2}(K[t, t^{-1}])$.

Lemma 9. *Let $\mathfrak{g} \in \mathbf{Lie_T}$. Then there is a Lie algebra morphism*

$$\theta_{\mathfrak{g}} : TAG(T(\mathfrak{g})) \to \mathfrak{g}$$

which is the identity on $T(\mathfrak{g})$.

Proof. Set $\mathfrak{d} = \mathfrak{g}^{\mathfrak{sl}_2}$, so we have $\mathfrak{g} = \mathfrak{d} \oplus \mathfrak{sl}_2 \otimes T(\mathfrak{g})$. Since $\mathrm{Hom}_{\mathfrak{sl}_2}(\mathfrak{sl}_2^{\otimes 2}, K) = K.\kappa$, there is a bilinear map $\psi : \Lambda^2 T(\mathfrak{g}) \to \mathfrak{d}$ such that

$$[x(a), y(b)] = [x, y](a \circ b) + \kappa(x, y)\, \psi(a, b)$$

for any $x,\, y \in \mathfrak{sl}_2$ and $a,\, b \in J$. For $x,\, y\,,\, z \in \mathfrak{sl}_2$, we have

$$[x(a), [y(a),\, z(a)]] = [x,\, [y,\, z]](a^3) + \kappa(x, [y, z])\, \psi(a, a^2).$$

The map $(x, y, z) \mapsto \kappa(x, [y, z])$ has a cyclic symmetry of order 3. Since $\kappa(h, [e, f]) = 4 \neq 0$, the Jacobi identity for the triple $h(a)$, $e(a)$, $f(a)$ implies that

$$\psi(a, a^2) = 0 \text{ for any } a \in J.$$

Therefore the map $\psi : \Lambda^2 T(\mathfrak{g}) \to \mathfrak{d}$ factors trough $\mathcal{B}T(\mathfrak{g})$. A linear map $\theta_{\mathfrak{g}} : TAG(T(\mathfrak{g})) \to \mathfrak{g}$ is defined by requiring that $\theta_{\mathfrak{g}}$ is the identity on $\mathfrak{sl}_2 \otimes T(\mathfrak{g})$ and $\theta_{\mathfrak{g}} = \psi$ on $\mathcal{B}T(\mathfrak{g})$. It is easy to check that $\theta_{\mathfrak{g}}$ is a morphism of Lie algebras. □

It is clear that the map $TAG : J \in \mathbf{Jor} \mapsto TAG(J) \in \mathbf{Lie_T}$ is a functor, and more precisely we have:

Lemma 10. *The functor* $TAG : \mathbf{Jor} \to \mathbf{Lie_T}$ *is the left adjoint of the Tits functor* T*, namely:*

$$\mathrm{Hom}_{\mathbf{Lie_T}}(TAG(J), \mathfrak{g}) = \mathrm{Hom}_{\mathbf{Jor}}(J, T(\mathfrak{g}))$$

for any $J \in \mathbf{Jor}$ *anf* $\mathfrak{g} \in \mathbf{Lie_T}$.

Proof. Let $J \in \mathbf{Jor}$ and $\mathfrak{g} \in \mathbf{Lie_T}$. Since $T(TAG(J)) = J$, any morphism of Lie algebra $TAG(J) \to \mathfrak{g}$ restricts to a morphism of Jordan algebras $J \to T(\mathfrak{g})$, so we there is a natural map

$$\mu : \mathrm{Hom}_{\mathbf{Lie_T}}(TAG(J), \mathfrak{g}) \to \mathrm{Hom}_{\mathbf{Jor}}(J, T(\mathfrak{g})).$$

Since the Lie algebra $TAG(J)$ is generated by $\mathfrak{sl}_2 \otimes J$, it is clear that μ is injective. Let $\phi : J \to T(\mathfrak{g})$ be a morphism of Jordan algebras. By functoriality of the TAG-construction, we get a Lie algebra morphism

$$TAG(\phi) : TAG(J) \to TAG(T(\mathfrak{g}))$$

and by Lemma 9 there is a canonical Lie algebra morphism

$$\theta_{\mathfrak{g}} : TAG(T(\mathfrak{g})) \to \mathfrak{g}.$$

So $\theta_{\mathfrak{g}} \circ TAG(\phi) : TAG(J) \to \mathfrak{g}$ extends ϕ to a morphism of Lie algebras. Therefore μ is bijective.

□

3.4 Statement of Conjecture 2

Let $D \geq 1$ be an integer and let $J(D)$ be the free Jordan algebra on D generators.

Lemma 11. *The Lie algebra $\mathfrak{sl}_2\, J(D)$ is free in the category $\mathbf{Lie_T}$.*

The lemma follows from Lemma 10 and the formal properties of the adjoint functors.

Let k be a non-negative integer. Since $\Lambda^k\, \mathfrak{sl}_2\, J(D)$ is a direct sum of $\mathfrak{sl}_2$-isotypical components of type $L(0)$, $L(2), \ldots, L(2k)$ there is a similar isotypcal decomposition of $H_k(\mathfrak{g})$. For an ordinary free Lie algebra $\mathfrak{m}$, we have $H_k(\mathfrak{m}) = 0$ for any $k \geq 2$. Here $\mathfrak{sl}_2\, J(D)$ is free relative to category $\mathbf{Lie_T}$. Since only the trivial and adjoint $\mathfrak{sl}_2$-type occurs in the category $\mathbf{T}$, the following conjecture seems very natural

Conjecture 2. *We have*

$$H_k(\mathfrak{sl}_2\, J(D))^{\mathfrak{sl}_2} = 0, \text{ and}$$
$$H_k(\mathfrak{sl}_2\, J(D))^{ad} = 0,$$

for any $k \geq 1$.

3.5 Conjecture 2 implies Conjecture 1

Lemma 12. *Assume that $H_k(\mathfrak{sl}_2\, J(D))^{\mathfrak{sl}_2} = 0$ for any odd k. Then we have $\mathcal{B}J(D) = \mathrm{Inner}\, J(D)$.*

Proof. Assume otherwise, i.e. assume that the natural map $\phi : \mathcal{B}J(D) \to \mathrm{Inner}\, J(D)$ is not injective. Since $\mathcal{B}J(D)$ and $\mathrm{Inner}\, J(D)$ are analytic $GL(D)$-modules, they are endowed with the natural grading. Let z be a non-zero homogenous element $z \in \mathrm{Ker}\, \phi$ and let n be its degree. Set $G = \mathfrak{sl}_2\, J(D)/K.z$. Since z is a homogenous $\mathfrak{sl}_2$-invariant central element, G inherits a structure of $\mathbb{Z}$-graded Lie algebra.

Moreover z belongs to $[\mathfrak{sl}_2\, J(D), \mathfrak{sl}_2\, J(D)]$. Therefore $\mathfrak{sl}_2\, J(D)$ is a non-trivial central extension of G. Let $c \in H^2(G)$ be the corresponding cohomology class and let $\omega \in (\Lambda^2\, G)^*$ be a homogenous two-cocycle representing c. We have $\omega(G_i \wedge G_j) = 0$ whenever $i + j \neq n$. It follows that the bilinear map ω has finite rank, therefore there exists an integer $N \geq 1$ such that $c^N \neq 0$ but $c^{N+1} = 0$.

There is a long exact sequence of cohomology groups [8]

$$\ldots H^k(G) \xrightarrow{j^*} H^k(\mathfrak{sl}_2\, J(D)) \xrightarrow{i_z} H^{k-1}(G) \xrightarrow{\wedge c} H^{k+1}(G) \xrightarrow{j^*} \ldots$$

where j^* is induced by the natural map $j : \mathfrak{sl}_2\, J(D) \to G$, where i_z is the contraction by z and where $\wedge c$ is the mutiplication by c. Therefore there exists $C \in H^{2N+1}(\mathfrak{sl}_2\, J(D))$ such that $c^N = i_z C$. Since c^N is $\mathfrak{sl}_2$-invariant, we can assume that C is also $\mathfrak{sl}_2$-invariant, and therefore

$$H^{2N+1}(\mathfrak{sl}_2\, J(D))^{\mathfrak{sl}_2} \neq 0$$

which contradicts the hypothesis.

