Linearity and Nonlinearity of Groups of Polynomial Automorphisms of the Plane

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Abstract

Given a field K, we investigate which subgroups of the group $\mathrm{Aut}\,\mathbb{A}^2_K$ of polynomial automorphisms of the plane are linear or not.

The results are contrasted. The group $\mathrm{Aut}\,\mathbb{A}^2_K$ itself is nonlinear, except if K is finite, but it contains some large subgroups, of "codimension-five" or more, which are linear. This phenomenon is specific to dimension two: it is easy to prove that any natural "finite-codimensional" subgroup of $\mathrm{Aut}\,\mathbb{A}^3_K$ is nonlinear, even for a finite field K.

When $\mathrm{ch}\ K = 0$, we also look at a similar questions for f.g. subgroups, and the results are again disparate. The group $\mathrm{Aut}\,\mathbb{A}^2_K$ has a one-related f.g. subgroup which is not linear. However, there is a large subgroup, of "codimension-three", which is locally linear but not linear.

This paper is respectfully dedicated to the memory of Jacques Tits.

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Introduction

Let K be a field given once and for all, and let $\mathrm{Aut}\,\mathbb{A}^2_K$ be the group of polynomial automorphisms of the affine plane $\mathbb{A}^2_K$ over K.

0.1 General Introduction
A group Γ is called *linear over a ring*, respectively *linear over a field*, if there is an embedding $\Gamma \subset \mathrm{GL}(n, R)$, resp. $\Gamma \subset \mathrm{GL}(n, L)$, for some positive integer n and some commutative ring R, resp. some field L.

Various authors have shown that the automorphism groups of algebraic varieties share many properties with the linear groups, e.g. the Tits alternative holds in $\mathrm{Aut}\,\mathbb{A}^2_{\mathbb{C}}$ [16], see also [2][23][3]. However, these groups are not always linear [13][21]. In this paper, we will investigate the following related

Question: which subgroups of $\mathrm{Aut}\,\mathbb{A}^2_K$ *are indeed linear or not?*

Answer: roughly speaking, $\mathrm{Aut}\,\mathbb{A}^2_K$ contains large linear subgroups and small ones which are not.

In order to be more specific, let us consider the following subgroups

$$\mathrm{Aut}_0\,\mathbb{A}^2_K = \{\phi \in \mathrm{Aut}\,\mathbb{A}^2_K \mid \phi(\mathbf{0}) = \mathbf{0})\},$$
$$\mathrm{SAut}_0\,\mathbb{A}^2_K = \{\phi \in \mathrm{Aut}_0\,\mathbb{A}^2_K \mid \mathrm{Jac}(\phi) = 1\}, \text{ and}$$
$$\mathrm{Aut}_S\,\mathbb{A}^2_K = \{\phi \in \mathrm{Aut}_0\,\mathbb{A}^2_K \mid \mathrm{d}\phi|_{\mathbf{0}} \in S\},$$

where $\mathrm{Jac}(\phi) := \det \mathrm{d}\phi$ is the jacobian of ϕ and S is a subgroup of $\mathrm{GL}(2, K)$. Since $\mathrm{Aut}\,\mathbb{A}^2_K/\mathrm{Aut}_0\,\mathbb{A}^2_K$ is naturally isomorphic to $\mathbb{A}^2_K$, *informally speaking* $\mathrm{Aut}_0\,\mathbb{A}^2_K$ is a subgroup of codimension two. Similarly, since $\mathrm{Jac}(\phi)$ is a constant polynomial, $\mathrm{SAut}_0\,\mathbb{A}^2_K$ has codimension three. Let $U(K)$ be the group of linear transformations $(x, y) \mapsto (x, y + ax)$ for some $a \in K$. Similarly, the group $\operatorname{Aut}_{U(K)} \mathbb{A}^2_K$ has codimension five, and the group $\operatorname{Aut}_1 \mathbb{A}^2_K := \operatorname{Aut}_{\{1\}} \mathbb{A}^2_K$ has codimension six. Anyhow, they are viewed as large subgroups.

It was known that the Cremona group $Cr_2(\mathbb{C})$ and $\operatorname{Aut} \mathbb{A}^2_{\mathbb{C}}$ are not linear over a field, see [8][9]. For the large subgroups of $\operatorname{Aut} \mathbb{A}^2_K$, the linearity results are much more contrasted, as it is shown by the following

Theorem A. *(A.1) Whenever K is infinite, the group $\operatorname{SAut}_0 \mathbb{A}^2_K$ is not linear, even over a ring.*

(A.2) However, there is an embedding $\operatorname{Aut}_{U(K)} \mathbb{A}^2_K \subset \mathrm{SL}(2, K(t))$.

For a finite field K, the index $[\operatorname{Aut} \mathbb{A}^2_K : \operatorname{Aut}_{U(K)} \mathbb{A}^2_K]$ is finite. Therefore Theorem A.1 admits the following converse

Corollary. *If K is finite, the group $\operatorname{Aut} \mathbb{A}^2_K$ is linear over the field $K(t)$.*

The existence of large linear subgroups in $\operatorname{Aut} \mathbb{A}^n_K$ is specific to the dimension 2. The case $n = 3$ is enough to show this. For a finite-codimensional ideal $\mathbf{m}$ of $K[x, y, z]$, let $\operatorname{Aut}_{\mathbf{m}} \mathbb{A}^3_K$ be the group of all automorphisms

$$(x, y, z) \mapsto (x + f, y + g, z + h),$$

where f, g and h belong to $\mathbf{m}$. Equivalently, ϕ fixes some infinitesimal neighborhood of a finite subset in $\mathbb{A}^3_K$. E.g. for $\mathbf{n} = (x, y, z)^2$, we have

$$\operatorname{Aut}_{\mathbf{n}} \mathbb{A}^3_K = \{\phi \in \operatorname{Aut} \mathbb{A}^3_K \mid \phi(\mathbf{0}) = \mathbf{0} \text{ and } \mathrm{d}\phi|_{\mathbf{0}} = \mathrm{id}\}.$$

However the groups $\operatorname{Aut}_{\mathbf{m}} \mathbb{A}^3_K$ are not linear, even if K is finite, as shown by

Theorem B. *The group $\operatorname{Aut}_{\mathbf{m}} \mathbb{A}^3_K$ is not linear, even over a ring.*

We will now turn our attention to the small subgroups, namely the finitely generated (f.g. in the sequel) subgroups of $\operatorname{Aut} \mathbb{A}^2_K$. Let $\Gamma \subset \operatorname{Aut} \mathbb{A}^2_{\mathbb{Q}}$ be

$$\Gamma = \langle S, T \rangle, \text{ where } S(x, y) = (y, 2x) \text{ and } T(x, y) = (x, y + x^2).$$

In [9], Y. Cornulier asked about the existence of nonlinear f.g. subgroups in $\operatorname{Aut} \mathbb{A}^2_{\mathbb{C}}$. An answer is provided by the Assertion C.1 of the next

Theorem C. *Let K be a field of characteristic zero.*

(C.1) The subgroup $\Gamma \subset \operatorname{Aut}_0 \mathbb{A}^2_K$ is not linear, even over a ring.

(C.2) Any f.g. subgroup of $\operatorname{SAut}_0 \mathbb{A}^2_K$ is linear over $K(t)$.

It turns out that the group Γ, which is presented by

$$\langle \sigma, \tau \mid \sigma^2 \tau \sigma^{-2} = \tau^2 \rangle,$$

appears in [11] as the first example of a one-related group which is residually finite but not linear over a field.

By Theorem C.1, Γ is also the first example of a 1-related group which is not linear (even over a ring) but which is embeddable in the automorphism group of an algebraic variety. Residual finiteness of Γ follows from general principles [2], but the observation that Γ acts on the finite sets $\mathbb{F}_p^2$, for any odd p, and faithfully on their product, provides a concrete proof.

For an infinite field K, Theorem A.2 suggests to ask which groups $G \supset \mathrm{Aut}_1\,\mathbb{A}_K^2$ are linear or not. Indeed, either this group contains

$$\mathrm{SAut}\mathbb{A}_K^2 := \{\phi \in \mathrm{Aut}_0\mathbb{A}_K^2 \mid \mathrm{Jac}(\phi) = 1\},$$

which is not linear, or it is ismorphic to $\mathrm{Aut}_S\mathbb{A}_K^2$, for some subgroup S of $\mathrm{GL}(2, K)$. Therefore we ask

For a given subgroup $S \subset \mathrm{GL}(2, K)$, is the group $\mathrm{Aut}_S\mathbb{A}_K^2$ linear?

For the subgroups S of $\mathrm{SL}(2, K)$, three criteria provide an almost complete answer, see Sections 8 and 9. Some examples of application are

Example A. *Let q be a quadratic form on K^2 and $S = \mathrm{SO}(q)$.*
If q is anisotropic, $\mathrm{Aut}_S\mathbb{A}_K^2$ is linear over a field extension of K.
Otherwise $\mathrm{Aut}_S\mathbb{A}_K^2$ is not linear, even over a ring.

Example B. *For some cocompact lattices $S \subset \mathrm{SL}(2, \mathbb{R})$, $\mathrm{Aut}_S\,\mathbb{A}_{\mathbb{C}}^2$ is linear over $\mathbb{C}$.*
For any lattice $S \subset \mathrm{SL}(2, \mathbb{C})$, $\mathrm{Aut}_S\,\mathbb{A}_{\mathbb{C}}^2$ is not linear, even over a ring.

Example C. *Let d be a squarefree integer, let $\mathcal{O}$ be the ring of integers of $k := \mathbb{Q}(\sqrt{d})$. Set $K = k(x)((t))$ and $S = \mathrm{SL}(2, \mathcal{O}[x, x^{-1}, t])$.*
If $d > 0$, the group $\mathrm{Aut}_S\,\mathbb{A}_K^2$ is not linear, even over a ring.
Otherwise, $\mathrm{Aut}_S\,\mathbb{A}_K^2$ is linear over some field of characteristic zero.

0.2 About the main points of the paper

Since *the topics are not ordered* as in the general introduction, a summary has been provided. Also note that *the statements in the introduction are often weaker* than those in the main text.

For a group S, we define in Section 2 the notion of *a mixed product* of S as a semi-direct product $S \ltimes *_{p\in P} E_p$, where $(E_p)_{p\in P}$ is a family of groups and S acts by permuting the factors of the free product. Hence P is an S-set such that $E_p^s = E_{s.p}$ for any $s \in S$ and $p \in P$.

In section 3, it is shown that, under a mild assumption, a mixed product (or an amalgamated product) which is linear over a ring is automatically linear over a field. This explains the dichotomy *linear over a field/not linear,* *even over a ring* in our statements. Then, we can use the theory of algebraic groups to show that some mixed products are not linear, even over a ring.

Let S be a subgroup of $\mathrm{SL}(2, K)$. The main question of the paper is to decide if $\mathrm{Aut}_S \mathbb{A}^2_K$ is linear or not. Indeed, this group is a mixed product $S \ltimes *_{\delta\in\mathbb{P}^1_K} E_\delta(K)$. Hence, there are two obstructions for the linearity.

First, the groups $S_\delta \ltimes E_\delta(K)$ have to be linear with a uniform bound on the degree. This problem is solved by using the notion of *semi-algebraic characters* for subgroups $\Lambda \subset K^*$. It was inspired by the famous paper of Borel and Tits [4], proving that the abstract isomorphisms of simple algebraic groups are semi-algebraic. Strictly speaking, our paper only provides a partial answer, because otherwise it would had been too long.

The second obstruction is the possibility, or not, to glue together some representations of the groups S and $S_\delta \ltimes E_\delta(K)$ to get a representation of $\mathrm{Aut}_S \mathbb{A}^2_K$. Our linearity criterion is stronger in characteristic zero than in finite characteristics. In characteristic zero, we can use for the gluing process the following stronger version of

Theorem C.2. *Assume K of characteristic 0. There is an embbeding*
$$\mathrm{SAut}_0^{<n} \mathbb{A}^2_K \subset \mathrm{SL}(d, K(t)),\ \textit{where}\ d = 1 + \mathrm{lcm}(1, 2 \ldots, n).$$

Here $\mathrm{SAut}_0^{<n} \mathbb{A}^2_K$ denotes the subgroup of $\mathrm{SAut}_0 \mathbb{A}^2_K$ generated by the automorphisms of degree $< n$, for some $n \geq 3$.

The proof of the strong version of Theorem C.2 uses one trick, based on the Lie superalgebra $\mathfrak{osp}(1, 2)$ and one idea, the ping-pong lemma. This idea was originally invented by Fricke and Klein for the dynamic of groups with respect to the metric topologies, see [12]. Later, Tits used it in the context of the ultrametric topologies [25], and we follow the Tits idea. Here, the ping-pong setting requires a representation of very large dimension.

As a brief conclusion

- if $\mathrm{ch}\, K = 0$, then $\mathrm{Aut}_0 \mathbb{A}^2_K$ is not even locally linear, $\mathrm{SAut}_0 \mathbb{A}^2_K$ is locally linear but not linear, and $\mathrm{Aut}_1 \mathbb{A}^2_K$ is linear. Therefore, the main questions arise for the groups G with $\mathrm{Aut}_1 \mathbb{A}^2_K \subset G \subset \mathrm{SAut}_0 \mathbb{A}^2_K$.
- if $\mathrm{ch}\, K = p$, $\mathrm{SAut}_0 \mathbb{A}^2_K$ is linear iff K is finite, while $\mathrm{Aut}_1 \mathbb{A}^2_K$ is always linear. In particular, $\mathrm{Aut}\, \mathbb{A}^2_{\overline{\mathbb{F}}_p}$ is locally linear but not linear. The existence of nonlinear f.g. subgroups is an open question.
- These phenomena are specific to dimension two.

1 Main Definitions and Conventions

Througout the paper, K will denote a given field. Its *ground field* is $\mathbb{F} = \mathbb{F}_p$ if $\operatorname{ch} K = p$ or $\mathbb{F} = \mathbb{Q}$ otherwise.

1.1 Group theoretical notation
Let S be a group and let $x, y \in S$. The symbols y^x and (x, y) are defined by

$$y^x := xyx^{-1} \text{ and } (x, y) := xyx^{-1}y^{-1}.$$

By definition, an *S-set* is a set P endowed with an action of S. The *stabilizer* of a point $p \in P$ is the subgroup $S_p := \{s \in S \mid s.p = p\}$. The *core* $\operatorname{Core}_S(A)$ of a subgroup A of S is the kernel of the action of S on S/A.

Similarly, an *S-group* is a group E endowed with a homomorphism $S \to \operatorname{Aut}(E)$. The corresponding semi-direct product of S by E is denoted by $S \ltimes E$. Given another S-group E', a homomophism, respectively an isomorphism, $\phi : E \to E'$ is called an *S-homomorphism*, resp. an *S-isomorphism* if it commutes with the S-action.

1.2 Commutative rings and group functors
Throughout the whole paper, a *commutative ring* means an associative commutative unital ring.

A *group functor* is a functor $G : R \mapsto G(R)$ from the category of commutative rings R to the category of groups (see e.g. [10] and [28] for the functorial approach to group theory). The standard example of a group functor is $R \mapsto \mathrm{GL}(n, R)$, where n is a given positive integer.

Given an ideal I of a commutative ring R, we denote by $G(I)$ the kernel of the homomorphism $G(R) \to G(R/I)$. It is called the *congruence subgroup* associated to the ideal I. For example, $\mathrm{GL}(n, I)$ is the subgroup of $\mathrm{GL}(n, R)$ of all matrices of the form $\mathrm{id} + A$, where all entries of A are in I.

In most cases, *we will only define the group $G(K)$ when K is a field* and the reader should understand that the definition over a ring is similar.

For the group functor $K \mapsto \operatorname{Aut} \mathbb{A}^2_K$ and its consorts, our notation is not consistent, since K is an index.

1.3 The group functors $\mathrm{Elem}_*(K)$, $\mathrm{Aff}(2, K)$, *and their subgroups*
By definition, an *elementary automorphism* of $\mathbb{A}^2_K$ is an automorphism

$$\phi : (x, y) \mapsto (z_1 x + t, z_2 y + f(x))$$

for some $z_1, z_2 \in K^*$, some $t \in K$ and some $f \in K[x]$. The group of elementary automorphisms of $\mathbb{A}^2_K$ is denoted $\mathrm{Elem}(K)$. Set

$$\mathrm{Elem}_0(K) = \mathrm{Elem}(K) \cap \operatorname{Aut}_0 \mathbb{A}^2_K,$$

$$\mathrm{SElem}_0(K) = \mathrm{Elem}(K) \cap \mathrm{SAut}_0\, \mathbb{A}^2_K, \text{ and}$$
$$\mathrm{Elem}_1(K) = \mathrm{Elem}(K) \cap \mathrm{Aut}_1\, \mathbb{A}^2_K.$$

However, *we will use the simplified notation* $E(K)$ *for* $\mathrm{Elem}_1(K)$.