□

Corollary 1. *Conjecture 2 implies Conjecture 1.*

Proof. Assume Conjecture 2 holds. In $\mathcal{R}_{an}(GL(D) \times PSL(2))$, the identity

$$[\Lambda^{even}\mathfrak{sl}_2\, J(D)] - [\Lambda^{odd}\mathfrak{sl}_2\, J(D)] = [H_{even}(\mathfrak{sl}_2\, J(D))] - [H_{even}(\mathfrak{sl}_2\, J(D))]$$

is Euler's characteristic formula. By definition of the λ-operation, we have $[\Lambda^{even}\mathfrak{sl}_2\, J(D)] - [\Lambda^{odd}\mathfrak{sl}_2\, J(D)] = \lambda([\mathfrak{sl}_2\, J(D)])$. Moreover by Lemma 12, we have $[\mathfrak{sl}_2\, J(D)] = [J(D) \otimes L(2)] + [\text{Inner}\, J(D)]$, therefore we get

$$\lambda([J(D) \otimes L(2)] + [\text{Inner}\, J(D)]) = [H_{even}(\mathfrak{sl}_2\, J(D))] - [H_{even}(\mathfrak{sl}_2\, J(D))].$$

It is clear that $H_0(\mathfrak{sl}_2\, J(D)) = K$ and

$$H_1(\mathfrak{sl}_2\, J(D)) = \mathfrak{sl}_2\, J(D)/[\mathfrak{sl}_2\, J(D), \mathfrak{sl}_2\, J(D)] \simeq K^D \otimes L(2).$$

Moreover, by hypothesis, the higher homology groups $H_k(\mathfrak{sl}_2\, J(D))$ contains no trivial or adjoint component. It follows that

$$\lambda([J(D) \otimes L(2)] + [\text{Inner}\, J(D)]) : [L(0)] = 1, \text{ and}$$
$$\lambda([J(D) \otimes L(2)] + [\text{Inner}\, J(D)]) : [L(2)] = -[K^D].$$

So by Lemma 1, we get $[J(D)] = A(D)$ and $[\text{Inner}\, J(D)] = B(D)$. □

4. Proved Cases of Conjecture 2

This section shows three results supporting Conjecture 2:

1. The conjecture holds for $D = 1$,
2. As a $\mathfrak{sl}_2$-module, $H_2(\mathfrak{sl}_2\, J(D))$ is isotypical of type $L(4)$, and
3. The trivial component of the $\mathfrak{sl}_2$-module $H_3(\mathfrak{sl}_2\, J(D))$ is trivial.

4.1 The $D = 1$ case

Proposition 1. *Conjecture 2 holds for $J(1)$.*

For $D = 1$, we have $J(1) = tK[t]$. So Conjecture 2 is an obvious consequence of the following :

Garland-Lepowski Theorem. *[4] For any $k \geq 0$, we have*

$$H_k(\mathfrak{sl}_2(tk[t])) \simeq L(2k).$$

Conversely, Garland-Lepowski Theorem can be used to prove that $J(1) = tK[t]$. Of course, it is a complicated proof of a very simple result!

4.2 Isotypical components of $H_2(\mathfrak{sl}_2\, J(D))$.
Let $D \geq 1$ be an integer. Let $\mathcal{M}_{\mathbf{T}}(\mathfrak{sl}_2\, J(D))$ be the category of $\mathfrak{sl}_2\, J(D)$-modules in category $\mathbf{T}$. As an analytic $GL(D)$-module, $\mathfrak{sl}_2\, J(D)$ is endowed with the natural grading. Let $\mathcal{M}^{gr}_{\mathbf{T}}(\mathfrak{sl}_2\, J(D))$ be the category of all $\mathbb{Z}$-graded $\mathfrak{sl}_2\, J(D)$-modules $M \in \mathcal{M}_{\mathbf{T}}(\mathfrak{sl}_2\, J(D))$ such that

1. $\dim M_n < \infty$ for any n, and
2. $M_n = 0$ for $n \gg 0$.

Lemma 13. *Let M be a $\mathfrak{sl}_2\, J(D)$-module. Assume that*

1. M belongs to $\mathcal{M}_{\mathbf{T}}(\mathfrak{sl}_2\, J(D))$ and $\dim M < \infty$, or

2. M belongs to $\mathcal{M}^{gr}_{\mathbf{T}}(\mathfrak{sl}_2\, J(D))$

Then we have $H_2(\mathfrak{sl}_2\, J(D), M)^{\mathfrak{sl}_2} = 0$.

Proof. 1) First assume M belongs to $\mathcal{M}(\mathfrak{sl}_2\, J(D))$ and $\dim M < \infty$. Let $c \in H^2(\mathfrak{sl}_2\, J(D), M^*)^{\mathfrak{sl}_2}$. Since $\mathfrak{sl}_2$ acts reductively, c is represented by a $\mathfrak{sl}_2$-invariant cocycle $\omega : \Lambda^2\, \mathfrak{sl}_2\, J(D) \to M^*$. This cocycle defines a Lie algebra structure on $L := M^* \oplus \mathfrak{sl}_2\, J(D)$. Let

$$0 \to M^* \to L \to \mathfrak{sl}_2\, J(D) \to 0$$

be the corresponding abelian extension of $\mathfrak{sl}_2\, J(D)$. Since ω is $\mathfrak{sl}_2$-invariant, it follows that L lies in $\mathbf{Lie}_{\mathbf{T}}$. By Lemma 11 $\mathfrak{sl}_2\, J(D)$ is free in this category, hence the previous abelian extension is trivial. Therefore we have

$$H^2(\mathfrak{sl}_2\, J(D), M^*)^{\mathfrak{sl}_2} = 0.$$

By duality, it follows that $H_2(\mathfrak{sl}_2\, J(D), M)^{\mathfrak{sl}_2} = 0$.

2) Assume now that M belongs to $\mathcal{M}^{gr}_{T}(\mathfrak{sl}_2\, J(D))$. The $\mathbb{Z}$-gradings of $\mathfrak{sl}_2\, J(D)$ and M induce a grading of $H_*(\mathfrak{sl}_2\, J(D), M)$. Relative to it, the degree n component is denoted by $H_*(\mathfrak{sl}_2\, J(D), M)|n$ and its $\mathfrak{sl}_2$-invariant part will be denoted by $H^0(\mathfrak{sl}_2, H_2(\mathfrak{sl}_2\, J(D), M)|n$.

For an integer n, set $M_{>n} = \oplus_{k>n} M_k$. Since the degree n-part of the complex $\Lambda\, \mathfrak{sl}_2\, J(D) \otimes M_{>n}$ is zero, we have

$$H_*(\mathfrak{sl}_2\, J(D), M)|n = H_*(\mathfrak{sl}_2\, J(D), M/M_{>n})|n.$$

Since $M/M_{>n}$ is finite dimensional, the first part of the lemma shows that $H^0(\mathfrak{sl}_2, H_2(\mathfrak{sl}_2\, J(D), M)|n = 0$. Since n is arbitrary, we have

$$H_2(\mathfrak{sl}_2\, J(D), M)^{\mathfrak{sl}_2} = 0.$$

□

Proposition 2. *The $\mathfrak{sl}_2$-module $H_2(\mathfrak{sl}_2\, J(D))$ is isotypical of type $L(4)$.*

Proof. It follows from Lemma 13 that $H_2(\mathfrak{sl}_2\, J(D))^{\mathfrak{sl}_2} = 0$.

The $PSL(2)$-module $L(2)$, with a trivial action of $\mathfrak{sl}_2\, J(D)$, belongs to $\mathcal{M}_{\mathbf{T}}(\mathfrak{sl}_2\, J(D))$. So it follows from Lemma 13 that $H_2(\mathfrak{sl}_2\, J(D), L(2))^{\mathfrak{sl}_2} = 0$. Since

$$H_2(\mathfrak{sl}_2\, J(D))^{ad} = H_2(\mathfrak{sl}_2\, J(D), L(2))^{\mathfrak{sl}_2} \otimes L(2)$$

we also have $H_2(\mathfrak{sl}_2\, J(D))^{ad} = 0$.

The only $PSL(2)$-types occurring in $\Lambda^2\, \mathfrak{sl}_2\, J(D)$ are $L(0)$, $L(2)$ and $L(4)$. Since the $L(0)$ and $L(2)$ types do not occur in $H_2(\mathfrak{sl}_2\, J(D))$, it follows that $H_2(\mathfrak{sl}_2\, J(D))$ is isotypical of type $L(4)$. □

4.3 Analytic functors

Let $Vect_K$ be the category of K-vector spaces and let $Vect_K^f$ be the subcategory of finite dimensional vector spaces. A functor $F : Vect_K \to Vect_K$ is called a *polynomial functor* [18] if

1. $F(Vect_K^f) \subset Vect_K^f$ and F commutes with the inductive limits,
2. There is some integer n such that the map

$$F : \mathrm{Hom}(U, V) \to \mathrm{Hom}(F(U), F(V))$$

is a polynomial of degree $\leq n$ for any U, $V \in Vect_K^f$. The polynomial functor F is called *a polynomial functor of degree n* if $F(z\, \mathrm{id}_V) = z^n\, \mathrm{id}_{F(V)}$ for any $V \in Vect_K^f$. It follows easily that $F(V)$ is a polynomial $GL(V)$-module of degree n, see [18]. Any polynomial functor can be decomposed as a finite sum $F = \oplus_{n\geq 0} F_n$, where F_n is a polynomial functor of degree n.

A functor $F : Vect_K \to Vect_K$ is called *analytic* if F can be decomposed as a infinite sum

$$F = \oplus_{n\geq 0} F_n$$

where each F_n is a polynomial functor of degree n. For an analytic functor F, it is convenient to set $F(D) = F(K^D)$. For example, for $V \in Vect_K$, let $J(V)$ be the free Jordan algebra generated by the vector space V. Then $V \mapsto J(V)$ is an analytic functor, and $J(D)$ is the previously defined free Jordan algebra on D generators.

4.4 Suspensions of analytic functors.

Let $D \geq 0$ be an integer. Let K^D be the space with basis x_1, $x_2 \dots x_D$. To emphasize the choice of x_0 as an additional vector, the vector space with basis x_0, $x_1 \dots x_D$ will be denoted by K^{1+D} and its linear group will be denoted by $GL(1 + D)$.

Lemma 14. *Let M be an analytic $GL(1+D)$-module. Let $\mathbf{m} = (m_0, \dots m_D)$ be a partition of some positive integer such that*

$$M_{\mathbf{m}} \neq 0 \text{ and } m_0 = 0.$$

Then there exists a partition $\mathbf{m'} = (m'_0, \dots m'_D)$ such that

$$M_{\mathbf{m'}} \neq 0 \text{ and } m'_0 = 1.$$

Proof. By hypotheses, there is an index $k \neq 0$ such that $m_k \neq 0$. Let $(e_{i,j})_{0 \leq i,j \leq D}$ the usual basis of $\mathfrak{gl}(1+D)$. Set $f = e_{0,k}$, $e = e_{k,0}$ and $h = [e, f]$. Then $(e,\, f,\, h)$ is a $\mathfrak{sl}_2$-triple in $\mathfrak{gl}(1+D)$. Let $\mathbf{m'}$ be the partition of n defined by $m'_i = m_i$ if $i \neq 0$ or k, $m'_k = m_k - 1$ and $m'_0 = 1$. The eigenvalue of h on $M_{\mathbf{m}}$ is the negative integer $-m_k$, so the map $e : M_{\mathbf{m}} \to M_{\mathbf{m'}}$ is injective, and therefore $M_{\mathbf{m'}}$ is not zero. □

Let F be an analytic functor. In what follows it will be convenient to denote by $K.x_0$ the one-dimensional vector space with basis x_0. Let $V \in Vect_K$. For $z \in K^*$, the element $h(z) \in GL(K.x_0 \oplus V)$ is defined by $h(z).x_0 = z\, x_0$ and $h(z).v = v$ for $v \in V$. There is a decomposition

$$F(K.x_0 \oplus V) = \oplus_n F(K.x_0 \oplus V)|_n$$

where $F(k \oplus V)|_n = \{v \in F(K.x_0 \oplus V) | F(h(z)).v = z^n v\}$. It is easy to see that $F(V) = F(K.x_0 \oplus V)|_0$. By definition, the *suspension* ΣF of F is the functor $V \mapsto F(K.x_0 \oplus V)_1$. A functor F is *constant* if $F(V) = F(0)$ for any $V \in Vect_K$.

Lemma 15. *1. Let F be an analytic functor. If $\Sigma F = \{0\}$, then F is constant.*

2. Let F, G be two analytic functors with $F(0) = G(0) = \{0\}$, and let $\Theta : F \to G$ be a natural transformation. If $\Sigma\Theta$ is an isomorphism, then Θ is an isomorphism.

Proof. 1) Let F be a non-constant analytic functor. Then for some integer D, there is a partition $\mathbf{m} = (m_1, \dots m_D)$ of a positive integer such that $F(D)_{\mathbf{m}} \neq 0$. By lemma 14, there exist a partition $\mathbf{m'} = (m'_0, \dots m'_D)$ with $m'_0 = 1$ such that $F(1+D)_{\mathbf{m'}} \neq 0$. Therefore we have $\Sigma F(D) \neq 0$, what proves the first assertion.

2) By hypothese have $\Sigma \mathrm{Ker}\Theta = \{0\}$ and $\mathrm{Ker}\Theta(0) = \{0\}$ (respectively $\Sigma \mathrm{Coker}\Theta\{0\}$ and $\mathrm{Coker}\Theta(0) = \{0\}$). It follows from the first assertion that $\mathrm{Ker}\Theta = \{0\}$ and $\mathrm{Coker}\Theta = \{0\}$, therefore Θ is an isomorphism.

□

4.5 Vanishing of $H_3(\mathfrak{sl}_2\, J(D))^{\mathfrak{sl}_2}$

Proposition 3. *We have*

$$H_3(\mathfrak{sl}_2\, J(D))^{\mathfrak{sl}_2} = 0.$$

Proof. We have

$$\Sigma\, \Lambda \mathfrak{sl}_2\, J(D) = \Lambda\, \mathfrak{sl}_2\, J(D) \otimes \Sigma \mathfrak{sl}_2\, J(D).$$

It follows that $\Sigma\, \Lambda \mathfrak{sl}_2\, J(D)$ is the complex computing the homology of $\mathfrak{sl}_2\, J(D)$ with value in the $\mathfrak{sl}_2\, J(D)$-module $\Sigma\, \mathfrak{sl}_2\, J(D)$. Taking into account the degree shift, it follows that

$$\Sigma H_3(\mathfrak{sl}_2\, J(D))^{\mathfrak{sl}_2} = H_2(\mathfrak{sl}_2\, J(D), \Sigma\, \mathfrak{sl}_2\, J(D))^{\mathfrak{sl}_2}.$$

Since $\Sigma\, \mathfrak{sl}_2\, J(D)$ belongs to $\mathcal{M}^{gr}_{\mathbf{T}}(\mathfrak{sl}_2\, J(D))$, it follows from Lemma 13 that $\Sigma H_3(\mathfrak{sl}_2\, J(D))^{\mathfrak{sl}_2} = 0$. It follows from Lemma 15 that $H_3(\mathfrak{sl}_2\, J(D))^{\mathfrak{sl}_2} = 0$. $\square$

5. Cyclicity of the Jordan Operads

In this section, we will prove that the Jordan operad $\mathcal{J}$ is cyclic, what will be used in the last Section to simplify Conjecture 2. Also there are compatible cyclic structures on the special Jordan operad $\mathcal{SJ}$ and the Cohn's Jordan operad $\mathcal{CJ}$. As a consequence, the degree D multilinear space of special identies or missing tetrads are acted by $\mathfrak{S}_{D+1}$.

5.1 Cyclic Analytic Functors

An analytic functor F is called *cyclic* if F is the suspension of some analytic functor G. We will now describe a practical way to check that an analytic functor is cyclic. In what follows, we denote by $x_1, \dots x_D$ a basis of K^D and we denote by K^{1+D} the vector space $K.x_0 \oplus K^D$.