Let $\mathrm{Aff}(2, K)$ be the subgroup of affine automorphisms of $\mathbb{A}^2_K$. Set

$$B_{\mathrm{Aff}}(K) := \mathrm{Aff}(2, K) \cap \mathrm{Elem}(K),$$
$$B_{\mathrm{GL}}(K) := B_{\mathrm{Aff}}(K) \cap \mathrm{GL}(2, K), \text{ and}$$
$$B(K) := B_{\mathrm{Aff}}(K) \cap \mathrm{SL}(2, K).$$

Indeed $B_{\mathrm{Aff}}(K)$, $B_{\mathrm{GL}}(K)$ and $B(K)$ are the standard Borel subgroups of $\mathrm{Aff}(2, K)$, $\mathrm{GL}(2, K)$ and $\mathrm{SL}(2, K)$. We have

$$\mathrm{Aff}(2, K)/B_{\mathrm{Aff}}(K) = \mathrm{GL}(2, K)/B_{\mathrm{GL}}(K) = \mathrm{SL}(2, K)/B(K) \simeq \mathbb{P}^1_K, \text{ and}$$
$$\mathrm{Elem}_0(K) = B_{GL}(K) \ltimes E(K) \text{ and } \mathrm{SElem}_0(K) = B(K) \ltimes E(K).$$

1.4 An informal definition of finite-codimensional group subfunctors

Informally speaking, a group subfunctor $H \subset G$ has *finite codimension* if the functor $R \mapsto G(R)/H(R)$ is "represented" by a scheme X of finite type. In particular, it means that there is a natural transform $G/R \to X$ wich induces a bijection $G(K)/H(K) \to X(K)$ whenever K is an algebraically closed field. Since this notion is used only for presentation purpose, we will not provide a formal definition.

For example, the subgroup $\mathrm{Aut}_0\, \mathbb{A}^2_K$ has codimension 2 in $\mathrm{Aut}\, \mathbb{A}^2_K$, since the quotient $\mathrm{Aut}\, \mathbb{A}^2_K/\mathrm{Aut}\, \mathbb{A}^2_K$ is naturally $\mathbb{A}^2_K$. This fact does not require to involve the elusive theory of ind-algebraic groups.

2 Mixed Products

Let S be a group and let $(E_p)_{p \in P}$ be a collection of groups indexed by some S-set P. Since we did not found a name in the literature, we will call *mixed product of* S any semi-direct product $S \ltimes *_P E_p$ where S acts on the free product $*_P E_p$ by permuting its factors, i.e. we have $E_p^s = E_{s.p}$ for any $s \in S$ and $p \in P$ (see [17] ch.4 for the definition of free products).

The connections between mixed products, amalgamated products and free products are investigated. As a consequence of van der Kulk's Theorem, we show that the groups $\mathrm{Aut}_S\, \mathbb{A}^2_K$ are mixed products and that the groups $\mathrm{Aut}_1\, \mathbb{A}^2_K$ and $\mathrm{Aut}_{U(K)}\, \mathbb{A}^2_K$ are free products.

2.1 Amalgamated products

Let A, G_1 and G_2 be groups and let $f_1 : A \to G_1$ and $f_2 : A \to G_2$ be group homomorphisms. Let $G_1 *_A G_2$ be the *amalgamated product* of G_1 and G_2

over A, see e.g. [22], ch. I. Since this product satisfies a universal property, it is often called a *free* amalgamated product, see [17] ch.8.

In what follows, we will *always assume that f_1 and f_2 are injective.* Hence the homomorphisms $G_1 \to G_1 *_A G_2$ and $G_2 \to G_1 *_A G_2$ are injective, by the Theorem 1 of ch. 1 in [22]. Therefore, we will use a less formal terminology. The groups G_1 and G_2 are viewed as subgroups of $G_1 *_A G_2$ and we will say that G_1 and G_2 *share A as a common subgroup.*

2.2 Reduced words

The usual definition [22] of reduced words is based on the right A-cosets. In order to avoid a confusion between the set difference notation $A \setminus X$ and the A-orbits notation $A \backslash X$, we will use a definition based on the left A-cosets.

Let G_1, G_2 be two groups sharing a common subgroup A, and let $\Gamma = G_1 *_A G_2$. Set $G_1^* = G_1 \setminus A$, $G_2^* = G_2 \setminus A$ and let $T_1^* \subset G_1^*$ (respectively $T_2^* \subset G_2^*$) be a set of representatives of G_1^*/A (resp. of G_2^*/A).

Let Σ be the set of all finite alternating sequences $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_n)$ of ones and twos. A *reduced word* of *type* $\boldsymbol{\epsilon}$ is a word $(x_1, \dots x_n, x_0)$ where x_0 is in A, and $x_i \in T_{\epsilon_i}^*$ for any $1 \le i \le n$. Let $\mathcal{R}$ be the set of all reduced words. The next Lemma is well-known, see e.g. [22], Theorem 1.

Lemma 1. *The map*

$$(x_1, \dots x_n, x_0) \in \mathcal{R} \mapsto x_1 \dots x_n x_0 \in G_1 *_A G_2$$

is bijective.

Set $\Gamma = G_1 *_A G_2$. For $\gamma \in \Gamma \setminus A$ there is some integer $n \ge 1$, some $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_n) \in \Sigma$ and some $g_i \in G_{\epsilon_i}^*$ such that $\gamma = g_1 ... g_n$. It follows from loc. cit. that $\gamma = x_1 ... x_n x_0$ for some reduced word $(x_1 ... x_n x_0)$ of type $\boldsymbol{\epsilon}$. Since it is determined by γ, the sequence $\boldsymbol{\epsilon}$ is called *the type* of γ.

2.3 Amalgamated product of subgroups

Let G_1, G_2 be two groups sharing a common subgroup A and set $\Gamma = G_1 *_A G_2$. Let $G_1' \subset G_1$, $G_2' \subset G_2$ and $A' \subset A$ be subgroups such that

$$G_1' \cap A = G_2' \cap A = A'.$$

Lemma 2. *(i) The natural map $G_1' *_{A'} G_2' \to \Gamma$ is injective.*

(ii) Let $\Gamma' \subset \Gamma$ be a subgroup such that $\Gamma'.A = \Gamma$. Then we have

$$\Gamma' = G_1' *_{A'} G_2',$$

where $G_1' = G_1 \cap \Gamma'$, $G_2' = G_2 \cap \Gamma'$ and $A' = A \cap \Gamma'$.

Proof. Proof of Assertion (i). For $i = 1, 2$ set $G_i^* = G_i \setminus A$, $G_i'^* = G_i' \setminus A'$. Let $T_i^* \subset G_i^*$ and $T_i'^* \subset G_i'^*$ be a set of representatives of G_i^*/A and $G_i'^*/A'$.

Since the maps $G_i'/A' \to G_i/A$ are injective, it can be assumed that $T_i'^* \subset T_i^*$. Let $\mathcal{R}$ and $\mathcal{R}'$ be the set of reduced words of $G_1 *_A G_2$, and respectively of $G_1' *_{A'} G_2'$. By definition, we have $\mathcal{R}' \subset \mathcal{R}$, thus by Lemma 1 the map $G_1' *_{A'} G_2' \mapsto G_1 *_A G_2$ is injective.

Proof of Assertion (ii). We will use the notations of the previous proof. Since $\Gamma'.A = \Gamma$, it follows that the maps $G_1'/A' \to G/A$ and $G_2'/A' \to G_2/A$ are bijective. Therefore $\mathcal{R}'$ is the set of all reduced words $(x_1, \ldots, x_n, x_0) \in \mathcal{R}$ such that $x_0 \in A'$. It follows easily that

$$G_1 *_A G_2 / G_1' *_{A'} G_2' \simeq A/A' = \Gamma/\Gamma',$$

and therefore we have $\Gamma' = G_1' *_{A'} G_2'$. □

2.4 The group $\operatorname{Aut} \mathbb{A}_K^2$ *is an amalgamated product*

Indeed, it is the classical

van der Kulk's Theorem. *[29] We have*

$$\operatorname{Aut} \mathbb{A}_K^2 \simeq \operatorname{Aff}(2, K) *_{B_{\operatorname{Aff}(K)}} \operatorname{Elem}(K).$$

2.5 Mixed products

Let S be a group, let P be an S-set and let $Q \subset P$ be a set of representatives of P/S. A mixed product $S \ltimes *_{p \in P} E_p$ satisfies the following universal property.

Lemma 3. *Let $\Gamma \supset S$ be a group. Assume given, for any $q \in Q$, a S_q-homomorphism $\phi_q : E_q \to \Gamma$. Then there is a unique group homomorphism*

$$\phi : S \ltimes *_{p \in P} E_p \to \Gamma$$

such that $\phi|_S = \mathrm{id}$ and $\phi|_{E_q} = \phi_q$ for any $q \in Q$.

Proof. Let us define, for any $p \in P$, a S_p-homomorphism $\phi_p : E_p \to \Gamma$ as follows. Let $s \in S$ such that $q := s.p$ belongs to Q. Set

$$\phi_p(u) = s^{-1} \phi_q(sus^{-1}) s,$$

for any $u \in E_p$. Since ϕ_q is a S_q-homomorphism, the defined homomorphism ϕ_p only depends on s modulo S_p. Moreover the collection of homomorphisms $(\phi_p)_{p \in P}$ induces an S-homomorphism from $*_{p \in P} E_p$ to Γ, which extends to the required homomorphism $\phi : S \ltimes *_{p \in P} E_p \to \Gamma$. □

It follows that a mixed product $S \ltimes *_{p \in P} E_p$ is entirely determined by S and the the S_q-groups E_q for $q \in Q$. For the record, let us state

Lemma 4. *Let $\Gamma = S \ltimes *_{p\in P} E_p$ and $\Gamma' = S \ltimes *_{p\in P} E'_p$ be two mixed groups.*

If for any $q \in Q$, the groups E_q and E'_q are S_q-isomorphic, then the groups Γ and Γ' are isomorphic.

2.6 Mixed products with a transitive action on P

In this subsection, we show that the mixed products with a transitive action of S on P are the amalgamated products $S *_A G$ where A is a retract in G.

First, let S, G be two groups sharing a common subgroup A with the additional assumption that A is a retract in G. Therefore, we have $G = A \ltimes E$, for some normal subgroup E of G. Set $\Gamma = S *_A G$ and let Γ_1 be the kernel of the map $\Gamma \to S$ induced by the retraction $G \to A \simeq G/E$. It is clear that

$$\Gamma = S \ltimes \Gamma_1.$$

Lemma 5. *Let P be a set of representatives of S/A. We have*

$$\Gamma_1 \simeq *_{\gamma\in P} E^\gamma.$$

In particular $S *_A G$ is isomorphic to the mixed product $S \ltimes *_{\gamma\in P} E^\gamma$.

Proof. We can assume that $1 \in P$. By Lemma 3, there is a unique homomorphism $\phi : S \ltimes *_{\gamma\in P} E^\gamma \to \Gamma$ such that its restriction to $E^1 = E$ and to S is the identity. Conversely, the group $S \ltimes *_{\gamma\in P} E^\gamma$ contains the subgroups S and $G \simeq A \ltimes E^1$ whose intersection is A. Hence, the universal property of amalgamated products provides a natural homomorphism $\psi : \Gamma \to \ltimes *_{\gamma\in P} E^\gamma$. Clearly, ϕ and ψ are inverses of each other, what shows the lemma. □

Conversely, let $\Gamma = S \ltimes *_{p\in P} E_p$ be a mixed product.

Lemma 6. *Assume that S acts transitively on P. Then we have*

$$S \ltimes (*_{p\in P} E_p) \simeq S *_{S_q} (S_q \ltimes E_q),$$

where q is any chosen point in P.

The proof of the Lemma 6 will be skipped. Indeed it is based on universal properties, as the previous proof.

2.7 The group $\mathrm{Aut}_S\, \mathbb{A}^2_K$ is a mixed product

For a subgroup S of $\mathrm{GL}(2, K)$, recall that

$$\mathrm{Aut}_S\, \mathbb{A}^2_K := \{\phi \in \mathrm{Aut}_0\, \mathbb{A}^2_K \mid \mathrm{d}\phi_{\mathbf{0}} \in S\}.$$

As usual, a line $\delta \in \mathbb{P}^1_K$ has *projective coordinates* $(a; b)$ if $\delta = K.(a, b)$. For such a δ, let $E_\delta(K) \subset \mathrm{Aut}\, \mathbb{A}^2_K$ be the subgroup

$$E_\delta(K) := \{(x, y) \mapsto (x, y) + f(bx - ay)(a, b) \mid f \in t^2K[t]\}.$$

Let $\gamma \in \mathrm{GL}(2, K)$ such that $\gamma.\delta_0 = \delta$ where $\delta_0 \in \mathbb{P}^1_K$ has coordinates $(0; 1)$. Then we have $E_{\delta_0}(K) = E(K)$ and $E_\delta(K) = E(K)^\gamma$.

Lemma 7. *We have*

$$\mathrm{Aut}_S\,\mathbb{A}^2_K \simeq S \ltimes *_{\delta\in\mathbb{P}^1_K} E_\delta(K).$$

Proof. Clearly, it is enough to prove the statement for $S = \mathrm{GL}(2,K)$. Since $B_{\mathrm{Aff}}(K)$ contains the translations, we have $B_{\mathrm{Aff}}(K).\mathrm{Aut}_0\,\mathbb{A}^2_K = \mathrm{Aut}\,\mathbb{A}^2_K$. Therefore by van der Kulk's Theorem and Lemma 2, we have

$$\mathrm{Aut}_0\,\mathbb{A}^2_K \simeq \mathrm{GL}(2,K) *_{B_{\mathrm{GL}}(K)} \mathrm{Elem}_0(K).$$

Since $\mathrm{Elem}_0(K) = B_{\mathrm{GL}}(K) \ltimes E(K)$, it follows from Lemma 5 that

$$\mathrm{Aut}_0\,\mathbb{A}^2_K \simeq \mathrm{GL}(2,K) \ltimes *_{\delta\in\mathbb{P}^1_K} E_\delta(K).$$ □

2.8 Mixed product with an almost free transitive action on P

The action of S on a set P is called *almost free transitive* if P consists of a fixed point and a free orbit under S. (It will be tacitly assumed that $S \neq 1$, so the fixed point and the free orbit are well defined.) In this subsection we show that the mixed products with an almost free transitive action of S on P are the free products $G * G'$, where S is a retract in G.

First let $\Gamma = S \ltimes *_{p\in P} E_p$ be a mixed product.

Lemma 8. *Assume that the action of S on P is almost free transitive. Then Γ is isomorphic to the free product*

$$(S \ltimes E_{p_0}) * E_{p_\infty},$$

where $p_0 \in P$ is the fixed point and $p_\infty \in P$ is any point of the free orbit.

Conversely let $\Gamma = (S \ltimes E) * F$ be a free product, where E is an S-group and F is another group.

Lemma 9. *The group Γ is isomorphic to the the mixed product*

$$S \ltimes (E * (*_{s\in S} F_s)),$$

where F_s denotes a copy of F, for any $s \in S$.

The easy proofs of the previous two lemmas, which follow the same pattern as Lemma 4, will be skipped.

2.9 The group $\mathrm{Aut}_{U(K)}\,\mathbb{A}^2_K$ is a free product

Recall that $U(K)$ is the group of linear transforms $(x,y) \mapsto (x, y+ax)$, for some $a \in K$. Let δ_0, $\delta_\infty \in \mathbb{P}^1_K$ be the points with projective coordinates $(0;1)$ and $(1;0)$. The group $E_{\delta_0}(K) = E(K)$ commutes with $U(K)$.

Lemma 10. *We have*

$$\mathrm{Aut}_{U(K)}\,\mathbb{A}^2_K \simeq (U(K) \times E_{\delta_0}(K)) * E_{\delta_\infty}(K).$$

Proof. By Lemma 7, we have $\mathrm{Aut}_{U(K)}\,\mathbb{A}^2_K \simeq U(K) \ltimes *_{\delta\in\mathbb{P}^1_K} E_\delta(K)$. Since the action of $U(K)$ on $\mathbb{P}^1_K$ is almost free transitive, the assertion follows from Lemma 8. □

2.10 A corollary

Corollary 1. *Let K, L be fields such that* $\mathrm{Card}\,K = \mathrm{Card}\,L$ *and* $\mathrm{ch}\,K = \mathrm{ch}\,L$. *We have*

$$\mathrm{Aut}_1\,\mathbb{A}^2_K \simeq \mathrm{Aut}_1\,\mathbb{A}^2_L \textit{ and } \mathrm{Aut}_{U(K)}\,\mathbb{A}^2_K \simeq \mathrm{Aut}_{U(L)}\,\mathbb{A}^2_L.$$

Proof. It can be assumed that K is infinite. Let $\mathbb{F}$ be its prime subfield and let E be a $\mathbb{F}$-vector space with $\dim_{\mathbb{F}} E = \aleph_0\,[K : \mathbb{F}] = \mathrm{Card}\ K$.

By Lemma 7, $\mathrm{Aut}_1\,\mathbb{A}^2_K$ is a free product of $\mathrm{Card}\,K$ copies of E, from which its follows that $\mathrm{Aut}_1\,\mathbb{A}^2_K$ only depends on the cardinality and the characteristic of the field K, hence we have $\mathrm{Aut}_1\,\mathbb{A}^2_K \simeq \mathrm{Aut}_1\,\mathbb{A}^2_L$.