Let F, G be two analytic functors and let $\Theta : F \otimes \mathrm{Id} \to G$ be a natural transform, where Id is the identity functor. Note that

$$\Sigma(F \otimes \mathrm{Id})(D) = \Sigma F(D) \otimes K^D \oplus F(D) \otimes x_0.$$

The triple (F, G, Θ) will be called a *cyclic triple* if the induced map

$$\Sigma F(D) \otimes K^D \to \Sigma G(D)$$

is an isomorphism, for any integer $D \geq 0$.

Lemma 16. *Let (F, G, Θ) be a cyclic triple. There is a natural isomorphism*

$$F \simeq \Sigma \mathrm{Ker}\, \Theta.$$

In particular, F is cyclic.

Proof. We have

$$\Sigma(F \otimes \mathrm{Id})(D) = \Sigma F(D) \otimes K^D \oplus F(D) \otimes x_0, \text{ and}$$
$$\Sigma(F \otimes \mathrm{Id})(D) = \Sigma F(D) \otimes K^D \oplus \mathrm{Ker}\Sigma\Theta(D).$$
Therefore $F(D) \simeq F(D) \otimes x_0$ is naturally identified with $\mathrm{Ker}\,\Sigma\Theta(D)$, i.e. the functor F is isomorphic to $\Sigma\mathrm{Ker}\,\Theta$. Therefore F is cyclic. $\square$

5.2 $\mathfrak{S}$-modules

Let $D \geq 1$. For any Young diagram $\mathbf{Y}$ of size D, let $\mathbf{S}(\mathbf{Y})$ be the corresponding simple $\mathfrak{S}_D$-module. Indeed $\mathfrak{S}_D$ is identified with the group of monomial matrices of $GL(D)$, and $\mathbf{S}(\mathbf{Y}) \simeq L(\mathbf{Y}; D)_{\mathbf{1^D}}$. It will be convenient to denote its class in $K_0(\mathfrak{S}_n)$ by $[\mathbf{Y}]$.

By definition a $\mathfrak{S}$-*module* is a vector space $\mathcal{P} = \oplus_{n\geq 0}\,\mathcal{P}(n)$ where the component $\mathcal{P}(n)$ is a finite dimensional $\mathfrak{S}_n$-module. An *operad* is a $\mathfrak{S}$-module $\mathcal{P}$ with some operations, see [5] for a precise definition. Set $K_0(\mathfrak{S}) = \prod_{n\geq 0} K_0(\mathfrak{S}_n)$, see [18]. The class $[\mathcal{E}] \in K_0(\mathfrak{S})$ of a $\mathfrak{S}$-module is defined by $[\mathcal{E}] = \sum_{n\geq 0} [\mathcal{E}(n)]$.

For a $\mathfrak{S}$-module $\mathcal{E}$, the $\mathfrak{S}$-modules $\mathrm{Res}\mathcal{E}$ and $\mathrm{Ind}\mathcal{E}$ are defined by
$$\mathrm{Res}\,\mathcal{E}(n) = \mathrm{Res}_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}}\mathcal{E}(n+1),$$
$$\mathrm{Ind}\,\mathcal{E}(n+1) = \mathrm{Ind}_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}}\mathcal{E}(n)$$
for any $n \geq 0$. The functors Res and Ind gives rise to additive maps on $K_0(\mathfrak{S})$ and they are determined by
$$\mathrm{Res}[\mathbf{Y}] = \textstyle\sum_{\mathbf{Y}'\in \mathrm{Res}\,\mathbf{Y}} [\mathbf{Y}'],$$
$$\mathrm{Ind}[\mathbf{Y}] = \textstyle\sum_{\mathbf{Y}'\in \mathrm{Ind}\,\mathbf{Y}} [\mathbf{Y}']$$
where $\mathrm{Res}\,\mathbf{Y}$ (respectively $\mathrm{Ind}\,\mathbf{Y}$) is the set of all Young diagrams obtained by deleting one box in Y (respectively by adding one box to $\mathbf{Y}$).

A $\mathfrak{S}$-module $\mathcal{E}$ is called *cyclic* if $\mathcal{E} = \mathrm{Res}\mathcal{F}$ for some $\mathfrak{S}$-module $\mathcal{F}$.

5.3 Schur-Weyl duality

The Shur-Weyl duality is an equivalence of the categories between the analytic functors and the $\mathfrak{S}$-modules.

For an analytic functor F, the corresponding $\mathfrak{S}$-module $\mathcal{F} = \oplus_{n\geq 0}\,\mathcal{F}(n)$ is defined by
$$\mathcal{F}(n) = F(n)_{\mathbf{1^n}}.$$
If $F = \Sigma E$ for some analytic functor E, it is clear that
$$F(n)_{\mathbf{1^n}} = E(1+n)_{\mathbf{1^{1+n}}}.$$
Therefore the cyclic analytic functors gives rise to cyclic $\mathfrak{S}$-modules. Conversely, for any $\mathfrak{S}$-module $\mathcal{E}$, the corresponding analytic functor $Sh_{\mathcal{E}}$, which is called a *Shur functor*, is defined by:

$$Sh_{\mathcal{E}}(V) = \oplus_{n\geq 0} H_0(\mathfrak{S}_n, \mathcal{E}(n) \otimes V^{\otimes n})$$

for any $V \in Vect_K$. E.g., $Sh_{\mathbf{S}(\mathbf{Y})}(D) = L(\mathbf{Y}; D)$ for any Young diagram $\mathbf{Y}$.

The class of an analytic functor F is $[F] = \sum_{n\geq 0} [F(n)_{\mathbf{1^n}}] \in K_0(\mathfrak{S})$.

Lemma 17. *Let (F, G, Θ) be a cyclic triple. Then we have*

$$[\operatorname{Ker}\Theta] + [G] = \operatorname{Ind}\circ\operatorname{Res}[\operatorname{Ker}\Theta] = \operatorname{Ind}[F]$$

Proof. It follows from the fact that the Schur-Weyl duality establishes the following correspondences:

Categories	Analytic functors	$\mathfrak{S}$-modules
	Cyclic analytic functors	Cyclic $\mathfrak{S}$-modules
Functor	$\otimes\,\mathrm{Id}$	Ind
Functor	Suspension Σ	Res

□

5.4 A list of analytic functors and $\mathfrak{S}$-modules

We will now provide a list of analytic functors P. For those, the analytic $GL(D)$-module $P(D)$ has been defined, so the definition of the corresponding functor is easy. This section is mostly about notations.

For example, for $V \in Vect_K$, let $T(V)$ be the free non-unital associative algebra over the vector space V. The functor $[T, T]$ is the subfunctor defined by $[T, T,](V) = [T(V), T(V)]$. Similarly, there are functors $J : V \mapsto J(V)$, $SJ : V \mapsto SJ(V)$ and $CJ : V \mapsto CJ(V)$ which provide, respectively, the free Jordan algebras, the free special Jordan algebras and the free Cohn-Jordan algebras.

Concerning the derivations, we will consider the analytic functors $\mathcal{B}J$, $\mathcal{B}SJ$, InnerSJ and InnerCJ. The last two are functors by Lemma 5.

For the missing spaces, we will consider the analytic functors of missing tetrads $M = CJ/SJ$, of missing derivations $MD = \text{Inner}CJ/\text{Inner}SJ$, which is a functor by Lemma 5. Also we will consider the functor of special identities $SI = \text{Ker}J \to SJ$.

Since it is a usual notation, denote by Ass the associative operad. The other $\mathfrak{S}$-modules will be denoted with calligraphic letters. The Jordan operad is denoted by $\mathcal{J}$. As a $\mathfrak{S}$-module, it is defined by $\mathcal{J}(D) = J(D)_{\mathbf{1_D}}$. The special Jordan operad $\mathcal{SJ}$ and the Cohn-Jordan operad $\mathcal{CJ}$ are defined similarly. The $\mathfrak{S}$-modules $\mathcal{M}$, $\mathcal{MD}$ and $\mathcal{SI}$ are the $\mathfrak{S}$-modules corresponding to the analytic functors M, MD and SI.

5.5 The cyclic structure on T and CJ

We will use Lemma 16 to describe the cyclic structure on the tensor algebras. It is more complicated than usual [5], because we are looking at a cyclic structure which is compatible with the free Jordan algebras. The present approach is connected with [19].

The natural map $TV \otimes V \to [TV, TV]$, $u \otimes v \mapsto [u, v]$ for any $V \in Vect_K$ is a natural transformation $\Theta_T : T \otimes \mathrm{Id} \to [T, T]$.

Lemma 18. *The triple $(T, [T, T], \Theta_T)$ is cyclic.*

Proof. Let's begin with a simple observation. Let n be an integer, let $M = \oplus_{0\leq k\leq n} M_k$ be a vector space and let $t : M \to M$ be an automorphism of order $n + 1$ such that $t(M_k) \subset M_{k+1}$ for any $0 \leq k < n$ and $t\, M_n \subset M_0$. Then it is clear that the map

$$\oplus_{0\leq k<n} M_k \to (1 - t)(M), u \mapsto u - t(u)$$

is an isomorphism

To prove the lemma, it is enough to prove that the triple $(T_n, [T, T]_{n+1}, \Theta_T)$ is cyclic for any integer n. Let $V \in Vect_k$ and set $W = k.x_0 \oplus V$. Since we have $[TW, TW] = [TW, W]$, it follows

$$\Sigma\Theta_T(T_n \otimes \mathrm{Id})(V)) = \Sigma[T, T]_{n+1}(V).$$

Once $T_nW \otimes W$ is identified with $W^{\otimes n+1}$, the map

$$\Theta_T : T_nW \otimes W \to T_{n+1}\, W,\, u \otimes w \mapsto [u, w]$$

is identified with the map $1-t$, where t is the automorphism of $W^{\otimes n+1}$ defined by $t(w_0 \otimes w_1 \otimes w_n) = w_n \otimes w_0 \otimes ... \otimes w_{n-1}$. Set $M_k = V^{\otimes k} \otimes x_0 \otimes V^{\otimes n-k}$ for any k. We have

$$\Sigma(T_n \otimes \mathrm{Id})(V) = \oplus_{0\leq k\leq n} M_k, \text{ and } \Sigma\, T_nV \otimes V = \oplus_{0\leq k<n} M_k.$$

Since $t(M_k) \subset M_{k+1}$ for any $0 \leq k < n$ and $t\, M_n \subset M_0$, it follows from the previous observation that Θ_T induces an isomorphism from $\Sigma\, T_nV \otimes V$ to $\Sigma[T, T]_{n+1}(V)$, so the triple $(T, [T, T], \Theta_T)$ is cyclic. □

Let $V \in Vect_K$. It follows from Lemma 5 that $\mathrm{Inner}\, CJV = [CJV, CJV]$. So the natural map $CJV \otimes V \to \mathrm{Inner}\, SV$, $u \otimes v \mapsto [u, v]$ is a natural transformation $\Theta_{CJ} : CJ \otimes \mathrm{Id} \to \mathrm{Inner}\, CJ$.

Lemma 19. *The triple $(CJ, \mathrm{Inner}\, CJ, \Theta_{CJ})$ is cyclic.*

Proof. It is clear that the triple $(CJ, \mathrm{Inner}CJ, \Theta_CJ)$ is a direct summand of the previous one, so it is cyclic. □

5.6 A preliminary result

According to [15], Schreier first proved a statement similar to the next Theorem in the more difficult context of the free group algebras. Next it has been proved by Kurosh [13] and Cohn [3] in the context of the free monoid algebras, or, equivalently for the enveloping algebra of a free Lie algebra.

Schreier-Kurosh-Cohn Theorem. *Let F be a free Lie algebra, and let M be a free module. Then any submodule $N \subset M$ is free.*

Let $D \geq 1$ be an integer and let F be the free Lie algebra generated by $F_1 := \mathfrak{sl}_2 \otimes K^D$, i.e. F is a free Lie algebra on $3D$ generators on which $PSL(2)$ acts by automorphism. Let $\mathcal{M}(F, PSL(2))$ be the category of $PSL(2)$-equivariant F-modules. The F-action on a module $M \in \mathcal{M}(F, PSL(2))$ is a $PSL(2)$-equivariant map $M \otimes F \to M, m \otimes g \mapsto g.m$. It induces the map

$$\mu_M : H_0(\mathfrak{sl}_2, M^{ad} \otimes F_1) \to H_0(\mathfrak{sl}_2, M).$$

Recall that $X = X^{\mathfrak{sl}_2} \oplus \mathfrak{sl}_2.X$ for any $PSL(2)$-module X.

Lemma 20. *Let $0 \to Y \to X \to M \to 0$ be a short exact sequence in $\mathcal{M}(F, PSL(2))$. Assume that*

1. the F-module X is free and generated by $\mathfrak{sl}_2.X$,

2. Y is generated by $\mathfrak{sl}_2.Y$.

Then the map the map μ_M is an isomorphism.

Proof. Since X is free, the action $X \otimes F_1 \to F.X, m \otimes g \mapsto g.m$ is an isomorphism, therefore the map

$$H_0(\mathfrak{sl}_2, X \otimes F_1) \to H_0(\mathfrak{sl}_2, F.X)$$

is an isomorphism. Since F_1 is of adjoint type, we have $H_0(\mathfrak{sl}_2, X \otimes F_1) = H_0(\mathfrak{sl}_2, X^{ad} \otimes F_1)$. As X is generated by $\mathfrak{sl}_2.X$ we have $H_0(\mathfrak{sl}_2, X/F.X) = 0$, so we have $H_0(\mathfrak{sl}_2, F.X) = H_0(\mathfrak{sl}_2, X)$. It follows that μ_X is an isomorphism.

By Schreier-Kurosh-Cohn Theorem, Y is also free, and therefore μ_Y is also an isomorphism. By the snake lemma, it follows that μ_M is an isomorphism. □

Similarly, for $M \in \mathcal{M}_{\mathbf{T}}(\mathfrak{sl}_2\, J(D))$, the action induces a map

$$\mu_M : H_0(\mathfrak{sl}_2, M^{ad} \otimes (\mathfrak{sl}_2 \otimes J_1(D))) \to H_0(\mathfrak{sl}_2, M).$$

Lemma 21. *Let M be the free $\mathfrak{sl}_2\, J(D)$-module in category $\mathcal{M}_{\mathbf{T}}(\mathfrak{sl}_2\, J(D))$ generated by one copy of the adjoint module $L(2)$. Then the map μ_M is an isomorphism.*

Proof. Let F be the free Lie algebra of the previous lemma. Any $PSL(2)$-equivariant isomorphism $\phi : F_1 \to \mathfrak{sl}_2 \otimes J_1(D)$ gives rise to a Lie algebra morphism $\psi : F \to \mathfrak{sl}_2\, J(D)$, so M can be viewed as a PSL(2)-equivariant F-module.

Let $X \in \mathcal{M}(F, PLS(2))$ be the free F-module generated by $L(2)$ and let P be the free F-module in category $\mathcal{M}_{\mathbf{T}}(F, PLS(2))$ generated by $L(2)$. There are natural surjective maps of F-modules

$$X \xrightarrow{\pi} P \xrightarrow{\sigma} M.$$

It is clear that $\operatorname{Ker}\pi$ is the F-submodule of X generated by its $L(4)$-component. Let K be the $L(4)$-component of F. It is clear that $\mathfrak{sl}_2\, J(D) = F/R$ where R is the ideal of F generated by K. Therefore $\operatorname{Ker}\sigma$ is the F-submodule of P generated by $K.P$. Since P is in $\mathbf{T}$, we have $K.P \subset P^{ad}$, therefore $\operatorname{Ker}\sigma$ is generated by its adjoint component.

Set $Y = \operatorname{Ker}\sigma \circ \pi$. It follows from the descriptions of $\operatorname{Ker}\pi$ and $\operatorname{Ker}\sigma$ that Y is generated by its $L(2)$ and its $L(4)$ components. Thus the short exact sequence

$$0 \to Y \to X \to M \to 0$$

satisfies the hypotheses of Lemma 20. It follows that μ_M is an isomorphism. □

5.7 Cyclic structures on J and SJ

The natural map $J(V) \otimes V \to \mathcal{B}J(V), a \otimes v \mapsto \{a, v\}$, defined for all $V \in Vect_K$ is indeed a natural transformation $\Theta_J : J \otimes \mathrm{Id} \to \mathcal{B}J$.