The proof that $\mathrm{Aut}_{U(K)}\,\mathbb{A}^2_K \simeq \mathrm{Aut}_{U(L)}\,\mathbb{A}^2_L$ is identical. □

3 Linearity over Rings vs. over Fields

In this section, we show Corollaries 2 and 3. They state that, under a mild assumption, a mixed product, or an amalgamated product, which is linear over a ring, is automatically linear over a field.

3.1 Linearity Properties

For a group, the linearity over a field is the strongest linearity property. Besides the case of *prime* rings, i.e. the subrings of a field, a group which is linear over a ring R is not necessarily linear over a field. Two relevant examples are provided in the subsection 8.6.

On the opposite, there are groups containing a f.g. subgroup which is not linear, even over a ring. These groups are nonlinear in the strongest sense.

3.2 Minimal embeddings

Let R be a commutative ring and let Γ be a subgroup of $\mathrm{GL}(n, R)$ for some $n \geq 1$. The embedding $\Gamma \subset \mathrm{GL}(n, R)$ is called *minimal* if for any ideal $J \neq \{0\}$ we have $\Gamma \cap \mathrm{GL}(n, J) \neq \{1\}$.

Lemma 11. *Let Γ be a subgroup of $\mathrm{GL}(n, R)$. For some ideal J, the induced homomorphism $\Gamma \to \mathrm{GL}(n, R/J)$ is a minimal embedding.*

Proof. Since R could be non-noetherian, the proof requires Zorn's Lemma.

Let $\mathcal{S}$ be the set of all ideals J of R such that $\Gamma \cap \mathrm{GL}(n, J) = \{1\}$. With respect to the inclusion, $\mathcal{S} \ni \{0\}$ is a nonempty poset. For any chain $\mathcal{C} \subset \mathcal{S}$, the ideal $\cup_{I \in \mathcal{C}} I$ belongs to $\mathcal{S}$. Therefore Zorn's Lemma implies that $\mathcal{S}$ contains a maximal element J. It follows that the induced homomorphism $\Gamma \to \mathrm{GL}(n, R/J)$ is a minimal embedding. □

3.3 Groups with trivial normal centralizers

By definition, a group Γ has *the trivial normal centralizers property* if, for any subset $S \not\subset \{1\}$, its centralizer $C_\Gamma(S)$ is not normal, except if $C_\Gamma(S)$ is the trivial group. Equivalently, if H_1 and H_2 are commuting normal subgroups of Γ, then one of them is trivial.

Lemma 12. *Let Γ be a group with the trivial normal centralizers property.*

If Γ is linear over a ring, then Γ is also linear over a field.

Proof. By hypothesis and Lemma 11, there exists a minimal embedding $\rho : \Gamma \subset \mathrm{GL}(n, R)$ for some commutative ring R. The case $\Gamma = \{1\}$ can be excluded, so we will assume that $R \neq \{0\}$.

Let I_1, I_2 be ideals of R with $I_1.I_2 = 0$. Since $H_1 := \Gamma \cap \mathrm{GL}(n, I_1)$ and $H_2 := \Gamma \cap \mathrm{GL}(n, I_2)$ are commuting normal subgroups of Γ, one of them is trivial. Since ρ is minimal, I_1 or I_2 is the zero ideal. Thus R is prime.

It follows that $\Gamma \subset \mathrm{GL}(n, K)$, where K is the fraction field of R. □

3.4 Amalgamated products with a trivial core

Let G_1, G_2 be two groups sharing a common subgroup A and set $\Gamma = G_1 *_A G_2$.

Let Σ be the set of all finite alternating sequences $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_n)$ of ones and twos. For i, $j \in \{1, 2\}$, let $\Sigma_{i,j}$ be be the subset of all $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_n) \in \Sigma$ starting with i and ending with j and let $\Gamma_{i,j}$ be the set of all $\gamma \in \Gamma$ of type $\boldsymbol{\epsilon}$ for some $\boldsymbol{\epsilon} \in \Sigma_{i,j}$. Therefore we have

$$\Gamma = A \sqcup \Gamma_{1,1} \sqcup \Gamma_{2,2} \sqcup \Gamma_{1,2} \sqcup \Gamma_{2,1}.$$

By definition, the amalgamated product $G_1 *_A G_2$ is called *nondegenerate* if $G_1 \neq A$ and $G_2 \neq A$. It is called *dihedral* if $G_1 = G_2 = \mathbb{Z}/2\mathbb{Z}$, and $A = \{1\}$, and *nondihedral* otherwise.

Lemma 13. *Let $\Gamma = G_1 *_A G_2$ be a nondegenerate and nondihedral amalgamated product such that $\mathrm{Core}_\Gamma(A)$ is trivial.*

For any element $g \neq 1$ of Γ, there are γ_1, $\gamma_2 \in \Gamma$ such that

$$g^{\gamma_1} \in \Gamma_{1,1} \text{ and } g^{\gamma_2} \in \Gamma_{2,2}.$$

In particular Γ *has the trivial normal centralizers property.*

Proof. First it should be noted that A cannot be simultaneously a subgroup of index 2 in G_1 and in G_2. Otherwise the core hypothesis implies that $A = \{1\}$ and Γ would be the dihedral group. Hence we can assume that G_2/A contains at least 3 elements.

Next it is clear that $G_i^*.\Gamma_{j,k} \subset \Gamma_{i,k}$ and $\Gamma_{k,j}.G_i^* \subset \Gamma_{k,i}$ whenever $i \neq j$.

Proof that the conjugacy class of any $g \neq 1$ *intersects both* $\Gamma_{1,1}$ *and* $\Gamma_{2,2}$. Let $\gamma_1 \in G_1^*$ and $\gamma_2 \in G_2^*$. We have $\Gamma_{2,2}^{\gamma_1} \subset \Gamma_{1,1}$ and $\Gamma_{1,1}^{\gamma_2} \subset \Gamma_{2,2}$. Therefore the claim is proved for any $g \in \Gamma_{1,1} \cup \Gamma_{2,2}$. Moreover it is now enough to prove that the conjugacy class of any $g \neq 1$ intersects $\Gamma_{2,2}$.

Assume now $g \in \Gamma_{2,1}$. We have $g = u.v$ for some $u \in G_2^*$ and $v \in \Gamma_{1,1}$. Since $[G_2 : A] \geq 3$, there is $\gamma \in G_2^*$ such that $\gamma.u \notin A$. It follows that $\gamma.g$ belongs to $\Gamma_{2,1}$, and therefore g^γ belongs to $\Gamma_{2,2}$.

For $g \in \Gamma_{1,2}$, the claim follows from the fact that g^{-1} belongs to $\Gamma_{2,1}$.

Last, let $g \in A \setminus \{1\}$. Since $\mathrm{Core}_A(\Gamma)$ is trivial, there is $\gamma \in \Gamma$ such that g^γ is not in A. Thus g^γ belongs to $\Gamma_{i,j}$ for some i, j. So g is conjugate to some element in $\Gamma_{2,2}$ by the previous considerations.

Proof that Γ *has the trivial normal centralizers property.* Let H_1, H_2 be nontrivial normal subgroups. By the previous point, there are elements g_1, g_2 with

$$g_1 \in H_1 \cap \Gamma_{1,1} \text{ and } g_2 \in H_2 \cap \Gamma_{2,2}.$$

Since we have $g_1 g_2 \in \Gamma_{1,2}$ and $g_2 g_1 \in \Gamma_{2,1}$, it follows that $g_1 g_2 \neq g_2 g_1$. Therefore H_1 and H_2 do not commute. □

Corollary 2. *Let* $\Gamma = G_1 *_A G_2$ *be a nondegenerate amalgamated product such that* $\mathrm{Core}_\Gamma(A)$ *is trivial* [1].

If Γ *is linear over a ring, then* Γ *is linear over a field.*

Proof. Since the infinite dihedral group is linear over a field, we will assume that the amalgamated product $\Gamma = G_1 *_A G_2$ is also nondihedral. Thus the result is an obvious corollary of Lemmas 12 and 13. □

3.5 Mixed products with trivial core

Let S be a group, and let $S \ltimes *_{p\in P} E_p$ be a mixed product of S.

[1]As noticed by the referee, this is equivalent to the faithfulness of the action of Γ on the associated Bass-Serre tree.

Let Σ be the set of all finite sequences $\boldsymbol{\pi} = (p_1, \dots, p_m)$ of elements of P with $p_i \neq p_{i+1}$, for any $i < m$. Set $\Gamma_1 = *_{p\in P} E_p$ and $E_p^* = E_p \setminus \{1\}$. Any element $u \in \Gamma_1 \setminus \{1\}$ is uniquely written as $u = u_1 \dots u_m$, where $u_i \in E_{p_i}^*$ for some $m \geq 1$ and some sequence $\boldsymbol{\pi} = (p_1, \dots, p_m) \in \Sigma$. The decomposition $u = u_1 \dots u_m$ is called the *reduced decomposition* of u, $\boldsymbol{\pi}$ is called its *type* and m is called its *length*. For $p, p' \in P$, let $E_{p,p'}$ be the set of all elements $u \in \Gamma_1 \setminus \{1\}$ whose type is a sequence $\boldsymbol{\pi}$ starting with p and ending with p'.

By definition, the free product $*_{p\in P} E_p$, or, by extension, the mixed product $S \ltimes *_{p\in P} E_p$, is called *nondegenerate* if Card $P \geq 2$ and $E_p \neq \{1\}$ for any $p \in P$. For a nondegenerate mixed product $S \ltimes *_{p\in P} E_p$, we have

$$\mathrm{Core}_\Gamma(S) = \mathrm{Core}_\Gamma(\cap_P S_p).$$

The mixed product $S \ltimes *_{p\in P} E_p$ is called *dihedral* if Card $P = 2$, if $E_p = \mathbb{Z}/2\mathbb{Z}$ for any $p \in P$ and if

$$S \simeq \mathbb{Z}/2Z \text{ permutes the two factors, or } S = \{1\}.$$

It is called *nondihedral* otherwise.

Lemma 14. *Let $\Gamma = S \ltimes *_{p\in P} E_p$ be a nondegenerate and nondihedral mixed product such that $\mathrm{Core}_\Gamma(\cap_P S_p) = \{1\}$. Let $p \in P$.*
(i) For any element $\gamma \in \Gamma_1 \setminus \{1\}$, there is $v \in \Gamma_1$ such that γ^v belongs to $E_{p,p}$.
(ii) For any element $\gamma \in \Gamma \setminus \Gamma_1$, there is $v \in \Gamma_1$ such that $(\gamma, v) \neq 1$.

In particular Γ has the trivial normal centralizers property.

Proof. Proof of Assertion(i). For Card $P = 2$, the group Γ_1 is not the infinite dihedral group $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ and the assertion follows from Lemma 13. Therefore, we will assume that Card $P \geq 3$.

The element γ belongs to E_{p_1,p_2} for some p_1, $p_2 \in P$. Let $p_3 \in P \setminus \{p_1, p_2\}$ and let $v \in E_{p,p_3}$. Thus the element γ^v belongs to $E_{p,p}$.

Proof of Assertion (ii). Let $\gamma = su$, where $s \in S \setminus \{1\}$ and $u \in \Gamma_1$.

Obviously Γ/S and Γ_1 are isomorphic S-sets. Since $\mathrm{Core}_\Gamma(S)$ is trivial, there is $t \in \Gamma_1$ such that $(s,t) \neq 1$. Thus, we can assume that $u \neq 1$.

Let $u = u_1 \dots u_m$ be its reduced decomposition, let $(p_1, \dots, p_m)$ be its type. Let $v \in E_{p'}^*$ with $p' \neq p_m$. By definition, $u_1 \dots u_m.v.u_m^{-1} \dots u_1^{-1}$ is a reduced decomposition of the element $w := uvu^{-1}$. Hence w and w^s have lenght $2m + 1 > 1$. Thus we have $v^\gamma = w^s \neq v$, or equivalently $(\gamma, v) \neq 1$.

Proof that Γ has the trivial normal centralizers property. Let H, H' be two nontrivial normal subgroups of Γ. Let $p \neq p'$ be elements of P. By Assertions (i) and (ii), there are elements g, g' with

$$g \in H \cap E_{p,p} \text{ and } g' \in H' \cap E_{p',p'}.$$

Since we have $gg' \in E_{p,p'}$ and $g'g \in E_{p',p}$, it follows that $gg' \neq g'g$. Therefore H and H' do not commute. □

Corollary 3. *Let* $\Gamma = S \ltimes *_{p\in P} E_p$ *be a nondegenerate mixed product such that* $\mathrm{Core}_\Gamma(\cap_P S_p) = \{1\}$.

If Γ *is linear over a ring, then* Γ *is linear over a field.*

Proof. Since the infinite dihedral group is linear over a field, we can assume that the mixed product $\Gamma = S \ltimes *_{p\in P} E_p$ is also nondihedral. Then the result is an obvious corollary of Lemmas 12 and 14. □

4 A Nonlinear f.g. Subgroup of $\mathrm{Aut}_0\, \mathbb{A}^2_{\mathbb{Q}}$

Let Γ be the group with presentation

$$\langle \sigma, \tau \mid \sigma^2 \tau \sigma^{-2} = \tau^2 \rangle.$$

In [11], C. Drutu and M. Sapir showed that Γ is not linear over a field[2]. We show that Γ is not linear either over a ring, and that Γ is isomorphic to an explicit subgroup of $\mathrm{Aut}_0\, \mathbb{A}^2_{\mathbb{Q}}$, which proves Theorem A.2.

4.1 The amalgamated decomposition $\Gamma = G_1 *_A G_2$

Let us consider the following subgroups of Γ

$$G_1 = \langle \sigma \rangle,\ G_2 = \langle \sigma^2, \tau \rangle \text{ and } A = \langle \sigma^2 \rangle.$$

The groups G_2 is isomorphic to $\mathbb{Z} \ltimes \mathbb{Z}[1/2]$ where any $n \in \mathbb{Z}$ acts over $\mathbb{Z}[1/2]$ by multiplication by 2^n. The group Γ is the amalgamated product

$$\Gamma \simeq G_1 *_A G_2.$$

Lemma 15. *The group* Γ *has the trivial normal centralizers property.*

Proof. Set $H = \mathbb{Z}[1/2].\tau$. The A-sets G_2/A and H are isomorphic, hence A acts faithfully on G_2/A. Therefore $\mathrm{Core}_\Gamma(A) \subset \mathrm{Core}_{G_2}(A)$ is trivial, and the assertion follows from Lemma 13. □

4.2 Quasi-unipotent endomorphisms

Let V be a finite-dimensional vector space over an algebraically closed field K. An element $u \in \mathrm{GL}(V)$ is called *quasi-unipotent* if all its eigenvalues are

[2] In the first version of this paper that appeared in the arXiv, I was unaware of [11]. I'm grateful to T. Delzant for providing this reference.

roots of unity. The *quasi-order* of a quasi-unipotent endomorphism u is the smallest positiver integer m such that u^m is unipotent.

If u is unipotent and ch $K = 0$, set

$$\log\, u = \log(1 - (1 - u)) := \sum_{k\geq 1} (1 - u)^k/k,$$

which is well-defined since $1 - u$ is nilpotent.

Lemma 16. *Let $h, u \in \mathrm{GL}(V)$. Assume that u has infinite order and $huh^{-1} = u^2$. Then u is quasi-unipotent of quasi-order m for some odd integer m. Moreover K has characteristic zero, and*

$$heh^{-1} = 2e,$$

where $e := \log\, u^m$.

Proof. Let Spec u be the spectrum of u. By hypothesis the map $\lambda \mapsto \lambda^2$ is bijective on Spec u. Hence all eigenvalues of u are odd roots of unity, what proves that u is quasi-unipotent of odd quasi-order m.

Over any field of finite characteristic, the unipotent endomorphisms have finite order. Hence we have ch $K = 0$. Moreover, we have

$$heh^{-1} = h(\log u^m)h^{-1} = \log(hu^mh^{-1}) = \log(u^{2m}) = 2e. \qquad \square$$

4.3 Nonlinearity of Γ

Drutu-Sapir's Lemma. *The group Γ is not linear over a field.*

The result is a particular case of Corollary 4 in [11]. Since their proof is based on an earlier result of [30], we shall provide a direct proof.

Proof. Assume otherwise and let $\rho' : \Gamma \to \mathrm{GL}(V)$ be an embedding, where V is a finite-dimensional vector space over an algebraically closed field K. Since τ has infinite order and $\sigma^2\tau\sigma^{-2} = \tau^2$, it follows from Lemma 16 that K has characteristic zero, $\rho'(\tau)$ is quasi-unipotent of odd quasi-order m.

Step 1: there is another embedding $\rho : \Gamma \to \mathrm{GL}(V)$ such that $\rho(\tau)$ is unipotent. Let $\psi : \Gamma \to \Gamma$ be the group homomorphism defined by $\psi(\sigma) = \sigma$, and $\psi(\tau) = \tau^m$. Since $\psi(G_1) \cap \psi(G_2) = A$, it follows from Lemma 2 that the natural homomorphism $\psi(G_1) *_A \psi(G_2) \to \Gamma$ is injective. Hence ψ is injective and $\rho := \rho' \circ \psi$ is an embedding such that $\rho(\tau) = \rho'(\tau)^m$ is unipotent.

Step 2: the unipotent subgroup $U \subset \mathrm{GL}(V)$. Set $h = \rho(\sigma^2)$, let $\Pi =$ Spec h be its spectrum, and for each $\lambda \in \Pi$, let $V_{(\lambda)}$ be the corresponding generalized eigenspace. For any $k \geq 0$, set

$$\Pi_{\geq} k = \{\lambda \in \Pi \mid \lambda \in 2^l\, \Pi \text{ for some } l \geq k\},$$

The filtration $\Pi = \Pi_{\geq 0} \supset \Pi_{\geq 1} \supset \ldots$ of the set Π induces a filtration of V

$$V = V_{\geq 0} \supset V_{\geq 1} \supset \dots,$$

where $V_{\geq k} = \oplus_{\lambda\in\Pi_{\geq k}} V_{(\lambda)}$. Let U be the group of all $g \in \mathrm{GL}(V)$ such that $(g - \mathrm{id})V_{\geq k} \subset V_{\geq k+1}$ for all $k \geq 0$. For some suitable basis, U is a group of upper triangular matrices. Therefore U is nilpotent.

Step 3: $\rho(\Gamma)$ is nilpotent by commutative. Since $\rho(G_1)$ commutes with h, we have $\rho(G_1).V_{\geq k} = V_{\geq k}$ for any integer k. Therefore $\rho(G_1)$ normalizes U.

Set $u = \rho(\tau)$ and $e = \log u$. By Lemma 16, we have $heh^{-1} = 2e$ and therefore we have $e.V_{\geq k} \subset V_{\geq k+1}$. It follows that $u = \exp e$ belongs to U. Since $\rho(\Gamma) = \langle \rho(G_1), u\rangle$ we have

$$\rho(\Gamma) \subset \rho(G_1) \ltimes U,$$

and therefore $\rho(\Gamma)$ is nilpotent by commutative. Hence $\rho(\Gamma)$ contains a non-trivial normal abelian subgroup. This contradicts Lemma 15, which states that Γ has the trivial normal centralizers property. □

Lemma 17. *The group Γ is not linear, even over a ring.*

Proof. By Lemma 15 the group Γ has the trivial normal centralizers property. It follows from Lemmas 12 and 4 that Γ is not linear, even over a ring. □

4.4 Proof of Theorem C.1

Theorem C.1. *The subgroup $\langle S, T\rangle$ of $\mathrm{Aut}_0\, \mathbb{A}^2_{\mathbb{Q}}$ is not linear, even over a ring, where*

$$S(x, y) = (y, 2x) \text{ and } T(x, y) = (x, y + x^2).$$

Proof. Set $H_1 = \langle S\rangle$, $H_2 = \langle S^2, T\rangle$, $C = \langle S^2\rangle$.

We have $S^2 = 2.\mathrm{id}$, therefore we have $H_1 \cap B_{\mathrm{Aff}}(K) = C$. Moreover H_2 is the group of automorphisms of the form

$$(x, y) \mapsto (2^k x, 2^k y + rx^2),$$

for $k \in \mathbb{Z}$ and $r \in \mathbb{Z}[1/2]$, therefore $H_2 \cap B_{\mathrm{Aff}}(K) = C$. It follows from Lemma 2 and van der Kulk's Theorem that the natural homomorphism $H_1 *_C H_2 \to \mathrm{Aut}_0\, \mathbb{A}^2_{\mathbb{Q}}$ is injective.

There is a group isomorphism $\Gamma \to H_1 *_C H_2$ sending σ to S^{-1} and τ to T. Thus, by Lemma 17, the subgroup of $\mathrm{Aut}_0\, \mathbb{A}^2_K$ generated by S and T is not linear, even over a ring. □

5 The Linear Representation of $\mathrm{Aut}_1\, \mathbb{A}^2_K$

We will prove Theorem A.2, in a way which is useful for Section 9.

5.1 Nagao's Theorem

For a subgroup S of $\mathrm{GL}(2,K)$, set

$$\mathrm{GL}_S(2,K[t]) = \{G(t) \in \mathrm{GL}(2,K[t]) \mid G(0) \in S\}.$$

Nagao's Theorem [18]. *We have*

$$\mathrm{GL}(2,K[t]) \simeq \mathrm{GL}(2,K) *_{B_{\mathrm{GL}(k)}} \mathrm{GL}_{B_{\mathrm{GL}(k)}}(2,K[t]).$$

5.2 The group $\mathrm{GL}_S(2,K[t])$ *is a mixed product*

For any $\delta \in \mathbb{P}^1_K$, let $e_\delta \in \mathrm{End}(K^2)$ be a nilpotent element with $\mathrm{Im}\, e_\delta = \delta$. For any commutative K-algebra R, set

$$U_\delta(R) := \{\mathrm{id} + re_\delta \mid r \in R\}.$$

Obviously, $U_\delta(R)$ is a subgroup of $\mathrm{SL}(2,R)$. Let $\gamma \in \mathrm{GL}(2,K)$ such that $\gamma.\delta_0 = \delta$ where $\delta_0 \in \mathbb{P}^1_K$ has coordinates $(0;1)$. We have

$$U_{\delta_0}(R) = U(R) \text{ and } U_\delta(R) = U(R)^\gamma.$$

Lemma 18. *Let S be a subgroup of* $\mathrm{GL}(2,K)$. *We have*

$$\mathrm{GL}_S(2,K[t]) \simeq S \ltimes *_{\delta\in\mathbb{P}^1_K} U_\delta(tK[t]).$$

Proof. Clearly, it is enough to prove the lemma for $S = \mathrm{GL}(2,K)$. Since

$$\mathrm{GL}_{B_{\mathrm{GL}(k)}(k)}(2,K[t]) = B_{\mathrm{GL}(k)}(k) \ltimes U(tK[t])$$

the lemma follows from Nagao's Theorem and Lemma 5. □

5.3 The groups $\mathrm{SL}(2,tK[t])$ *and* $\mathrm{SL}_{U(K)}(2,K[t])$ *are free products*

For $S \subset \mathrm{SL}(2,K)$, the group $\mathrm{GL}_S(2,K[t])$ lies in $\mathrm{SL}_S(2,K[t])$. Thus set

$$\mathrm{SL}_S(2,K[t]) := \mathrm{GL}_S(2,K[t]).$$

Let δ_0, δ_∞ be the points in $\mathbb{P}^1_K$ with coordinates $(0;1)$ and $(1;0)$.

Lemma 19. *We have*

$$\mathrm{SL}(2,tK[t]) = *_{\delta\in\mathbb{P}^1_K} U_\delta(tK[t], \text{ and}$$
$$\mathrm{SL}_{U(K)}(2,K[t]) = U([K[t]) * U_{\delta_\infty}(tK[t]).$$

Proof. The first assertion follows from Lemma 18. Since the action of $U(K)$ on $\mathbb{P}^1_K$ is almost free transitive, the second point follows from Lemma 8. □

Remark. The group $\mathrm{SL}_{U(K)}(2,K[t])$ is the "lower nilradical" of the affine Kac-Moody group $\mathrm{SL}(2,K[t,t^{-1}])$. In [26], Tits defined the "lower nilradical" of any Kac-Moody group in term of an inductive limit, which is essentialy equivalent to the previous lemma for $\mathrm{SL}_{U(K)}(2,K[t])$.

Since the notes [26] are not widely distributed, let us mention that an equivalent result is stated in [28], Section 3.2 and 3.2, see also [27].

5.4 Proof of Theorem A.2

Lemma 20. *There are isomorphisms*

$$\mathrm{Aut}_1 \mathbb{A}^2_K \simeq \mathrm{SL}(2, tK[t]) \text{ and } \mathrm{Aut}_{U(K)} \mathbb{A}^2_K \simeq \mathrm{SL}_{U(K)}(2, K[t]).$$

Proof. By Lemmas 7 and 19, $\mathrm{Aut}_1 \mathbb{A}^2_K$ and $\mathrm{SL}(2, tK[t])$ are free products of $\mathrm{Card}\, \mathbb{P}^1_K$ copies of a K-vector space of dimension $\aleph_0$. Therefore these two groups are isomorphic.

The proof of the isomorphism $\mathrm{Aut}_{U(K)} \mathbb{A}^2_K \simeq \mathrm{SL}_{U(K)}(2, K[t])$ follows similarly from Lemmas 10 and 19. □

Theorem A.2. *The groups* $\mathrm{Aut}_1 \mathbb{A}^2_K$ *and* $\mathrm{Aut}_{U(K)} \mathbb{A}^2_K$ *embed in* $\mathrm{SL}(2, K(t))$.

Moreover if $K \supset k(t)$ for some infinite field k, then there exists an embedding $\mathrm{Aut}_1 \mathbb{A}^2_K \subset \mathrm{Aut}_{U(K)} \mathbb{A}^2_K \subset \mathrm{SL}(2, K)$.

Proof. It follows from Lemma 20 that $\mathrm{Aut}_1 \mathbb{A}^2_K$ and $\mathrm{Aut}_{U(K)} \mathbb{A}^2_K$ are subgroups of $\mathrm{SL}(2, K(t)$, and therefore they are linear over $K(t)$.

Assume now that $K \supset k(t)$ for some infinite field k. We claim that there exists a field L with $L(t) \subset K$ and $\mathrm{Card}\, L = \mathrm{Card}\, K$. If $\mathrm{Card}\, K = \aleph_0$, then the subfield k satisfies the claim. Otherwise, we have $\mathrm{trdeg}\, K > \aleph_0$ and there is an embedding $L(t) \subset K$ for some subfield L with $\mathrm{trdeg}\, L = \mathrm{trdeg}\, K$. Since $\mathrm{trdeg}\, L = \mathrm{Card}\, L = \mathrm{Card}\, K$, the claim is proved.

It follows from Corollary 1 that

$$\mathrm{Aut}_{U(K)} \mathbb{A}^2_K \simeq \mathrm{Aut}_{U(K)} \mathbb{A}^2_L \subset \mathrm{SL}(2, L(t)) \subset \mathrm{SL}(2, K),$$

therefore $\mathrm{Aut}_1 \mathbb{A}^2_K$ and $\mathrm{Aut}_{U(K)} \mathbb{A}^2_K$ are subgroups of $\mathrm{SL}(2, K)$. □

5.5 A Corollary

For a a finite field K, $\mathrm{Aut}_1 \mathbb{A}^2_K$ has finite index in $\mathrm{Aut}\, \mathbb{A}^2_K$, hence

Corollary 4. *For a a finite field K, the group* $\mathrm{Aut}\, \mathbb{A}^2_K$ *is linear over $K(t)$.*

6 The Linear Representation of $\mathrm{SAut}_0^{<n} \mathbb{A}^2_K$

For $n \geq 3$, let $\mathrm{SAut}_0^{<n} \mathbb{A}^2_K$ be the subgroup of $\mathrm{SAut}_0 \mathbb{A}^2_K$ generated by all automorphisms $\phi \in \mathrm{SAut}_0 \mathbb{A}^2_K$ of degree $< n$. In this section, *we will assume that K has characteristic zero*, in order to show that the group $\mathrm{SAut}_0^{<n} \mathbb{A}^2_K$

is linear, what proves Theorem C.2. Unfortunately, our approach does not extend to fields of finite characteristic.

For a nonzero vector-valued polynomial $v(t) = \sum v_i t^i$, let $\deg v$ be its degree and let $\mathrm{hdc}(v) := v_{\deg v}$ be its *highest degree component*.

6.1 A ping-pong lemma

Let S be a group and let $\Gamma = S \ltimes *_{p\in P} F_p$ be a mixed product of S.

Lemma 21. *Let Ω be a Γ-set, and let $(\Omega_p)_{p\in P}$ be a collection of subsets in Ω. Set $F_p^* = F_p \setminus \{1\}$ and assume*

(i) the free product $*_{p\in P} F_p$ is nondegenerate and nondihedral,

(ii) $\mathrm{Core}_\Gamma(\cap_P S_p)$ is trivial,

(iii) the subsets Ω_p are nonempty and disjoint, and

(iv) we have $F_p^*.\Omega_q \subset \Omega_p$ whenever $p \neq q$.

Then Γ acts faithfully on Ω.

Proof. Let $p \neq p'$ be two elements in P. By Lemma 14, Assertion (i) any nontrivial normal subgroup of Γ contains some $\gamma \in E_{p,p}$. We have $\gamma.\Omega_{p'} \subset \Omega_p$, therefore γ acts nontrivially on Ω. Since any nontrivial normal subgroup acts nontrivially, the action of Γ on Ω is faithful. □

6.2 The group $\mathrm{SAut}_0^{<n} \mathbb{A}_K^2$ is a mixed product with trivial core

Let $\delta \in \mathbb{P}^1$ with coordinates $(a; b)$. For $n \geq 3$, let $E_\delta^{<n}(K) \subset E_\delta(K)$ be the subgroup of all automorphisms of the form $(x, y) \mapsto (x, y) + f(bx - ay)(a, b)$ where $f(t) \in t^2K[t]$ and $\deg f(t) < n$.

Lemma 22. *For any $n \geq 3$, the group $\mathrm{SAut}_0^{<n} \mathbb{A}_K^2$ is isomorphic to the nondegenerate mixed product*

$$\Gamma := \mathrm{SL}(2, K) \ltimes *_{\delta\in\mathbb{P}_K^1} E_\delta^{<n}(K).$$

Moreover $\mathrm{Core}_\Gamma(\mathrm{SL}(2, K))$ is trivial.

Proof. Let $u \in *_{\delta\in\mathbb{P}_K^1} E_\delta(K)$ with reduced decomposition $u_1 \dots u_m$, where $u_i \in E_{\delta_i}(K)$. By induction over n, it is easy to prove simultaneously that $\deg u = \prod \deg u_i$ and that $\mathrm{hdc}(u)$ is of the form $(x, y) \mapsto (bx - ay)^{\deg u}(c, d)$, where $(c; d)$ and $(a; b)$ are some coordinates of δ_1 and δ_n.

By Lemma 7, $\mathrm{SAut}_0 \mathbb{A}_K^2$ is isomorphic to the mixed product

$$\mathrm{SL}(2, K) \ltimes *_{\delta\in\mathbb{P}_K^1} E_\delta(K).$$

Any $\phi \in \mathrm{SAut}_0 \mathbb{A}_K^2$ decomposes uniquely as $\phi = su_1 \dots u_m$, where $s \in \mathrm{SL}(2, K)$ and $u_1 \dots u_m$ is a reduced decomposition in $*_{\delta\in\mathbb{P}_K^1} E_\delta(K)$. Since $\deg \phi = \prod \deg u_i$, we have $\mathrm{SAut}_0^{<n} \mathbb{A}_K^2 = \langle \mathrm{SL}(2, K), E_\delta^{<n}(K)\rangle$. Thus

$$\mathrm{SAut}_0^{<n}\,\mathbb{A}^2_K \simeq \mathrm{SL}(2,K) \ltimes *_{\delta\in\mathbb{P}^1_K} E_\delta^{<n}(K).$$

It has been noticed that $\mathrm{Core}_\Gamma(\mathrm{SL}(2,K))$ acts trivially on $\mathbb{P}^1_K$, hence it is included in $\{1,\sigma\}$, where $\sigma(x,y) = (-x,-y)$. Set $\tau(x,y) := (x, y+x^2)$. Since $\tau^\sigma(x,y) = (x, y-x^2)$, it follows that $\mathrm{Core}_\Gamma(\mathrm{SL}(2,K))$ is trivial. □

6.3 The square root η of e

Let ϵ be an odd variable. For an integer $N \geq 1$, let $L(N) \subset K[x,y]$ and $\hat{L}(N) \subset K[x,y,\epsilon]$ be the subspaces of homogenous polynomials of degree N. Let (e,h,f) be the usual basis of $\mathfrak{sl}(2,K)$. As an $SL(2,K)$-module, we have $\hat{L}(N) = L(N) \oplus L(N-1)$ and e acts as the derivation $x\frac{\partial}{\partial y}$. Set

$$\eta = x\tfrac{\partial}{\partial \epsilon} + \epsilon\tfrac{\partial}{\partial y}.$$

It is clear that $\eta^2 = e$. Indeed $\hat{L}(N)$ is a representation of the Lie superalgebra $\mathfrak{osp}(1,2)$, and $\eta \in \mathfrak{osp}(1,2)$ is an odd element such that $\eta^2 = e$.

For any $\delta \in \mathbb{P}^1_K$ with projective coordinates $(a;b)$, set $L_\delta := K.(ax+by)^N$ and $L^*_\delta = L_\delta \setminus \{0\}$. Let $\delta_0, \delta_\infty \in \mathbb{P}^1_K$ be the points with projective coordinates $(0;1)$ and $(1;0)$. Since $\eta^{2N}.y^N = (x\frac{\partial}{\partial y})^N y^N = N!\,x^N$, it follows that

Lemma 23. *We have $\eta^{2N}.L^*_{\delta_\infty} \subset L^*_{\delta_0}$.*

6.4 The representation ρ_N of $\mathrm{SAut}_0\,\mathbb{A}^2_K$ on $\hat{L}(N)\otimes K[t]$

We will extend the natural representation of of $\mathrm{SL}(2,K)$ on $\hat{L}(N)\otimes K[t]$ to $\mathrm{SAut}_0\,\mathbb{A}^2_K$ as follows. For any automorphism $\tau \in E(K)$, set

$$\rho_N(\tau) = \exp(t\eta f(\eta)),$$

if $\tau(x,y) = (x, y+f(x))$, where $f(x) \in x^2K[x]$. Since $[e,\eta] = 0$ and $[h,\eta] = \eta$, the homomorphism ρ_N is $B(K)$-equivariant. By Lemma 3, ρ_N extends to a $K[t]$-linear action of $\mathrm{SAut}_0\,\mathbb{A}^2_K$.