Lemma 22. *The triple $(J, \mathcal{B}J, \Theta_J)$ is cyclic.*

Proof. Let $D \geq 0$ be an integer. Let M be the free $\mathfrak{sl}_2\, J(D)$-module in category $\mathcal{M}_{\mathbf{T}}(\mathfrak{sl}_2\, J(D))$ generated by one copy L of the adjoint module. Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{sl}_2\, J(D) \ltimes M$. Let ϕ be a $PSL(2)$-equivariant map $\phi : J(D) \otimes K^{1+D} \to \mathfrak{g}$ defined by the requirement that ϕ is the identity on $J(D) \otimes K^D$ and $\phi|_{J(D)\otimes x_0}$ is an isomorphism to L.

By Lemma 11, $\mathfrak{sl}_2\, J(D)$ is free in the category $\mathbf{Lie}_{\mathbf{T}}$. Therefore ϕ extends to a Lie algebra morphism $\Phi : \mathfrak{sl}_2\, J(D) \to G$. Note that Φ sends $\Sigma J(D)$ to M. Since $\Sigma J(D)$ a the $\mathfrak{sl}_2\, J(D)$-module generated by $J(D) \otimes x_0$, it follows that

$$\Sigma \mathfrak{sl}_2\, J(D) \simeq M \text{ as a } \mathfrak{sl}_2\, J(D)\text{-module.}$$

By Lemma 21, μ_M is an isomorphism, which amounts to the fact that

$$\Sigma J(J) \otimes K^D \to \Sigma\mathcal{B}J(D),\ a \otimes v \mapsto \{a, v\}$$

is an isomorphism. Therefore the triple $(J, \mathcal{B}(J), \Theta_J)$ is cyclic.

□

The natural transformation Θ_J induces a natural transformation Θ_{SJ} : $SJ \otimes \mathrm{Id} \to \mathrm{Inner}\, SJ$. Similarly, we have

Lemma 23. *The triple $(SJ, \mathrm{Inner}\, SJ, \Theta_{SJ})$ is cyclic.*

Moreover the natural map $\Sigma\mathcal{B}\, SJ(D) \to \Sigma\, \mathrm{Inner}\, SJ(D)$ is an isomorphism for all D.

Proof. For any D, there is a commutative diagram

$$
\begin{array}{ccccc}
\Sigma J(D) \otimes K^D & \overset{a}{\twoheadrightarrow} & \Sigma SJ(D) \otimes K^D & \overset{b}{\hookrightarrow} & \Sigma CJ(D) \otimes K^D \\
\alpha' \downarrow & \alpha \swarrow & & \searrow \beta & \downarrow \beta' \\
\Sigma\mathcal{B}SJ(D) & \twoheadrightarrow \Sigma\mathcal{B}SJ(D) & \twoheadrightarrow & \Sigma\mathrm{Inner}\, SJ(D) & \hookrightarrow \Sigma\mathrm{Inner}\, CJ(D)
\end{array}
$$

In the diagram, the horizontal arrows with two heads are obviously surjective maps, and those with a hook are obviously injective maps. By Lemma 22 the map α' is onto and by Lemma 19 the map β' is one-to-one. By diagram chasing, α and β are isomorphisms. Both assertions follow.

□

5.8 Cyclicity Theorem

There is a commutative diagram of natural transformations:

$$
\begin{array}{ccccccc}
J \otimes \mathrm{Id} & \to & SJ \otimes \mathrm{Id} & \to & S \otimes \mathrm{Id} & \to & T \otimes \mathrm{Id} \\
\downarrow \Theta_J & & \downarrow \Theta_{SJ} & & \downarrow \Theta_S & & \downarrow \Theta_T \\
\mathcal{B}J & \to & \mathrm{Inner} SJ & \to & \mathrm{Inner} S & \to & [T, T]
\end{array}
$$

Theorem 1. *The four triples $(J, \mathcal{B}J, \mathrm{Inner}_{SJ}, \Theta_J)$, $(SJ, \mathrm{Inner}_{SJ}, \Theta_{SJ})$, $(CJ, \mathrm{Inner}_{CJ}\Theta_{CJ})$ and $(T, [T,T], \ \Theta_T)$ are cyclic. Moreover the operads $\mathcal{J}$, $\mathcal{SJ}$, $\mathcal{CJ}$ and $\mathcal{T}$ are cyclic.*

Proof. The first Assertions follows from Lemmas 18, 19, 22 and 23. It follows that the $\mathfrak{S}$-modules $\mathcal{J}$, $\mathcal{SJ}$, $\mathcal{S}$ and $\mathcal{T}$ are cyclic. For an operad, the definition of cyclicity requires an additional compatibility condition for the action of the cycle, see [5]. Since this fact will be of no use here, the proof will be skipped. It is, indeed, formally the same as the proof for the associative operad, see [5].

□

5.9 Consequences for the free special Jordan algebras

Corollary 2. *We have* $\mathcal{B}SJ(D) = \operatorname{Inner} SJ(D)$ *for any* D.

Proof. Lemma 23 shows that the natural map $\Sigma\mathcal{B}SJ = \Sigma\operatorname{Inner} SJ$ is an isomorphism. Thus the corollary follows from Lemma 15. □

For any D, set $\mathcal{M}(D) = M(D)_{\mathbf{1^D}}$.

Corollary 3. *The space* $\mathcal{M}(D)$ *of multilinear missing tetrads is a* $\mathfrak{S}_{D+1}$ *-module*

Proof. By Theorem 1, $\mathcal{SJ}$ and $\mathcal{S}$ are compatibly cyclic. Therefore $\mathcal{M}(D)$ is a $\mathfrak{S}_{D+1}$-module □

For a Young diagram $\mathbf{Y}$, denote by $c_i(\mathbf{Y})$ the height of the i^{th} column.

Lemma 24. *Let* $\mathbf{Y}$ *be a Young diagram of size* $D+1$. *Assume that* $\mathbf{S}(\mathbf{Y})$ *occurs in the* $\mathfrak{S}_{D+1}$*-module* $\mathcal{M}(D)$.

1. *We have* $c_1(\mathbf{Y}) \geq 5$ *or* $c_1(\mathbf{Y}) = c_2(\mathbf{Y}) = 4$.
2. *If moreover* $D = 2$ *or* 3 *modulo* 4, *then we have* $c_1(\mathbf{Y}) \leq D - 1$.

Proof. Recall that

$$\mathbf{S}(\mathbf{Y})|_{\mathfrak{S}_D} = \oplus_{\mathbf{Y}' \in \operatorname{Res}\mathbf{Y}} \mathbf{S}(\mathbf{Y}').$$

Since $M(3) = 0$ by Cohn's reversible Theorem, $\operatorname{Res}\mathbf{Y}$ contains no Young diagram of heigth < 4. So it is proved that $c_1(Y) \geq 4$. Moreover if $c_1(\mathbf{Y}) = 4$, removing the bottom box on the first colum does not give rise to a Young diagram, what forces that $c_2(\mathbf{Y}) = 4$. Assertion 1 is proved.

Note that the signature representation of $\mathfrak{S}_D$ occurs with multiplicity one in $\mathcal{T}(D)$. So if $D = 2$ or $D = 3$ modulo 4, this representation occurs in the multilinear part of $A(D)$, so it does not occur in $\mathcal{M}(D)$. It follows easily that $c_1(\mathbf{Y}) \leq D - 1$. □

The Jordan multiplication induces the maps $L : CJ_1(D) \otimes M_n(D) \to M_{n+1}(D)$. On the multilinear part, it provides a natural map:

$$L_D : \operatorname{Ind}_{\mathfrak{S}_D}^{\mathfrak{S}_{D+1}} \mathcal{M}(D) \to \mathcal{M}(D+1).$$

Lemma 25. *For* D *even, the map* L_D *is onto.*

Proof. In the course of the proof of Cohn's Reversible Theorem [16], it appears that $CJ_1(D).CJ_n(D) = CJ_{n+1}(D)$ when n is even. Therefore the map L_D is onto for D even. □

Corollary 4. *1. As a $\mathfrak{S}_5$-module, we have $\mathcal{M}(4) = \mathbf{S}(\mathbf{1^5})$.*

2. As a $\mathfrak{S}_6$-module, we have $\mathcal{M}(5) = \mathbf{S}(\mathbf{2}, \mathbf{1^4})$.

3. As a $\mathfrak{S}_7$-module, we have $\mathcal{M}(6) = \mathbf{S}(\mathbf{3}, \mathbf{1^4})^2$

4. As a $\mathfrak{S}_8$-module, we have

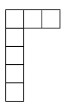
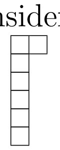
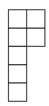
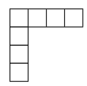
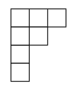
$$\mathcal{M}(7) = \mathbf{S}(\mathbf{4}, \mathbf{1^4})^2 \oplus \mathbf{S}(\mathbf{3}, \mathbf{2}, \mathbf{1^3}) \oplus \mathbf{S}(\mathbf{2^2}, \mathbf{1^4}) \oplus \mathbf{S}(\mathbf{3}, \mathbf{1^5}).$$

Proof. The cases $D = 4$ or $D = 5$ are easy and the proof for those cases is skipped. We have $\dim \mathcal{M}(D) = D!/2 - \dim \mathcal{SJ}(D)$ for any $D \geq 1$. In [6], it is proved that $\dim \mathcal{SJ}(6) = 330$ and $\dim \mathcal{SJ}(7) = 2345$. Therefore we have

$$\dim \mathcal{M}(6) = 30 \text{ and } \dim \mathcal{M}(7) = 175.$$

Let's consider the case $D = 6$. The two Young diagrams of size 7 and heigth 5 are $\mathbf{Y_1} = (\mathbf{3}, \mathbf{1^4})$ and $\mathbf{Y_2} = (\mathbf{2^2}, \mathbf{1^3})$. By Lemma 24, $\mathbf{S}(\mathbf{Y_1})$ and $\mathbf{S}(\mathbf{Y_2})$ are the only possible simple submodules of the $\mathfrak{S}_7$-module $\mathcal{M}(6)$. We have dim $\mathbf{S}(\mathbf{Y_1}) = 15$ and dim $\mathbf{S}(\mathbf{Y_2}) = 14$. Since dim $\mathcal{M}(6) = 30$, we have $\mathcal{M}(6) \simeq \mathbf{S}(\mathbf{3}, \mathbf{1^4})^2$.

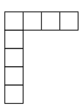
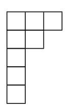
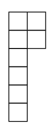
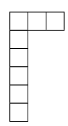
For $D = 7$, let's consider the following Young diagrams of size 7

$\mathbf{K_1} =$  $\mathbf{K_2} =$  $\mathbf{K_3} =$  $\mathbf{K_4} =$  $\mathbf{K_5} =$ 

We have $\mathrm{Ind}\,\mathrm{Res}\,\mathbf{S}(\mathbf{3}, \mathbf{1^4}) = \mathbf{S}(\mathbf{K_1})^2 \oplus \mathbf{S}(\mathbf{K_2}) \oplus \mathbf{S}(\mathbf{K_3}) \oplus \mathbf{S}(\mathbf{K_4}) \oplus \mathbf{S}(\mathbf{K_5})$. It follows from Lemma 25 that

$$\mathcal{M}(7) = \oplus_{1 \leq i \leq 5} \mathbf{S}(\mathbf{K_i})^{k_i}$$

where $k_1 \leq 4$ and $k_i \leq 2$ for $2 \leq i \leq 5$. The list of Young diagrams Y such that $\mathrm{Res}\, Y \subset \{K_1, K_2, K_3, K_4, K_5\}$ is

$\mathbf{Y_1} =$  $\mathbf{Y_2} =$  $\mathbf{Y_3} =$  $\mathbf{Y_4} =$ 

If follows that the $\mathfrak{S}_8$-module $\mathcal{M}(7)$ can be decomposed as

$$\mathcal{M}(7) = \oplus_{1 \leq i \leq 4} \mathbf{S}(\mathbf{Y_i})^{m_i}.$$

Since $\dim \mathcal{M}(7) = 175$ while $\dim \mathbf{S}(\mathbf{Y_1}) = 35$, $\dim \mathbf{S}(\mathbf{Y_2}) = 64$, $\dim \mathbf{S}(\mathbf{Y_3}) = 20$, and $\dim \mathbf{S}(\mathbf{Y_4}) = 21$, it follows that

$$35m_1 + 64m_2 + 20m_3 + 21.m_4 = 175.$$

The inequality $k_i \leq 2$ for $i \geq 2$ adds the constraint $m_i \leq 2$ for any i. Thus the only possibility is $m_1 = 2$, $m_2 = 1$, $m_3 = 1$, $m_4 = 1$, and therefore

$$\mathcal{M}(7) = \mathbf{S}(\mathbf{Y_1})^2 \oplus \mathbf{S}(\mathbf{Y_2}) \oplus \mathbf{S}(\mathbf{Y_3}) \oplus \mathbf{S}(\mathbf{Y_4})$$

.

$\square$

Corollary 5. *We have* $\mathcal{MD}(D) = 0$ *for* $D \leq 4$, *and*

$\mathcal{MD}(5) = \mathbf{S}(\mathbf{2}, \mathbf{1^3})$,

$\mathcal{MD}(6) = \mathbf{S}(\mathbf{1^6}) \oplus \mathbf{S}(\mathbf{2}, \mathbf{1^4}) \oplus \mathbf{S}(\mathbf{3}, \mathbf{1^3}) \oplus \mathbf{S}(\mathbf{2^2}, \mathbf{1^2})$

$\mathcal{MD}(7) = [\mathbf{S}(\mathbf{2}, \mathbf{1^5}) \oplus \mathbf{S}(\mathbf{2^2}, \mathbf{1^3}) \oplus \mathbf{S}(\mathbf{3}, \mathbf{1^4}) \oplus \mathbf{S}(\mathbf{3}, \mathbf{2}, \mathbf{1^2}) \oplus \mathbf{S}(\mathbf{4}, \mathbf{1^3})]^2$, *and*

$$[\mathcal{MD}(8)] = 4\,[\mathbf{4}, \mathbf{1^4}] + 6\,[\mathbf{3}, \mathbf{2}, \mathbf{1^3}] + [\mathbf{2^2}, \mathbf{4}] + 5\,[\mathbf{3}, \mathbf{1^5}] + 2\,[\mathbf{2}, \mathbf{1^6}]$$
$$+2\,[\mathbf{2}, \mathbf{1^6}] + 2\,[\mathbf{2^3}, \mathbf{1^2}] + [\mathbf{3^2}, \mathbf{1^2}] + 3\,[\mathbf{4}, \mathbf{2}, \mathbf{1^2}] + 2\,[\mathbf{5}, \mathbf{1^3}],$$

where $[\mathbf{Y}]$ *stands for the class of* $\mathbf{S}(\mathbf{Y})$, *for any Young diagram* $\mathbf{Y}$.

Proof. The natural transformation $\Theta_{CJ} : CJ \otimes \mathrm{Id} \to \mathrm{Inner}\, CJ$ gives rise to a natual transformation $\Theta_M : M \otimes \mathrm{Id} \to MD$. By Lemmas 23 and 19, the triple $(M \otimes \mathrm{Id}, MD, \Theta_M)$ is cyclic. Therefore the following equality

$$[\mathcal{MD}(D+1)] = [\mathrm{Ind} \circ \mathrm{Res}\, \mathcal{M}(D)] - [\mathcal{M}(D)]$$

holds in $K_0(\mathfrak{S}_{D+1})$ by Lemma 17. Since Corollary 4 provides the character of the $\mathfrak{S}_{D+1}$-module $\mathcal{M}(\mathcal{D})$ for $D \leq 7$, it is possible to compute the character of $\mathcal{MD}(D)$ for any $D \leq 8$. The other case being simpler, some details will be provided for $\mathcal{MD}(8)$.