Lemma 24. *Assume that $2N$ is divisible by* $\mathrm{lcm}(1,2,\dots,n)$.
Then the restriction of ρ_N to $\mathrm{SAut}_0^{<n}\,\mathbb{A}^2_K$ is faithful.

Proof. For any $\delta \in \mathbb{P}^1_K$, set $F^*_\delta = E^{<n}_\delta(K) \setminus \{1\}$ and

$$\Omega_\delta = \{v(t) \in \hat{L}(N)\otimes K[t] \setminus \{0\} \mid \mathrm{hdc}(v(t)) \in L^*_\delta\}.$$

First step. Let $\delta_0, \delta_\infty \in \mathbb{P}^1_K$ be as in Lemma 23. We claim that

$$F^*_{\delta_0}.\Omega_{\delta_\infty} \subset \Omega_{\delta_0}.$$

Let $\tau(x,y) = (x, y+f(x))$ be in $F^*_{\delta_0}$. We have $f(x) = ax^k+$ higher terms, for some $a \in K^*$ and some k with $2 \leq k < n$. By definition, we have

$$\rho_N(\tau) = \exp t\eta f(\eta) = \textstyle\sum_{m\geq 0} \frac{\eta^m f(\eta)^m}{m!}\, t^m.$$

Since $\eta f(\eta)$ is divisible by η^{k+1} and $\eta^{2N+1} = 0$, it follows that $\eta^m f(\eta)^m = 0$ for $m > 2N/(k+1)$. Since $k+1$ divides $2N$, $\rho_N(\tau)$ is a polynomial of degree $d := 2N/(k+1)$ and we have

$$\mathrm{hdc}(\rho_N(\tau)) = \tfrac{a^d}{d!}\,\eta^{2N}.$$

Let $v(t) \in \Omega_{\delta_\infty}$. By Lemma 23, $\mathrm{hdc}(\rho_N(\tau)).\mathrm{hdc}(v(t))$ is nonzero and belongs to $L^*_{\delta_0}$. It follows that $\rho_N(\tau).\Omega_{\delta_\infty} \subset \Omega_{\delta_0}$, what proves the claim.

Second step: use of the ping-pong lemma. Let $\delta \neq \delta'$ in $\mathbb{P}^1_K$. Since $F^*_\delta \times \Omega_{\delta'}$ is conjugate under $\mathrm{SL}(2, K)$ to $F^*_{\delta_0} \times \Omega_{\delta_\infty}$, the previous result implies that

$$F^*_\delta.\Omega_{\delta'} \subset \Omega_\delta.$$

By Lemmas 21 and 22, the restriction of ρ_N to $\mathrm{SAut}_0^{<n}\,\mathbb{A}^2_K$ is faithful. $\square$

6.5 Proof of Theorem C.2

Since $\dim \hat{L}(N) = 2N + 1$, Lemma 24 implies that

Theorem C.2. *For any $n \geq 3$, there is an embedding*

$$\mathrm{SAut}_0^{<n}\,\mathbb{A}^2_K \subset \mathrm{SL}(1 + \mathrm{lcm}(1, 2, \ldots, n), K(t)).$$

In particular, any f.g. subgroup of $\mathrm{SAut}_0\,\mathbb{A}^2_K$ is linear over $K(t)$.

7 Semi-algebraic Characters

Let $\Lambda \subset K^*$ be a subgroup. For any $n \geq 1$, let $K_n \subset K$ be the subfield generated by Λ^n. Let L be an algebraically closed field, which contains at least one subfield isomorphic to K_1 and let $\mathbb{F}$ be the ground field of K.

For $n \geq 1$, a group homomorphism $\chi : \Lambda \to L^*$ is called a *semi-algebraic character* of degree n if $\chi(z) = \mu(z^n)$ for some field embedding $\mu : K_n \to L$. Let $\mathcal{X}_n(\Lambda)$ be the set of all semi-algebraic characters of Λ of degree n. The degree of a semi-algebraic character is *not uniquely defined*. Given $n \neq m$, we will show a criterion for the disjointness of $\mathcal{X}_n(\Lambda)$ and $\mathcal{X}_m(\Lambda)$.

7.1 The invariant $I_n(\Lambda)$

Let $\mathbb{F}[\Lambda]$ be the group algebra of Λ. Given a field $E \supset \mathbb{F}$, any homomorphism $\chi : \Lambda \to E^*$ extends to an algebra homomorphism $\hat{\chi} : \mathbb{F}[\Lambda] \to E$. Set

$$\mathrm{Ker}\,\hat{\chi} := \{\textstyle\sum_\lambda a_\lambda \lambda \in \mathbb{F}[\Lambda] \mid \sum_\lambda a_\lambda \chi(\lambda) = 0\}.$$

For $n \geq 1$, let χ_n be the homomorphism $\chi_n : \lambda \in \Lambda \mapsto \lambda^n \in K^*_n$. Set

$$I_n(\Lambda) = \mathrm{Ker}\,\hat{\chi}_n.$$

Lemma 25. *A group homomorphism $\chi : \Lambda \to L^*$ is a semi-algebraic character of degree n iff* $\operatorname{Ker} \hat{\chi} = I_n(\Lambda)$.

In particular, we have $\mathcal{X}_n(\Lambda) = \mathcal{X}_m(\Lambda)$ or $\mathcal{X}_n(\Lambda) \cap \mathcal{X}_m(\Lambda) = \emptyset$, for any positive integers $n \neq m$.

Proof. By definition, the fraction field of the prime ring $\mathbb{F}[\Lambda]/I_n(\Lambda)$ is K_n. Hence $\hat{\chi}$ factors through K_n, i.e. $\hat{\chi} = \mu \circ \hat{\chi}_n$ for some field embedding $\mu : K_n \to L$. The first point follows, as well as the second. □

7.2 Minimally bad subgroups of K^*

Let $\Lambda \subset K^*$ be a subgroup. By definition, the *transcendental degree* of Λ is $\operatorname{trdeg} \Lambda := \operatorname{trdeg} K_1$ and its *rank* is $\operatorname{rk} \Lambda := \dim \Lambda \otimes \mathbb{Q}$. Both are cardinals and we have $\operatorname{trdeg} \Lambda \leq \operatorname{rk} \Lambda$. We say that Λ is a *good subgroup* of K^* if $\operatorname{trdeg} \Lambda' = \operatorname{rk} \Lambda'$, for any f.g. subgroup Λ' of Λ and a *bad subgroup* otherwise.

Assume now that Λ is a free abelian group of rank $r < \infty$, with basis $x_1, \dots, x_r$. The ring $\mathbb{F}[\Lambda]$ is isomorphic to the ring $\mathbb{F}[x_1^{\pm 1}, \dots x_r^{\pm 1}]$ of Laurent polynomials. For $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{Z}^r$, set $x^\alpha = x_1^{\alpha_1}, \dots x_r^{\alpha_r}$. The *support* of a Laurent polynomial $P = \sum_{\alpha \in \mathbb{Z}^r} a_\alpha x^\alpha$ is the set

$$\operatorname{Supp} P := \{\alpha \in \mathbb{Z}^r \mid a_\alpha \neq 0\}.$$

Assume now that $\operatorname{trdeg} \Lambda = r - 1$. Since $\mathbb{F}[x_1^{\pm 1}, \dots x_r^{\pm 1}]$ is a unique factorization domain, $I_1(\Lambda)$ is a principal ideal. If P be one of its generator, the other generators are the polynomials $a x^\gamma P$, for $a \in \mathbb{F}^*$ and $\gamma \in \mathbb{Z}^r$. Hence the subgroup $X(\Lambda) \subset \mathbb{Z}^r$ generated by $\alpha - \beta$ for $\alpha, \beta \in \operatorname{Supp} P$ only depends on Λ. Moreover, if $0 \in \operatorname{Supp} P$, then we have $X(\Lambda) = \langle \operatorname{Supp} P \rangle$.

A subgroup $\Lambda \subset K^*$ is called *minimally bad* if

(i) Λ is a f.g. free abelian group, and

(ii) we have $\operatorname{rk} \Lambda = 1 + \operatorname{trdeg} \Lambda$ and $X(\Lambda) = \mathbb{Z}^r$, where $r = \operatorname{rk} \Lambda$.

Lemma 26. *Let $\Lambda \subset K^*$ be a bad subgroup of K^*.*

Then Λ contains a minimally bad subgroup Λ'.

Proof. By definition, Λ contains a f.g. bad subgroup Λ_0. Moreover, we can assume that Λ_0 is torsion free.

Let $\mathcal{C}$ be the set of all subgroups $\Pi \subset \Lambda_0$ such that $\operatorname{rk} \Pi > \operatorname{trdeg} \Pi$. Let us pick one element Π' of $\mathcal{C}$ of minimal rank r. It is clear that $\operatorname{rk} \Pi' = 1 + \operatorname{trdeg} \Pi'$. Let $x_1, \dots x_r$ be a basis of Π' and let $P = \sum_{x \in \mathbb{Z}^r} a_\alpha x^\alpha$ be a generator of the ideal $I_1(\Pi')$ of $\mathbb{F}[\Pi'] \simeq \mathbb{F}[x_1^{\pm 1}, \dots x_r^{\pm 1}]$ such that $a_0 \neq 0$.

It is clear that $\Lambda' := \langle x^\alpha \mid \alpha \in \operatorname{Supp} P \rangle$ is a minimally bad subgroup. □

7.3 The Newton polygon of P_n

Let $r \geq 1$ be an integer. Let P be a generator of a principal ideal I of $\mathbb{F}[x_1^{\pm 1}, \dots x_r^{\pm 1}]$. The *Newton polygon* $\mathrm{Newton}(P)$ of P is the convex closure of $\mathrm{Supp}\, P$ in $\mathbb{R} \otimes \mathbb{Z}^r$, and let $\mathrm{Ext}(P)$ be its set of extremal points. Up to translation by $\mathbb{Z}^r$, $\mathrm{Ext}(P)$ is an invariant of I. Hence the largest integer $e(P)$ such that $\alpha - \beta \in e(P).\mathbb{Z}^r$ for any α, $\beta \in \mathrm{Ext}(P)$ only depends on I.

It will convenient to choose an ordering of $\mathbb{Z}^r$. A Laurent polynomial $P \in \mathbb{F}[x_1^{\pm 1}, \dots x_r^{\pm 1}]$ is *normalized* if $a_0 = 1$ and any $\alpha \in \mathrm{Supp}\, P$ is nonnegative. Any principal ideal I of $\mathbb{F}[x_1^{\pm 1}, \dots x_r^{\pm 1}]$ has a unique normalized generator P. Since 0 belongs to $\mathrm{Ext}(P)$, we have $\mathrm{Ext}(P) \subset e(P).\mathbb{Z}^r$.

Let $\Lambda \subset K^*$ be a minimally bad subgroup and let $x_1, \dots, x_r$ be a basis of Λ. For any $n \geq 1$, let P_n be the normalized generator of $I_n(\Lambda)$.

Lemma 27. *Assume that $n \geq 1$ is not divisible by* ch K. *Then we have*

$$\mathrm{Newton}(P_n) = \frac{n^{r-1}}{f_n} \mathrm{Newton}(P_1),$$

for some integer f_n dividing $e(P_1)$.

Proof. In $\overline{\mathbb{F}}[x_1^{\pm 1}, \dots x_r^{\pm 1}]$ the polynomial P_1 decomposes uniquely as

$$P_1 = Q_1 \dots Q_k$$

where Q_1, $Q_2, \dots$ are normalized irreducible polynomials in $\overline{\mathbb{F}}[x_1^{\pm 1}, \dots x_r^{\pm 1}]$. Since they are permuted by $\mathrm{Gal}(\mathbb{F})$, we have $\mathrm{Supp}\, Q_1 = \mathrm{Supp}\, Q_2 ...$, hence

(i) $\mathrm{Supp}\, Q_1$ generates $\mathbb{Z}^r$, and

(ii) $\mathrm{Newton}(Q_1) = \frac{1}{k} \mathrm{Newton}(P_1)$ and k divides $e(P_1)$.

Let $\mu_n \subset \overline{\mathbb{F}}^*$ be the group of all nth root of one. For various $(\zeta_1, \dots \zeta_r) \in \mu_n^r$, the normalized polynomials $Q_1(\zeta_1 x_1, \dots, \zeta_r x_r)$ are pairwise distinct by the assertion (i). Thus the polynomial $R = \prod_{\zeta_1, \dots, \zeta_r \in \mu_n^r} Q_1(\zeta_1 x_1, \dots, \zeta_r x_r)$ is irreducible in $\overline{\mathbb{F}}[x_1^{\pm n}, \dots x_r^{\pm n}]$.

Set $G_1 = \{\sigma \in \mathrm{Gal}(\mathbb{F}) \mid Q_1^\sigma = Q_1\}$ and $G = \{\sigma \in \mathrm{Gal}(\mathbb{F}) \mid R^\sigma = R\}$. Since R is normalized, the polynomials R^σ for $\sigma \in \mathrm{Gal}(\mathbb{F})/G$ are pairwise distinct, and therefore $S = \prod_{\sigma \in \mathrm{Gal}(\mathbb{F})/G} R^\sigma$ is a normalized irreducible polynomial in $\mathbb{F}[x_1^{\pm n}, \dots x_r^{\pm n}]$. It follows that

$$P_n(x_1^n, \dots, x_r^n) = S(x_1, \dots, x_r).$$

Since $[\mathrm{Gal}(\mathbb{F}) : G] = k$, the integer $f_n := [G : G_1]$ divides $e(P_1)$, and

$$\mathrm{Newton}(P_n) = 1/n \, \mathrm{Newton}(S) = \frac{n^{r-1}}{f_n} \mathrm{Newton}(P_1).$$

□

7.4 A criterion for the disjointness of $\mathcal{X}_n(\Lambda)$ and $\mathcal{X}_m(\Lambda)$

Lemma 28. *Let $\Lambda \subset K^*$ be a bad subgroup of K^*. Then there is an integer $e \geq 1$ such that*

$$\mathcal{X}_n(\Lambda) \cap \mathcal{X}_m(\Lambda) = \emptyset,$$

whenever the integers $n \neq m$ are coprime to e.

Proof. By Lemma 26, Λ contains a minimally bad subgroup Λ'. Moreover by Lemma 25, it is enough to show the lemma for Λ'. Therefore we can assume that Λ itself is minimally bad. For any $n \geq 1$, let P_n be the normalized generator of $I_n(\Lambda)$, relative to some order of Λ.

Proof when $\operatorname{rk}\Lambda = 1$. Thus $\mathbb{F} = \mathbb{Q}$ and a generator λ of Λ is an algebraic number of infinite order. Let n, $m \geq 1$ be two integers such that $P_n = P_m$. Since $P_n(\lambda^m) = 0$, there is $\sigma \in \operatorname{Gal}(\mathbb{Q})$ such that $\lambda^m = \sigma(\lambda^n)$. Let $k \geq 1$ be an integer such that $\sigma^k(\lambda) = \lambda$. Therefore, we have

$$\lambda^{n^k} = \sigma^k(\lambda^{n^k}) = \lambda^{m^k}.$$

Since λ has infinite order, it follows that $n = m$. In this case, the lemma is proved for $e = 1$.

Proof when $r := \operatorname{rk}\Lambda \geq 2$. Set $e = pe(P_1)$ if $\operatorname{ch} K = p$ and $e = e(P_1)$ ortherwise. Let n, $m \geq 1$ be two integers coprime to e such that $P_n = P_m$. By Lemma 27, we have $\frac{n^{r-1}}{f_n} = \frac{m^{r-1}}{f_m}$ for some integers f_n and f_m dividing e. It follows that $n = m$. □

8 A Nonlinearity Criterion for $\operatorname{Aut}_S \mathbb{A}^2_K$

Let S_0 be a subgroup of $B(K)$. We will use Lemma 28 to give a necessary condition for the linearity over a field of $S_0 \ltimes E(K)$. Then we derive a nonlinearity criterion for the groups $\operatorname{Aut}_S \mathbb{A}^2_K$.

From now on, let $\rho : S_0 \ltimes E(K) \to \operatorname{GL}(V)$ be a a given embedding, where V is a finite-dimensional vector space over an algebraically closed field L. Let $W(\rho) \subset \operatorname{End} V$ be the linear subspace generated by $\rho(E(K))$.

The commutative group structure on $E(K)$ will be denoted additively. Indeed $E(K)$ has a natural structure of a graded vector space over K, namely $E(K) = \oplus_{n\geq 3} K.T_n$ where $T_n(x, y) = (x, y + x^{n-1})$.

For any $g \in B(K)$, set $\chi_B(g) = \lambda$ if $g(x, y) = (\lambda^{-1}x, \lambda y + tx)$, for some $t \in K$. The group of all eigenvalues of elements in S_0 is $\Lambda := \chi_B(S_0) \subset K^*$. We have $gT_n g^{-1} = \chi_B(g)^n T_n$. Since the action of S_0 on $E(K)$ factors through Λ, it follows that $E(K)$ and $W(\rho)$ are Λ-modules.

8.1 An obvious estimate

A integer $n \geq 1$ is called a *divisor of* Λ if Λ contains a primitive nth root of one. Let $d(\Lambda)$ be the number, finite or infinite, of divisors of Λ.

Lemma 29. *We have* $d(\Lambda) \leq 2 + (\dim V)^2$.

Proof. For each divisor n with $n \geq 3$, set $t_n = \rho(T_n)$. Since $g t_n g^{-1} = t_n$ iff $\chi(g)^n = 1$, it follows that the elements t_n are linearly independant. Thus we have $\dim \operatorname{End} V \geq d(\Lambda) - 2$, from which the assertion follows. $\square$

8.2 Unipotent representations

Lemma 30. *Assume that* $\operatorname{rk}\Lambda \geq 1$.
Then $\operatorname{ch} L = \operatorname{ch} K$ *and* $\rho(E(K))$ *is a unipotent group.*

Proof. If $\operatorname{ch} K = p$, the assertions follow from the fact that $E(K)$ is an elementary p-group of infinite rank.

We will now assume that $\operatorname{ch} K = 0$. Let $V = \oplus_{\chi \in \Omega} V_{(\chi)}$ be the generalized weight decomposition of the $E(K)$-module V, where Ω is the set of group homomorphisms $\chi : E(K) \to L^*$ such that $V_{(\chi)} \neq 0$.

Let $\chi \in \Omega$. Since Ω is finite, the group $S_0' = \{s \in S_0 \mid \chi^s = \chi\}$ has finite index in S_0. There is some $s \in S_0'$ such that $\chi_B(s)$ has infinite order. It follows that the map $e \in E(K) \mapsto e - s e s^{-1} \in E(K)$ is invertible. Therefore χ is trivial and $\rho(E(K))$ is a unipotent group.

Since $E(K)$ is torsion-free, L has characteristic 0. $\square$

8.3 A linearity criterion for $S_0 \ltimes E(K)$

Lemma 31. *Assume again that* ρ *is a faithful representation of* $S_0 \ltimes E(K)$.
Then Λ *is a good subgroup of* K^*.

Proof. Since any zero-rank subgroup of K^* is good, we can assume that $\operatorname{rk}\Lambda \geq 1$. For $n \geq 3$, set $E_n = K.T_n$. By Lemma 30, K and L have the same ground field $\mathbb{F}$ and, $\rho(E_n)$ is a unipotent group.

We claim that, for any $n \geq 3$, *there is a* $L[\Lambda]$*-submodule* $W' \subset W(\rho)$ *such that* $W(\rho)/W'$ *contains a* $\mathbb{F}[\Lambda]$*-submodule* X *isomorphic to* $\mathbb{F}[\Lambda]/I_n(\Lambda)$.

Proof for $\mathbb{F} = \mathbb{Q}$. In that case $W' = \{0\}$ and $X := \log \rho(\mathbb{Q}[\Lambda].T_n)$ is a $\mathbb{Q}[\Lambda]$-submodule of $W(\rho)$ isomorphic to $\mathbb{Q}[\Lambda]/I_n(\Lambda)$.

Proof for $\mathbb{F} = \mathbb{F}_p$. As a substitute for the log, set $\theta(a) = 1 - \rho(a)$ for $a \in E_n$. Let $M \subset W(\rho)$ be the linear space generated by $\theta(E_n)$. Since

$$\theta(a)\theta(b) = \theta(a) + \theta(b) - \theta(a+b),$$

M is a nonunital algebra and $M^p = \{0\}$. Since ρ is injective, we have $\theta^{-1}(M^p) = \{0\}$. Thus there exists a unique integer $m < p$ such that

$$Y := \theta^{-1}(M^m) \neq \{0\} \text{ but } \theta^{-1}(M^{m+1}) = \{0\}.$$

Set $W' = M^{m+1}$. It follows from the previous formula that Y is a subgroup of E_n and the induced map $\overline{\theta} : Y \to W(\rho)/W'$ is additive. Thus $\overline{\theta}$ is a homomorphism of $\mathbb{F}_p[\Lambda]$-modules. Since $\operatorname{Ker} \overline{\theta}$ is trivial, $\overline{\theta}$ is injective. Any cyclic $\mathbb{F}_p[\Lambda]$-sumodule of Y is isomorphic to $\mathbb{F}_p[\Lambda]/I_n(\Lambda)$, therefore $W(\rho)/W'$ contains a $\mathbb{F}_p[\Lambda]$-submodule X isomorphic to $\mathbb{F}_p[\Lambda]/I_n(\Lambda)$.

We claim now that, for any $n \geq 3$, $W(\rho)_{(\chi)} \neq 0$ for some $\chi \in \mathcal{X}_n(\Lambda)$. Let Z be the L-vector space generated by X, and let $Z = \oplus_{\chi \in \Omega_Z} Z_{(\chi)}$ be the decomposition of the $L[\Lambda]$-module Z into generalized weight spaces, where Ω_Z is the set of group homomorphisms $\chi : \Lambda \to L^*$ such that $Z_{(\chi)} \neq 0$. For each $\chi \in \Omega_Z$, let I_χ be the annihilator in $\mathbb{F}[\Lambda]$ of $Z_{(\chi)}$. It follows that

$$I_n(\Lambda) = \cap_{\chi \in \Omega_Z} I_\chi.$$

Since Ω_Z is finite and $I_n(\Lambda)$ is a prime ideal, we have $I_n(\Lambda) = I_\chi$, for some $\chi \in \Omega_Z$. Moreover the radical of I_χ is $\operatorname{Ker}\hat{\chi}$, hence $I_n(\Lambda) = \operatorname{Ker}\hat{\chi}$. Thus by Lemma 25, χ is a semi-algebraic character of degree n. Moreover we have $W(\rho)_{(\chi)} \neq 0$, what proves the claim.

End of the proof. Assume otherwise, namely that Λ is a bad subgroup of K^*. Then by Lemma 28, there is an infinite set T of integers $n \geq 3$ such that the family $(\mathcal{X}_n(\Lambda))_{n \in T}$ consists of mutually disjoint sets. This would contradict that the finite set of generalized weights of $W(\rho)$ intersects each of them. □

8.4 Proof of the Nonlinearity Criterion

Let S be a subgroup of $\mathrm{SL}(2, K)$. For $\delta \in \mathbb{P}^1_K$, let $\Lambda_\delta \subset K^*$ be the subgroup of all eigenvalues of elements $g \in S_\delta$.

Nonlinearity Criterion. *Assume one of the following two hypotheses*

(i) Λ_δ is a bad subgroup of K^* for some $\delta \in \mathbb{P}^1_K$, or

(ii) the function $\delta \in \mathbb{P}^1_K \to d(\Lambda_\delta)$ is unbounded.

Then the group $\operatorname{Aut}_S \mathbb{A}^2_K$ is not linear, even over a ring.

Proof. By contraposition, we will assume that $\operatorname{Aut}_S \mathbb{A}^2_K$ is linear over a ring, and we will show that neither Assertion (i) nor Assertion (ii) holds.

By Lemma 7, $\operatorname{Aut}_S \mathbb{A}^2_K$ is the mixed product $\Gamma := S \ltimes *_{\delta \in \mathbb{P}^1_K} E_\delta(K)$. Since $\cap_{\delta \in \delta} S_\delta \subset \{\pm 1\}$ and $T_3^{-\mathrm{id}} = -T_3$, it follows that $\operatorname{Core}_\Gamma(\cap_{\delta \in \delta} S_\delta)$ is trivial. Hence by Corollary 3, $\operatorname{Aut}_S \mathbb{A}^2_K$ is linear over a field. Let $\rho : \operatorname{Aut}_S \mathbb{A}^2_K \to$

$\mathrm{GL}(n, L)$ be an injective homomorphism, for some algebraically closed field L and some positive integer n.

Since ρ provides a faithful representation of $B_\delta \ltimes E_\delta(K)$, it follows from Lemma 31 that Λ_δ is a good subgroup of K^*. Moreover by Lemma 29, we have $d(\Lambda_\delta) \leq 2 + n^2$.

Therefore neither Assertion (i) nor Assertion (ii) holds. $\square$

8.5 Proof of the Theorem A.1 of the introduction.

Theorem A.1. *For any infinite field,* $\mathrm{SAut}_0\, \mathbb{A}^2_K$ *is not linear, even over a field.*

Proof. With the previous notations, we have $\mathrm{Aut}_0\, \mathbb{A}^2_K = \mathrm{Aut}_{\mathrm{SL}(2,K)}\, \mathbb{A}^2_K$, and $\Lambda_\delta = K^*$, for any $\delta \in \mathbb{P}^1_K$. Therefore it is enough to check that K^* satifies one of the two assertions of the Nonlinearity Criterion.

If K is an infinite subfield of $\overline{\mathbb{F}}_p$, then $d(K^*)$ is infinite. Otherwise K contains $\mathbb{Q}$ or a transcendental element t. For the subgroup $\Lambda := \langle 2 \rangle$ in the first case or $\Lambda := \langle t, t+1 \rangle$ in the second case, it is clear that $\mathrm{rk}\, \Lambda > \mathrm{trdeg}\, \Lambda$. Hence K^* itself is a bad group. $\square$

8.6 Comparison with Cornulier's Theorem

Let G_{Cor} be the group of all automorphisms of $\mathbb{A}^2_{\mathbb{C}}$ of the form

$(x, y) \mapsto (x + u, y + f(x))$, for some $u \in \mathbb{C}$ and $f(t) \in \mathbb{C}[t]$.

The group G_{Cor} is locally nilpotent but not nilpotent [9], hence

Cornulier's Theorem. *Neither* G_{Cor} *nor* $\mathrm{Aut}\, \mathbb{A}^2_{\mathbb{C}}$ *is linear over a field.*

Set $R := \mathbb{C}[[t]] \oplus I$, where $I := \mathbb{C}((t))/\mathbb{C}[[t]]$ is a square-zero ideal. Let Γ be the subgroup of $\mathrm{SL}(2, R)$ generated by the matrices

$$\begin{pmatrix} 1+t & 0 \\ 0 & (1+t)^{-1} \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \text{ when } a \text{ runs over } I.$$

The group Γ is isomorphic to $G_{\mathrm{Cor}} \simeq \mathbb{C} \ltimes \mathbb{C}[x]$, where $\mathbb{C}$ acts by translation on $\mathbb{C}[x]$, hence G_{Cor} is linear over R. Similarly, the subgroup $\mathrm{SElem}_0(K)$ is isomorphic to $K^* \ltimes x^2 K[x]$, where K^* acts on $x^2 K[x]$ by multiplication. Since $\mathrm{SElem}_0(K)$ embeds into $\prod_{n \geq 2} (K^* \ltimes K x^n)$, it is a subgroup of $\mathrm{GL}(2, K^\infty)$.

Therefore both G_{Cor} and $\mathrm{SElem}_0(K)$ (for K infinite) are linear over some rings but not over a field. This explains the motivation of Section 3.

9 Two Linearity Criteria for $\mathrm{Aut}_S\,\mathbb{A}^2_K$

For a subgroup S of $\mathrm{SL}(2,K)$, there are two linearity criteria for $\mathrm{Aut}_S\,\mathbb{A}^2_K$. The second one, stronger, is only proved for a field K of characteristic zero.

Let $\Lambda \subset K^*$ be a subgroup. For any $\mathbb{F}[\Lambda]$-module M and any $n \geq 1$, set $M^{(n)} = (\rho_n)_* M$, where ρ_n is the group homomorphism $x \in \Lambda \mapsto x^n \in \Lambda$.

9.1 The standard module for torsion-free good subgroups of K^*
Assume now that Λ is a torsion-free good subgroup of K^*. By definition, the *standard $\mathbb{F}[\Lambda]$-module* is K_1, and it is denoted by $\mathrm{St}(\Lambda)$.

Given a K-vector space V and an integer $n \geq 1$, it is clear that $V^{(n)}$ is a direct sum of standard modules, and its multiplicity is the cardinal

$$[V^{(n)} : \mathrm{St}(\Lambda)] = [\Lambda : \Lambda^n]\, \dim_{K_1} V = \dim_{K_n} V.$$

9.2 The First Linearity Criterion
Let S be a subgroup of $\mathrm{SL}_S(2,K)$ and let Λ_δ be the set of all eigenvalues of S_δ, for any $\delta \in \mathbb{P}^1_K$. Set $\mathrm{SL}_S(2,K[t]) := \{G \in \mathrm{SL}(2,K[t]) \mid G(1) \in S\}$.

Linearity Criterion 1. *Assume that Λ_δ is a torsion-free good subgroup of K^*, for any $\delta \in \mathbb{P}^1_K$. Then, for some field extension L of K, there is an embedding*

$$\mathrm{Aut}_S\,\mathbb{A}^2_K \subset \mathrm{SL}(2, L(t)).$$

Moreover if $\mathrm{rk}\,\Lambda_\delta \leq \aleph_0$ *for any $\delta \in \mathbb{P}^1_K$, then we have*

$$\mathrm{Aut}_S\,\mathbb{A}^2_K \simeq \mathrm{SL}_S(2,K[t]) \subset \mathrm{SL}(2,K(t)).$$

Proof. Set $M = \mathrm{Sup}\;\mathrm{rk}\,\Lambda_\delta$, where δ runs over $\mathbb{P}^1_K$. There exists a field extension $L \supset K$, which satisfies one of the following two hypotheses

($\mathcal{I}_1$) $[L:K] \geq M$ if $M > \aleph_0$, or

($\mathcal{I}_2$) $L = K$ if $M \leq \aleph_0$

It follows from Lemmas 7 and Lemma 19 that

$$\mathrm{Aut}_S\,\mathbb{A}^2_K = S \ltimes *_{\delta\in\mathbb{P}^1_K} E_\delta(K), \text{ and}$$
$$\mathrm{SL}_S(2,L[t]) = S \ltimes *_{\delta\in\mathbb{P}^1_L} U_\delta(zL[z]) \supset S \ltimes *_{\delta\in\mathbb{P}^1_K} U_\delta(zL[z]).$$

Let $\delta \in \mathbb{P}^1_K$. The $\mathbb{F}[\Lambda_\delta]$-modules $E_\delta(K)$ and $U_\delta(zL[z])$ are copies of standard $\mathbb{F}[\Lambda]$-module. Since $E_\delta(K) \simeq \oplus_{n\geq 3} K^{(n)}$, we have

$$[E_\delta(K) : \mathrm{St}(\Lambda)] = \textstyle\sum_{n\geq 3} [\Lambda : \Lambda^n][K : K_1].$$

On the other hand, $U_\delta(zL[z])$ is isomorphic to $\aleph_0$ copies of $L^{(2)}$, therefore we have

$$[U_\delta(zL[z]) : \mathrm{St}(\Lambda)] = \aleph_0\,[\Lambda : \Lambda^2][L:K][K:K_1].$$

Hence ($\mathcal{I}_1$) implies the existence of a S_δ-equivariant embedding

$$\psi_\delta : U_\delta(zL[z]) \to E_\delta(K),$$

and $(\mathcal{I}_2)$ implies the existence a S_δ-equivariant isomorphism

$$\psi_\delta : U_\delta(zK[z]) \to E_\delta(K).$$

Therefore by Lemma 3, $(\mathcal{I}_1)$ implies the existence of a an embedding

$$\mathrm{Aut}_S\, \mathbb{A}^2_K \subset S \ltimes *_{\delta\in\mathbb{P}^1_K} U_\delta(zL[z]) \subset \mathrm{SL}_S(2, L[t]) \subset \mathrm{SL}(2, L(t)),$$

and $(\mathcal{I}_2)$ implies the existence an isomorphism

$$\mathrm{Aut}_S\, \mathbb{A}^2_K \simeq S \ltimes *_{\delta\in\mathbb{P}^1_K} U_\delta(zK[z]) \simeq \mathrm{SL}_S(2, K[t]) \subset \mathrm{SL}(2, L(t)). \qquad \square$$

Theorem D. *Let R be a f.g. subring of K and let $S \subset \mathrm{SL}(2, R)$.*

If $\mathrm{rk}\,\Lambda_\delta = \mathrm{trdeg}\ \Lambda_\delta$ *for any $\delta \in \mathbb{P}^1_K$, then* $\mathrm{Aut}_S\, \mathbb{A}^2_K$ *is linear over $K(t)$.*

Otherwise, $\mathrm{Aut}_S\, \mathbb{A}^2_K$ *is not linear, even over a ring.*

Proof. The second assertion follows from the Nonlinearity Criterion.

In order to prove the first one, assume now that $\mathrm{rk}\,\Lambda_\delta = \mathrm{trdeg}\ \Lambda_\delta$ for any $\delta \in \mathbb{P}^1_K$. Let L be the field of fraction of R and set $F = \overline{\mathbb{F}} \cap L$. Since R is a f.g. ring, F is a finite extension of $\mathbb{F}$. Therefore the set $\mu \subset K$ of roots of unity in F or in a quadratic extension of F is finite.