Let's consider the notations of Corollary 4. We have

$$[\mathcal{M}(7)] = 2\,[\mathbf{Y_1}] + [\mathbf{Y_2}] + [\mathbf{Y_3}] + [\mathbf{Y_4}].$$

It follows that

$$\mathrm{Res}[\mathcal{M}(7)] = 4\,[\mathbf{K_1}] + 2\,[\mathbf{K_2}] + 2\,[\mathbf{K_3}] + 2\,[\mathbf{K_4}] + [\mathbf{K_5}], \text{ and}$$
$$\mathrm{Ind} \circ \mathrm{Res}[\mathcal{M}(7)] = 6\,[\mathbf{Y_1}] + 7\,[\mathbf{Y_2}] + 2\,[\mathbf{Y_3}] + 6\,[\mathbf{Y_4}] + 2\,[\mathbf{2}, \mathbf{1^6}]$$
$$+2\,[\mathbf{2}, \mathbf{1^6}] + 2\,[\mathbf{2^3}, \mathbf{1^2}] + [\mathbf{3^2}, \mathbf{1^2}] + 3\,[\mathbf{4}, \mathbf{2}, \mathbf{1^2}] + 2\,[\mathbf{5}, \mathbf{1^3}]$$

from which the formula follows.

□

5.10 Consequence for the free Jordan algebras

Corollary 6. *We have* $\mathcal{B}_k(J(D)) = \mathrm{Inner}_k\, J(D) = \mathrm{Inner}_k\, SJ(D)$ *for any* $k \leq 8$ *and any* D.

Proof. By Theorem 1, we have

$$\Sigma\mathcal{B}_k(J(D)) \simeq \Sigma J_{k-1}(D) \otimes K^D, \text{ and } \Sigma\mathcal{B}_k(SJ(D)) \simeq \Sigma SJ_{k-1}(D) \otimes K^D.$$

By Glennie Theorem, $J_{k-1}(D)$ and $SJ_{k-1}(D)$ are isomorphic for $k \leq 8$. Therefore we have

$$\Sigma\mathcal{B}_k(J(D)) \simeq \Sigma\mathcal{B}_k(SJ(D))$$

for any $k \leq 8$ and any D. By Lemma 15, it follows that $\mathcal{B}_k(J(D)) \simeq \mathcal{B}_k(SJ(D))$ whenever $k \leq 8$.

Let's consider the commutative diagram

$$\begin{array}{ccc}
\mathcal{B}_k(J(D)) & \xrightarrow{\alpha} & \mathcal{B}_k(SJ(D)) \\
\downarrow a & & \downarrow b \\
\mathrm{Inner}_k\, J(D) & \xrightarrow{\beta} & \mathrm{Inner}_k\, SJ(D)
\end{array}$$

Observe that all maps are onto. By Corollary 2, b is an isomorphism, while it has been proved that α is an isomorphism for $k \leq 8$. Therefore, the maps a and α are also isomorphism, what proves Corollary 6.

□

Corollary 7. *The space $\mathcal{SI}(D)$ of special identities is a $\mathfrak{S}_{D+1}$-module.*

Proof. By Theorem 1, $\mathcal{J}$ and $\mathcal{SJ}$ are cyclic and the map $\mathcal{J}(D) \to \mathcal{SJ}(D)$ is $\mathfrak{S}_{D+1}$-equivariant. Therefore $\mathcal{SI}(D)$ is a $\mathfrak{S}_{D+1}$-module. □

For example, let G be the multilinear part of the Glennie Identity. As an element of $\mathcal{SI}(8)$, it generates a simple $\mathfrak{S}_8$ module $M \simeq \mathbf{S}(\mathbf{3^2})$. What is the $\mathfrak{S}_9$-module $\hat{M}$ generated by G in $\mathcal{SI}(8)$? It is clear that there are only two possibilities

A) $M \simeq \mathbf{S}(\mathbf{3^3})$. In such a case, $\hat{M} = M$.

B) $\hat{M} \simeq \mathbf{S}(\mathbf{3^2}, \mathbf{2}, \mathbf{1})$.

If so, $\mathrm{Res}\, \hat{M} \simeq \mathbf{S}(\mathbf{3^2}, \mathbf{2}) \oplus \mathbf{S}(\mathbf{3}, \mathbf{2^2}, \mathbf{1}) \oplus \mathbf{S}(\mathbf{3^2}, \mathbf{1^2})$. This would provide two independent new special identities in $J(4)$. When computing the simplest of these two identities, we found a massive expression. Unfortunately, it was impossible to decide if this special identity is zero or not.

6. The Conjecture 3

Conjecture 2 is quite natural. However, the vanishing of $H_*(\mathfrak{sl}_2\, J(D))^{ad}$ does not look very tractable. Conjecture 3 is a weaker and better version. As a consequence of Theorem 1, it will be proved that it is nevertheless enough to deduce Conjecture 1.

Conjecture 3. *We have $H_k(\mathfrak{sl}_2\, J(D))^{\mathfrak{sl}_2} = 0$ for any $k \geq 1$.*

Note that Conjecture 3 holds for $k = 1, 2$ and 3, as it was proved in section 4.

Theorem 2. *If Conjecture 3 holds for $\mathfrak{sl}_2\, J(1 + D)$, then Conjecture 1 holds for $\mathfrak{sl}_2\, J(D)$.*

Proof. The proof is similar to the proof of Corollary 1. Assume Conjecture 3 holds for $\mathfrak{sl}_2\, J(1+D)$.

Since $H_*(\mathfrak{sl}_2\, J(D))$ is a summand in $H_*(\mathfrak{sl}_2\, J(1+D)))$, it follows that $H_k(\mathfrak{sl}_2\, J(D))^{\mathfrak{sl}_2} = 0$ for any $k \geq 1$. By Lemma 12, this implies that $\mathcal{B}J(D) = \mathrm{Inner}J(D)$. As in the proof of Corollary 1, we get that

$$(\mathcal{E}_1) \qquad [\lambda[\mathfrak{sl}_2\, J(D)] : L(0)] = 1$$

where $[\mathfrak{sl}_2\, J(D)]$ denotes the class of $\mathfrak{sl}_2\, J(D)$ in $\mathcal{M}_{an}(GL(D) \times PSL(2))$.

Similarly $\Sigma H_*(\mathfrak{sl}_2\, J(D))$ it is a component of $H_*(\mathfrak{sl}_2\, J(1+D))$, and therefore $\Sigma H_*(\mathfrak{sl}_2\, J(D))^{\mathfrak{sl}_2}$ vanishes. The complex computing $\Sigma H_*(\mathfrak{sl}_2\, J(D))$ is $\Lambda\, \mathfrak{sl}_2\, J(D) \otimes \Sigma \mathfrak{sl}_2\, J(D)$. It follows that

$$[(\lambda[\mathfrak{sl}_2\, J(D)].[\Sigma\, \mathfrak{sl}_2\, J(D)]) : L(0)] = 0.$$

Using that $[\Sigma\, \mathfrak{sl}_2\, J(D)] = [\Sigma BJ(D)] + [J(D)].[L(2)]$, the previous equation can be rewritten as:

$$[\Lambda[\mathfrak{sl}_2\, J(D)] : L(0)][\Sigma BJ(D)] + [\lambda[\mathfrak{sl}_2\, J(D)] : L(2)][\Sigma J(D)] = 0$$

It has been proved that $[\lambda[\mathfrak{sl}_2\, J(D)] : L(0)] = 1$. Moreover by Theorem 1, we have $[\Sigma BJ(D)] = [K^D][\Sigma J(D)]$. Therefore the previous equation can be simplified into

$$([K^D] + [\lambda[\mathfrak{sl}_2\, J(D)] : L(2)]).[\Sigma J(D)] = 0.$$

The ring $\mathcal{R}_{an}(GL(D))$ is ring of formal series, see [18] or Section 1.4, and it has no zero divisors. It follows that

$$(\mathcal{E}_2) \qquad [\lambda[\mathfrak{sl}_2\, J(D)] : L(2)] = -[K^D].$$

Using that $\mathcal{B}J(D) = \mathrm{Inner}J(D)$, Equations $\mathcal{E}_1$ and $\mathcal{E}_2$ implies that:

$$\lambda([J(D)][L(2)] + [\mathrm{Inner}J(D)]) : [L(0)] = 1, \text{ and}$$
$$\lambda([J(D)][L(2)] + [\mathrm{Inner}J(D)]) : [L(2)] = -[K^D].$$

So by Lemma 1, Conjecture 3 implies Conjecture 1.

□

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