Let $\mathbf{m}$ be a maximal ideal of R such that $p := \mathrm{ch}\ R/\mathbf{m}$ is coprime to Card μ. Indeed if $\mathbb{F} = \mathbb{Q}$ the characteristic of $R/\mathbf{m}$ is arbitrarily large, while in the opposite case, ch $R/\mathbf{m}$ is automatically coprime to Card μ.

Set $S' = \{g \in S \mid g \equiv \mathrm{id} \bmod \mathbf{m}\}$. For any $\delta \in P^1_K$ let Λ'_δ be the eigenvalues of S'_δ. Since S' is residually p-group, Λ'_δ is torsion-free. Since $[S : S'] < \infty$, we can assume that $S = S'$.

We claim that Λ_δ is f.g. for any $\delta \in \mathbb{P}^1_K$. If δ is defined over a field F', where $F' = F$ or F' is a quadratic extension of F, then $\Lambda_\delta \subset \overline{R}^*$, where $\overline{R}$ is the integral closure of R in F'. Since $\overline{R}^*$ is f.g., so is Λ_δ. Otherwise, Λ_δ is trivial. Hence Λ_δ is a torsion-free good subgroup of K^* for any $\delta \in \mathbb{P}^1_K$.

The first Linearity Criterion implies that $\mathrm{Aut}_S\, \mathbb{A}^2_K$ is linear over $K(t)$. $\square$

9.3 The standard modules for finite-torsion good subgroups of K^*

From now on, K is a field of characteristic zero. Let Λ be a good subgroup of K^*, such that Card $\Lambda \cap \mu_\infty = n$ for some $n < \infty$.

The $\mathbb{Q}[\Lambda]$-module $\mathrm{St}_d(\Lambda) := K_d$, where d is a divisor of n, are called the *standard* $\mathbb{Q}[\Lambda]$*-modules*. By Baer Theorem [1], Λ is isomorphic to $\mu_n \times \overline{\Lambda}$, where $\overline{\Lambda} = \Lambda/\mu_n$ is torsion-free. It follows that $\mathrm{St}_d(\Lambda) \simeq \mathbb{Q}(\mu_{n/d}) \otimes \mathbb{Q}(\overline{\Lambda}^d)$.

Given a K-vector space V and an integer $m \geq 1$, it is clear that $V^{(m)}$ is a direct sum of the standard module $\mathrm{St}_{\gcd(n,m)}(\Lambda)$, and its multiplicity is the cardinal

$$[V^{(m)} : \mathrm{St}_{gcd(n,m)}(\Lambda)] = [\overline{\Lambda} : \overline{\Lambda}^n]\, \phi(n)/\phi(\gcd(n,m))\, \dim_{K_1} V.$$

Lemma 32. *Let S_0 be a subgroup of B such that $\chi_B(S_0) = \Lambda$ is a good subgroup of K^* such that $n := \mathrm{Card}\, \Lambda \cap \mu_\infty$ is finite. Let $l > n$ be a prime number and let $L \supset K$ be a field such that $[L : K] \geq \aleph_0 \mathrm{rk}(\Lambda)$.*

Then there is a S_0-equivariant embedding $E(K) \subset E^{<2l}(L)$.

Proof. Let D be the set of divisors of n. The $\mathbb{Q}[\Lambda]$-module $E(K)$ and $E^{<2l}(L)$ are direct sums of standard modules, therefore we have

$$E(K) = \oplus_{d\in D} \mathrm{St}_d(\Lambda)^{m_d}, \text{ and}$$
$$E^{<2l}(L) = \oplus_{d\in D} \mathrm{St}_d(\Lambda)^{n_d},$$

where the multiplicities m_d and n_d are cardinals. Therefore it is enough to prove that $n_d \geq m_d$ for any $d \in D$.

Since $E(K) = \oplus_{m\geq 3} K.T_m$, it is clear that

$$m_d := [E(K)(d) : \mathrm{St}_d(\Lambda)] \leq \aleph_0 \mathrm{rk}(\Lambda)[K : K_1].$$

Similarly, we have $E^{<2l}(L) = \oplus_{3\leq m\leq 2l} L.T_m$. Let $d \in D$. If $d \geq 3$, $L.T_d$ is a direct sum of standard modules $\mathrm{St}_d(\Lambda)$ and it is clear that $[L.T_d : \mathrm{St}_d(\Lambda)] \geq [L : K][K : K_1] \geq \aleph_0 \mathrm{rk}(\Lambda)[K : K_1]$, and therefore $n_d \geq m_d$. Since l is coprime to n, then $L.T_l$ is a direct sum of standard modules $\mathrm{St}_1(\Lambda)$, and we have $n_1 \geq m_1$.

If n is odd, the assertion is proved. Otherwise, $L.T_{2l}$ is a direct sum of standard modules $\mathrm{St}_2(\Lambda)$, and similarly we have $n_2 \geq m_2$. □

9.4 The Second Linearity Criterion

Linearity Criterion 2. *Let K be a field of characteristic zero. Assume that*

(i) Λ_δ is a good subgroup of K^*, for any $\delta \in \mathbb{P}^1_K$, and

(ii) the function $\delta \mapsto \mathrm{Card}\, \Lambda \cap \mu_\infty$ is bounded.

Then $\mathrm{Aut}_S\, \mathbb{A}^2_K$ is a linear group over some field extension L of K.

Proof. Let $L \supset K$ be a field such that $[L : K] \geq \aleph_0 M$, where $M = \mathrm{Sup}\, \mathrm{rk}\, \Lambda_\delta$, and let $Q \subset \mathbb{P}^1_K$ be a set of representatives of $\mathbb{P}^1_K/S_0$. By Lemma 32 there is a S_δ-embedding $\psi_\delta : E_\delta(K) \to E^{<2l}_\delta(L)$ where $l > \mathrm{Max}\, \mathrm{Card}\, \Lambda \cap \mu_\infty$ is a prime number. Therefore we get some embeddings

$$\mathrm{Aut}_S \mathbb{A}^2_K \simeq S \ltimes *_{\delta\in\mathbb{P}^1_K} E_\delta(K) \subset S \ltimes *_{\delta\in\mathbb{P}^1_K} E^{<2l}_\delta(L) \subset \mathrm{Aut}^{<2l}_0 \mathbb{A}^2_L.$$

So, the strong version of Theorem C.2 implies that $\mathrm{Aut}_S \mathbb{A}^2_K$ is linear. □

10 Examples of Linear or Nonlinear $\mathrm{Aut}_S\,\mathbb{A}^2_K$

We provide three examples using the Linearity/Nonlinearity Criteria.

10.1 Example A, with $S = \mathrm{SO}(q)$ and K infinite

Example A. *Let q be a quadratic form on K^2 and $S = \mathrm{SO}(q)$.*
If q is anisotropic, $\mathrm{Aut}_S\mathbb{A}^2_K$ is linear over a field extension of K.
Otherwise $\mathrm{Aut}_S\mathbb{A}^2_K$ is not linear, even over a ring.

As we will see, the proofs for ch $K = 0$ and for ch $K \neq 0$ are very different. In particular, the group $S = \mathrm{SO}(2, \mathbb{R})$ has no subgroups of finite index and the proof for $K = \mathbb{R}$ cannot be reduced to the first Linearity Criterion.

Proof. If q is isotropic or degenerate, we have $\Lambda_\delta = K^*$ for some $\delta \in \mathbb{P}^1_K$. The proof of Theorem A.1 shows that K^* itself is bad. Hence by the Nonlinearity Criterion, $\mathrm{Aut}_S\mathbb{A}^2_K$ is not linear, even over a ring.

Assume now that q is anisotropic. Let $L \supset K$ be the quadratic extension splitting q. Then $\mathrm{SO}(q)$ is isomorphic to $S := \{z \in L^* \mid N_{L/K}(z) = 1\}$. Let S^∞ be the subgroup of all $s \in S$ of order a power of 2.

1. Proof for $\mathrm{ch}\, K = p$. We claim that S^∞ is finite. So we can assume that $\mathrm{Card}\, S^\infty \geq 4$. Since $\sqrt{-1} \in S$ and $S \cap K^* = \{\pm 1\}$, it follows that $L = K(\sqrt{-1})$. Therefore $p \equiv -1 \bmod 4$, and $L = K.\mathbb{F}_{p^2}$. Let $s \in S^\infty$ of order > 2. There is an integer $n \geq 1$ such that $s \in \mathbb{F}_{p^{2n}}$ where $\mathbb{F}_{p^n} \subset K$. Since $\mathbb{F}_{p^2} \not\subset K$, the integer n is odd. Since $\mathrm{Card}\, \mathbb{F}^*_{p^{2n}}/\mathbb{F}^*_{p^2}$ is odd, s belongs to $\mathbb{F}^*_{p^2}$. Hence $S^\infty \subset \mathbb{F}^*_{p^2}$ is finite.

By Baer's theorem [1], we have $S = S^\infty \times S'$ for some subgroup $S' \subset S$. Since $S'_\delta = \{1\}$ for any $\delta \in \mathbb{P}^1_K$, the group $\mathrm{Aut}_{S'}\,\mathbb{A}^2_K$ embeds into $\mathrm{SL}(2, K(t))$ by the first Linearity Criterion. Since

$$[\mathrm{Aut}_S\,\mathbb{A}^2_K : \mathrm{Aut}_{S'}\,\mathbb{A}^2_K] = [S : S'] < \infty,$$

the group $\mathrm{Aut}_{\mathrm{SO}(q)}\,\mathbb{A}^2_K$ is also linear over $K(t)$.

2. Proof for $\mathrm{ch}\, K = 0$. Since $SO(q)_\delta = \{\pm 1\}$ for any $\delta \in \mathbb{P}^1_K$, the group $\mathrm{Aut}_{\mathrm{SO}(q)}$ is linear over a field extension of K by the second Linearity Criterion. □

10.2 A preparatory lemma for the example B

We did not found a reference for the next well-known result. The given proof that $l(\gamma_a)$ is arbitrarily large is due to Y. Benoist. Also it is implicit in [6] that $l(\gamma_a)$ is not constant, as pointed out by I. Irmer. A third proof is based on the fact that any loxodromic representation $\Pi_g \to \mathrm{PSL}(2, \mathbb{R})$ can be deformed to the trivial representation inside the character variety, i.e, the space of semi-simple complex representations $\Pi_g \to \mathrm{PSL}(2, \mathbb{C})$. All these proofs involves some geometric arguments.

Lemma 33. *There are cocompact lattices $S \subset \mathrm{SL}(2, \mathbb{R})$ such that $\mathrm{Tr}\,\gamma$ is a transcendental number for any infinite order element $\gamma \in S$.*

Proof. Let $g \geq 2$. Let Σ_g be the oriented Riemann surface of genus g, let $T(\Sigma_g)$ be its Teichmüller space and let Π_g be the group presented by

$$\langle \alpha_1, \beta_1 \dots \alpha_g, \beta_g \mid \prod_{1 \leq k \leq g} (\alpha_k, \beta_k) = 1 \rangle.$$

Let γ be a conjugacy class in Π_g. Any point $a \in T(\Sigma_g)$ determines a group homorphism $\rho_a : \Pi_g \to \pi_1(\Sigma_g)$ and an hyperbolic metric g_a on Σ_g, up to some equivalence [15], ch.5. Hence $\rho_a(\gamma)$ is represented by a unique closed g_a-geodesic $\gamma_a : S^1 \to \Sigma_g$, i.e. a geodesic relative to the metric g_a.

In elementary terms, γ is represented, modulo $\mathrm{PSL}(2, \mathbb{R})$-conjugacy, by a hyperbolic element $h_a \in \mathrm{PSL}(2, \mathbb{R})$. The complete geodesic $\Gamma \subset \mathbb{H}$ whose the extreme points in $\partial\mathbb{H}$ are the fixed points of h_a, is the locus of minima for the function $z \in \mathbb{H} \mapsto d_{\mathbb{H}}(z, h_a.z)$. The g_a-geodesic γ_a is, up to reparametrization, the image in Σ_g of any segment $[z, h_a.z]$ of Γ.

We claim that the length function $a \mapsto l(\gamma_a)$ is not constant. Let us pick another conjugacy class δ in Π_g such that $\rho_a(\delta)$ is represented by a simple geodesic δ_a which meets γ_a transversally (since $T(\Sigma_g)$ is connected, this condition is independent of a). By a corollary of the collar theorem,

$$\mathrm{sh}\,\frac{l(\gamma_a)}{2}\,\mathrm{sh}\,\frac{l(\delta_a)}{2} > 1,$$

see 4.1.2 in [7]. Since $l(\delta_a)$ can be arbitrarily small, $l(\gamma_a)$ is arbitrarily large.

Each $a \in T(\Sigma_g)$ determines an homomorphism $\pi_1(\Sigma_g) \to \mathrm{PSL}(2, \mathbb{R})$ up to some equivalence, which induces a homomorphism

$$\tilde{\rho}_a : \Pi_g \to \mathrm{PSL}(2, \mathbb{R})$$

defined up to conjugacy by $\mathrm{PSL}(2, \mathbb{R})$.

For $g \in \mathrm{PSL}(2, \mathbb{R})$, $\mathrm{Tr}\, g^2$ is well-defined. We have

$$\mathrm{Tr}\,\tilde{\rho}_a(\gamma)^2 = 2\,\mathrm{ch}\; l(\gamma_a).$$

Let $\mathcal{I}$ be the set of irreducible polynomials in $\mathbb{Q}[t]$. For $P \in \mathcal{I}$, set

$$\Omega(\gamma, P) = \{a \in T(\Sigma_g) \mid P(\mathrm{Tr}\,\tilde{\rho}_a(\gamma)^2) \neq 0\}.$$

Since the function $a \mapsto P(\mathrm{Tr}\,\tilde{\rho}_a(\gamma)^2)$ is nonconstant and analytic, $\Omega(\gamma, P)$ is a dense open subset of $T(\Sigma_g)$. By the Baire Theorem

$$\Omega := \cap_{\gamma \neq 1, P \in \mathcal{I}}\, \Omega(\gamma, P)$$

is dense. For any $a \in \Omega$, the lattice

$$S := \{s \in \mathrm{SL}(2,\mathbb{R}) \mid s \bmod \pm 1 \text{ belongs to } \tilde{\rho}_a(\Pi_g)\}$$

satisfies the required condition. (Indeed $S \simeq \Pi_g \times \{\pm 1\}$ by [20].) □

10.3 Example B, where S is a lattice

Example B. *For some cocompact lattices $S \subset \mathrm{SL}(2,\mathbb{R})$, the group $\mathrm{Aut}_S\,\mathbb{A}^2_{\mathbb{C}}$ is linear over $\mathbb{C}$.*

For any lattice $S \subset \mathrm{SL}(2,\mathbb{C})$, $\mathrm{Aut}_S\,\mathbb{A}^2_{\mathbb{C}}$ is not linear, even over a ring.

Proof. Let $S \subset \mathrm{SL}(2,\mathbb{R})$ be a cocompact lattice as in Lemma 33. Then for any $\delta \in \mathbb{P}^1_{\mathbb{C}}$, it is clear that Λ_δ has rank one and contains a transcendental element, or it is finite. Hence $\mathrm{Aut}_S\,\mathbb{A}^2_{\mathbb{C}}$ is linear over $\mathbb{C}$ by Theorem D.

Let S be a lattice of $\mathrm{SL}(2,\mathbb{C})$, let $g \in S$ be of infinite order and let $\delta \in \mathbb{P}^1_{\mathbb{C}}$ be a fixed point of g. By the Garland-Raghunathan rigidity theorem 0.11 of [14], the eigenvalues of g are algebraic numbers. Since $\mathrm{rk}\,\Lambda_\delta > \mathrm{trdeg}\ \Lambda_\delta$, Theorem D implies that $\mathrm{Aut}_S\,\mathbb{A}^2_{\mathbb{C}}$ is not linear, even over a ring. □

10.4 A preparatory lemma for example C

Lemma 34. *Let R be a prime normal ring with fraction field k and let $m \geq 1$. Let B be the integral closure of $R[t_1, \dots, t_m]$ in some quadratic extension $L_1 \subset k((t_1, \dots, t_m))$ of $k(t_1, \dots, t_m)$.*

Then B^*/R^* is isomorphic to $\{1\}$ or $\mathbb{Z}$.

Proof. Since R is normal, we have $B \cap k = R$, hence the map $B^*/R^* \to (k \otimes B)^*/k^*$ is one to one. So we can assume that $R = k$. Set $C = \mathrm{Spec}\ B$. There is a unique normal compactification $\overline{C}$ of C such that the finite map $C \to \mathbb{A}^m_k = \mathrm{Spec}\ k[t_1, \dots t_m]$ extends to a finite map $\pi : \overline{C} \to \mathbb{P}^m_k$.

Set $Z := \pi^{-1}(\mathbb{P}^{m-1}_k)$, where $\mathbb{P}^{m-1}_k := \mathbb{P}^m_k \setminus \mathbb{A}^m_k$. For any irreducible divisor D in $\overline{C}$, let v_D be the corresponding valuation.

If Z is irreducible, then $v_Z(f) \leq 0$ for any $f \in B$. Hence $v_Z(f) = 0$ for any $f \in B^*$, and therefore $B^* = k^*$.

Otherwise, Z is the union of two divisors Z_1 and Z_2. For any $f \in B^* \setminus k^*$, either $v_{Z_1}(f) < 0$ or $v_{Z_2}(f) < 0$. Hence the homomorphism $f \in B^* \mapsto (v_{Z_1}(f), v_{Z_2}(f)) \in \mathbb{Z}^2$ embeds B^*/k^* in a free $\mathbb{Z}$-module of rank ≤ 1. □

10.5 Example C, where $\operatorname{rk}\Lambda_\delta$ *is not constant*

Let A be a torsion-free additive group of any rank, let $d, m > 0$ be integers with d square-free and let $\mathcal{O}$ be the ring of integers of $k := \mathbb{Q}(\sqrt{d})$. Set $K = k(A)((t_1, \dots, t_m))$ where $k(A)$ is the field of fractions of $k[A]$, and

$$S := \mathrm{SL}(2, \mathcal{O}[A][t_1, \dots, t_m]).$$

The Example C of the introduction is the case $A = \mathbb{Z}$ and $m = 1$.

Example C. *If $d < 0$, then* $\operatorname{Aut}_S \mathbb{A}^2_K$ *is linear over a field extension of K.*

Otherwise, $\operatorname{Aut}_S \mathbb{A}^2_K$ *is not linear, even over a ring.*

Proof. Set $L_0 := k(A)(t_1, \dots, t_m)$.

Proof if $d > 0$. Then $\mathbb{Q}(\sqrt{d})$ is a real field, we have $\operatorname{rk}\mathcal{O}^* = 1 > \operatorname{trdeg}\mathcal{O}^* = 0$. For $\delta \in \mathbb{P}^1_{L_0}$, the group $\Lambda_\delta = \mathcal{O}^* \times A$ is a bad subgroup of K^*. So, by the Nonlinearity Criterion, $\operatorname{Aut}_S \mathbb{A}^2_K$ is not linear, even over a ring.

Proof if $d < 0$. Let $\delta \in \mathbb{P}^1_L$.

If δ belongs to $\mathbb{P}^1_{L_0}$, we have $\Lambda_\delta = \mathcal{O}^* \times A$. Since $\mathbb{Q}(\sqrt{d})$ is an imaginary field, $\Lambda_\delta = \mathcal{O}^* \times A$ is a good subgroup of K^* and $\operatorname{Card}\Lambda_\delta \cap \mu_\infty \leq 6$.

Assume now that δ belongs to $\mathbb{P}^1_{L_1} \setminus \mathbb{P}^1_{L_0}$, where L_1 is a quadratic extension of L_0. Let B be the algebraic closure of $\mathcal{O}[A][t_1, \dots, t_m]$ in L_1 and let $N := \{z \in L_1 \mid N_{L_1/L_0}(z) = 1\}$ be the norm group. It is clear that

$$\Lambda_\delta = N \cap B^*.$$

By Lemma 34, we have $\Lambda_\delta = \{\pm 1\}$ or $\Lambda_\delta = \{\pm 1\} \times \mathbb{Z}$. Since $\mathcal{O}$ is algebraically closed in L, it should be noted that when $\operatorname{rk}\Lambda_\delta = 1$, we also have $\operatorname{trdeg}\Lambda_\delta = 1$. Thus Λ_δ is a good subgroup of K^* and $\operatorname{Card}\Lambda_\delta \cap \mu_\infty = 2$.

Otherwise, we have $\Lambda_\delta = \{\pm 1\}$ and the same conclusion holds.

Therefore, by the second Linearity Criterion, $\operatorname{Aut}_S \mathbb{A}^2_K$ is linear over some field extension of K. □

11 Nonlinearity of Finite-Codimensional Subgroups of $\operatorname{Aut}\mathbb{A}^3_K$

Theorem A.2 shows that $\operatorname{Aut}\mathbb{A}^2_K$ contains some finite-codimensional subgroups, which are linear as abstract groups. However, this result does not extend to $\mathbb{A}^n_K$, for $n \geq 3$, as it will be shown in this section.

For our purpose, the case $n = 3$ is enough. Unlike in the introduction, *it will be convenient to use the coordinates (z, x, y) for* $\mathbb{A}^3_K$. Let $\operatorname{TAut}\mathbb{A}^3_K$ be the the subgroup of tame automorphisms of $\mathbb{A}^3_K$, see 11.5 for the definition.

By the famous result of Shestakov and Urmibaev [24], $\mathrm{TAut}\,\mathbb{A}^3_K$ is a *proper* subgroup of $\mathrm{Aut}\,\mathbb{A}^3_K$.

Let $\mathbf{m}$ be a finite-codimensional ideal in $K[z,x,y]$. Let $\mathrm{Aut}_{\mathbf{m}}\,\mathbb{A}^3_K$ be the group of all polynomial automorphisms ϕ of the form

$$(z,x,y) \mapsto (z+f, x+g, y+h),$$

where f, h and g belongs to $\mathbf{m}$. Set

$$\mathrm{TAut}_{\mathbf{m}}\,\mathbb{A}^3_K = \mathrm{TAut}\,\mathbb{A}^3_K \cap \mathrm{Aut}_{\mathbf{m}}\,\mathbb{A}^3_K.$$

The nonlinearity result for $n = 3$, valid even if K is finite, is unrelated with the existence of wild automorphisms in $\mathbb{A}^3_K$, as shown by

Theorem B. *For any finite-codimensional ideal* $\mathbf{m}$ *of* $K[z,x,y]$*, the groups* $\mathrm{Aut}_{\mathbf{m}}\,\mathbb{A}^3_K$ *and* $\mathrm{TAut}_{\mathbf{m}}\,\mathbb{A}^3_K$ *are not linear, even over a ring.*

The proof uses the folklore embedding Φ, likely known by Nagata [19], and used in [24]. The simplest obstruction for the linearity is due to a nonnilpotent locally nilpotent subgroup. In characteristic zero, the proof is easy, and it follows the line of [9] together with Corollary 2. In characteristic p, the proof involves the strange formula of Lemma 37.

11.1 Nilpotency class of some p-groups

The *nilpotency class* of a nilpotent group is the lenght of its ascending central series. Let p be a prime integer, and let E be an elementary p-group of rank r. Note that E acts by translation on $\mathbb{F}_p[E]$ and set $G(r) = E \ltimes \mathbb{F}_p[E]$.

Set $E = D_1 \times \cdots \times D_r$, where each D_i has rank 1. For each i, the socle filtration of the D_i-module $\mathbb{F}_p[D_i]$ has length p and we have $\mathbb{F}_p[E] = \mathbb{F}_p[D_1] \otimes \cdots \otimes \mathbb{F}_p[D_n]$. Hence the socle filtration of the E-module $\mathbb{F}_p[E]$ has lenght $1 + (p-1)r$. It follows that

Lemma 35. *The nilpotency class of the nilpotent group* $G(r)$ *is* $1+(p-1)r$.

Let M be a cyclic $\mathbb{F}_p[E]$-module generated by some $f \in M$.

Lemma 36. *If* $\sum_{u\in E} u.f \neq 0$*, then the* $\mathbb{F}_p[E]$*-module* M *is free of rank one.*

Proof. Set $N = \sum_{u\in E} e^u$, where $(e^u)_{u\in E}$ is the usual basis of $\mathbb{F}_p[E]$. Note that $\mathbb{F}_p.N = H^0(E, \mathbb{F}_p[E])$, hence any nonzero ideal of $\mathbb{F}_p[E]$ contains N. Since $N.f \neq 0$, M is freely generated by f. □

11.2 A formula

Lemma 37. *Let A be a commutative $\mathbb{F}_p$-algebra and let $E \subset A$ be a linear subspace of dimension r. We have*

$$\sum_{u\in E} u^{p^r-1} = \prod_{u\in E\setminus\{0\}} u.$$

Proof. It is enough to prove the claim for $A = \mathbb{F}_p[x_1, \ldots, x_r]$ and $E = \oplus_i \mathbb{F}_p.x_i$. Set $P(x_1, \ldots, x_n) := \sum_{u\in E} u^{p^r-1}$, set $H = \mathbb{F}_p.x_2 \oplus \mathbb{F}_p.x_3 \cdots \oplus \mathbb{F}_p.x_r$ and for $v \in H$, set $Q_v := \sum_{\lambda\in\mathbb{F}_p} (\lambda x_1 + v)^{p^r-1}$. For any integer $n \geq 0$, we have $\sum_{\lambda\in\mathbb{F}_p} \lambda^n = 0$ except if n is a positive multiple of $p - 1$. Hence

$$Q_v = \sum_{n>0} c_n x_1^{n(p-1)} v^{p^r-1-n(p-1)},$$

for some $c_n \in \mathbb{F}_p$ and the polynomial Q_v is divisible by x_1^{p-1}. Since

$$P(x_1, \ldots, x_n) = \sum_{v\in H} Q_v,$$

the polynomial $P(x_1, \ldots, x_n)$ is divisible by x_1^{p-1}. By $\mathrm{GL}(r, \mathbb{F}_p)$-invariance, $P(x_1, \ldots, x_n)$ is divisible by u^{p-1} for all $u \in E \setminus \{0\}$. Hence it is divisible by $\prod_{u\in E\setminus\{0\}} u$. Since both polynomials have degree $p^r - 1$, it follows that

$$P(x_1, \ldots, x_n) = c \prod_{u\in E\setminus\{0\}} u,$$

for some $c \in \mathbb{F}_p$.

As it is a universal constant, we can compute c for $A = E = \mathbb{F}_{p^r}$. Since

$$\sum_{\lambda\in\mathbb{F}_p^r} \lambda^{p^r-1} = -1, \text{ and } \prod_{\lambda\in\mathbb{F}_p^r} \lambda = -1,$$

it follows that $c = 1$. □

11.3 The locally nilpotent group $G(I)$

Let $\mathbb{F}$ be a prime field. For any ideal I of a commutative $\mathbb{F}$-algebra A, let us consider the semi-direct product $G(I) := I \ltimes I[t]$, where I acts by translation on the space $I[t]$ of polynomials with coefficients in I.

Let $E \subset I$ be an additive subgroup, let $f(t) \in I[t] \setminus \{0\}$ and let M be the additive subgoup generated by all polynomials $f(t + u)$ when u runs over E. The group $E \ltimes M$, which is a subgroup of $G(I)$, is obvioulsy nilpotent.

Lemma 38. *Assume that the algebra A is prime.*

(i) If $\mathbb{F} = \mathbb{Q}$, the nilpotency class of $E \ltimes M$ is $1 + \deg f$.

(ii) Assume that $\mathbb{F} = \mathbb{F}_p$, that $\dim_{\mathbb{F}_p} E = r$ and that $f(t) = ax^{p^r-1}$ for some $a \in I \setminus \{0\}$. Then the group $E \ltimes M$ has nilpotency class $1 + r(p - 1)$.

(iii) If $\dim_{\mathbb{F}_p} I = \infty$, the group $G(I)$ is locally nilpotent but not nilpotent.

Proof. Assertion (i) is obvious, and Assertion (iii) is a consequence of Assertions (i) and (ii). We will prove Assertion (ii), for which $\mathbb{F} = \mathbb{F}_p$.

Set $g(t) = \sum_{u\in E} f(t+u)$. We have $g(0) = \sum_{u\in E} au^{p^r-1} = a\prod_{u\in E\setminus\{0\}} u$ by Lemma 37. Since $g(0) \neq 0$, it follows from Lemma 36 that the $\mathbb{F}_p[E]$-module M is free of rank one, and $E \ltimes M$ is isomorphic to $G(r)$. Thus its nilpotency class is $1 + r(p-1)$ by Lemma 35. □

11.4 The amalgamated product $\mathrm{Aff}(2, I) *_{B_{Aff}(I)} \mathrm{Elem}(I)$

From now on, let I be a proper nonzero ideal in $K[z]$.

Lemma 39. *The group* $\mathrm{Elem}(I)$ *is not linear over a field.*

Proof. Since I is a proper ideal, $\mathrm{Elem}(I)$ is the group of all automorphisms

$$\phi : (x, y) \mapsto (x + u, y + f(x)),$$

for some $u \in I$ and $f \in I[x]$. It follows that $\mathrm{Elem}(I)$ is isomorphic to the group $G(I)$, and therefore $\mathrm{Elem}(I)$ contains subgroups of abitrarily large nilpotency class by Lemma 38. Hence $\mathrm{Elem}(I)$ is not linear over a field. □

Lemma 40. *The group* $\mathrm{Aff}(2, I) *_{B_{Aff}(I)} \mathrm{Elem}(I)$ *is not linear, even over a ring.*

Proof. By Lemma 39, the group $\mathrm{Elem}(I)$ is not linear over a field. Therefore by Corollary 1, it is enough to show that the amalgamated product $\Gamma := \mathrm{Aff}(2, I) *_{B_{Aff}(I)} \mathrm{Elem}(I)$ satisfies

$$\mathrm{Core}_\Gamma(B_{Aff}(I)) = \{1\}.$$

In order to do so, we first define two specific automorphisms γ and ϕ as follows. Let $r \in I \setminus \{0\}$ and let $n \geq 3$ be an integer coprime to ch K. Let $\gamma \in \mathrm{Aff}(2, I)$ be the linear map $(x, y) \mapsto (x + ry, y)$ and let $\phi \in \mathrm{Elem}(I)$ be the polynomial automorphisms $(x, y) \mapsto (x, y + rx^n)$.

Let g be an arbitrary element of $B_{Aff}(I) \setminus \{1\}$. By definition, g is an affine map $(x, y) \mapsto (x + u, y + v + wx)$ for some u, $v.w \in I$.

If $w \neq 0$, the linear part of g^γ is not lower triangular, therefore g^γ is not in $B_{Aff}(I)$. If $w = 0$ but $u \neq 0$, then the leading term g^ϕ, which is $(x, y) \mapsto (0, nrux^{n-1})$, has degree ≥ 2. Therefore g^ϕ is not in $B_{Aff}(I)$. Last if $u = w = 0$, then v is not equal to zero. It follows that the leading term of $g^{\gamma\phi}$, which is $(x, y) \mapsto (0, nr^2vx^{n-1})$, has degree ≥ 2. Therefore $g^{\gamma\phi}$ is not in $B_{Aff}(I)$.

Hence, for any $g \in B_{Aff}(I) \setminus \{1\}$ at least one of the three elements g^γ, g^ϕ or $g^{\gamma\phi}$ is not in $B_{Aff}(I)$. Therefore $\mathrm{Core}_\Gamma(B_{Aff}(I))$ is trivial. □

11.5 Proof of Theorem B

The group $\mathrm{TAut}\,\mathbb{A}^3_K \subset \mathrm{Aut}\,\mathbb{A}^3_K$ of *tame automorphisms* of $\mathbb{A}^3_K$ is

$$\mathrm{TAut}\,\mathbb{A}^3_K = \langle \mathrm{Aff}(3,K), T(3,K)\rangle,$$

where $\mathrm{Aff}(3,K)$ is the group of affine automorphisms of $\mathbb{A}^3_K$ and $T(3,K)$ is the group of all triangular automorphisms

$$(z,x,y) \mapsto (z, x+f(z), y+g(z,x)),$$

where f and g are polynomials. Note that $\mathrm{Aut}\,\mathbb{A}^2_{K[z]} = \mathrm{Aut}_{K[z]}\,K[z,x,y]$ is obviously the subgroup of $\mathrm{Aut}\,\mathbb{A}^3_K = \mathrm{Aut}_K\,K[z,x,y]$ of all automorphisms of the form

$$(z,x,y) \mapsto (z, f(z,x,y), g(z,x,z)),$$

where f and g are polynomials.

It is easy to see that the groups $\mathrm{Elem}(K[z])$ and $\mathrm{Aff}(2,K[z])$ are subgroups of $\mathrm{TAut}\,\mathbb{A}^3_K$. Therefore van der Kulk's Theorem for the field $K(z)$ and Lemma 2 imply that the homomorphism

$$\Phi : \mathrm{Aff}(2,K[z]) *_{B(K[z]} \mathrm{Elem}(K[z]) \to \mathrm{Aut}\,\mathbb{A}^2_{K[z]} \cap \mathrm{TAut}\,\mathbb{A}^3_K$$

is an embedding, likely known by Nagata. The hard and beautiful result of [24] states that Φ is onto, a result which is not needed here.

Now, we prove Theorem B.

Proof. Without loss of generality, we can assume that the ideal $\mathbf{m}$ is also a proper ideal. Hence the ideal $I := \mathbf{m} \cap K[z]$ is nonzero and proper.

The previously defined morphism Φ induces an embedding

$$\mathrm{Aff}(2,I) *_{B_{Aff}(I)} \mathrm{Elem}(I) \to \mathrm{Aut}_{\mathbf{m}}\,\mathbb{A}^3_K.$$

Hence by Lemma 40, the group $\mathrm{TAut}_{\mathbf{m}}\,\mathbb{A}^3_K$ is not linear, even over a ring. □

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