Exercises for the course “Groups acting on the circle”

Nicolás Matte Bon

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This file collects several exercises for the course. It is organised in sections according to the topic.

1 Actions of $\mathbb{Z}$, rotation number theory

Exercice 1. We denote by $\mathrm{Homeo}_+(\mathbb{R})$ the group of increasing homeomorphisms of $\mathbb{R}$. Let $f \in \mathrm{Homeo}_+(\mathbb{R})$ be an element without fixed points.

1. Let $f \in \mathrm{Homeo}_+(\mathbb{R})$ be an element without fixed points. Prove that that either $f(x) > x$ for every $x \in \mathbb{R}$ or $f(x) < x$ for every $x \in \mathbb{R}$. We say that f has *positive direction* in the first case, and *negative direction* otherwise.

2. Prove that if f has no fixed point and positive direction, then it is conjugate to the map $x \mapsto x + 1$. (Here $f, g \in \mathrm{Homeo}_+(\mathbb{R})$ are said to be *conjugate* if there exists $h \in \mathrm{Homeo}_+(\mathbb{R})$ such that $f \circ h = h \circ g$).

3. Deduce that two elements $f, g \in \mathrm{Homeo}_+(\mathbb{R})$ without fixed points are conjugate if and only if they have the same direction.

Exercice 2. Prove that if $f \in \mathrm{Homeo}_+(\mathbb{R}/\mathbb{Z})$ admits a periodic point, then all periodic points have the same minimal period (which is defined as the smallest $q > 1$ such that $f^q(x) = x$).

Exercice 3. Let $f, g \in \mathrm{Homeo}_+(\mathbb{R}/\mathbb{Z})$. Assume that $f \circ g = g \circ f$. Prove that the rotation numbers satisfy $\rho(f \circ g) = \rho(f) + \rho(g)$. Give an example of f, g such that $\rho(f \circ g) \neq \rho(f) + \rho(g)$.

Exercice 4. Let $f, g \in \mathrm{Homeo}_+(\mathbb{R}/\mathbb{Z})$ such that f is semi-conjugate to g. Prove that $\rho(f) = \rho(g)$.

Exercice 5. Let $R_\alpha \in \mathrm{Homeo}_+(\mathbb{R}/\mathbb{Z})$ be a rotation with α irrational. Prove that the only elements of $\mathrm{Homeo}_+(\mathbb{R}/\mathbb{Z})$ that commute with R_α are the rotations $R_\beta, \beta \in \mathbb{R}/\mathbb{Z}$.

Exercice 6. Let G be a finite subgroup of $\mathrm{Homeo}_+(\mathbb{R}/\mathbb{Z})$. Prove that G is conjugate to the cyclic group generated by a rational rotation (*Hint*: begin by finding an atomless probability measure invariant under all elements of G, and argue similarly as in the proof of Poincaré’s theorem given in class).

Exercice 7 (Denjoy counterexample)**.** A *Denjoy counteraxample* is a homeomorphisms $f \in \mathrm{Homeo}_+(\mathbb{R}/\mathbb{Z})$ which has irrational rotation number but is not conjugate to an irrational rotation. The construction of such an example was sketched in class. The purpose of this exercise is to fill-in the details of the construction. We follow the same idea explained in class, but implement it in a slightly different way.

Let $\alpha \in \mathbb{R}/\mathbb{Z}$ be irrational, and set $O = \{n\alpha \colon n \in \mathbb{Z}\}$ be the orbit of 0 under the rotation R_α. Choose positive real numbers $(u_n)n \in \mathbb{Z}$ such that $\sum_{n\in\mathbb{Z}} u_n = 1$. We define two maps $\varphi_-, \varphi_+ \colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ and as follows. For $x \neq 0$, we set

$$\varphi_-(x) = \sum_{n\colon n\alpha\in[0,x)} u_n \pmod{\mathbb{Z}}; \qquad \varphi_+(x) = \sum_{n: n\alpha\in[0,x]} u_n \pmod{\mathbb{Z}}$$

and we set $\varphi_-(0) = 0$ and $\varphi_+(0) = u_0 \pmod{\mathbb{Z}}$.

1. Prove that φ_- is not φ_+ are continuous at x if and only if $x \notin O$ if and only if $\varphi_-(x) = \varphi_+(x)$. For $x \in (\mathbb{R}/\mathbb{Z}) \setminus O$, we set $\varphi(x) := \varphi_-(x) = \varphi_+(x)$.

2. For every $n \in \mathbb{Z}$ consider the arc $I_n = [\varphi_-(n\alpha), \varphi_+(n\alpha)] \subset \mathbb{R}/\mathbb{Z}$. Prove that the arcs I_n are pairwise disjoint and we have

$$\mathbb{R}/\mathbb{Z} = \{\varphi(x) \colon x \in (\mathbb{R}/\mathbb{Z}) \setminus O\} \sqcup \Big(\bigsqcup_{n\in\mathbb{Z}} I_n\Big).$$

3. For every n choose an orientation–preserving homeomorphism $h_n \colon I_n \to I_{n+1}$. Define a map $f \colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ by

$$f(y) = \begin{cases} \varphi(x+\alpha) & \text{if } y = \varphi(x) \\ h_n(y) & \text{if } y \in I_n \end{cases}$$

   Prove that f is well defined element of $\mathrm{Homeo}_+(\mathbb{R}/\mathbb{Z})$.

4. Prove that f has no periodic orbit.

5. Prove that the set $C = \varphi\left((\mathbb{R}/\mathbb{Z}) \setminus O\right) \cup \varphi_-(O) \cup \varphi_+(O)$ is closed and f-invariant. Deduce that f is not conjugate to an irrational rotation.

6. Prove that f is semi-conjugate to the rotation R_α, and deduce that $\rho(f) = \alpha$.

7. Prove that the set C is homeomorphic to the Cantor set.

Exercice 8 (Continuity of rotation number)**.** The purpose of this exercise is to show that the translation number $\tau \colon \widetilde{\mathrm{Homeo}}_+(\mathbb{R}/\mathbb{Z}) \to \mathbb{R}$ is a continuous function if we endow $\widetilde{\mathrm{Homeo}}_+(\mathbb{R}/\mathbb{Z})$ with the topology of uniform convergence.

We fix $f \in \widetilde{\mathrm{Homeo}}(\mathbb{R}/\mathbb{Z})$. Our goal is to show that τ is continuous at f. Explicitly, this means that for every $\varepsilon > 0$, there exists δ such that every $g \in \widetilde{\mathrm{Homeo}}(\mathbb{R}/\mathbb{Z})$ such that $|f(x) - g(x)| < \delta$ for every $x \in \mathbb{R}$ satisfies $\tau(g) \in [\alpha - \varepsilon, \alpha + \varepsilon]$.

1. Assume first that $f(x) = x + \alpha$ for every $x \in \mathbb{R}$. Show that τ is continuous at f.

2. Suppose that $\tau(f)$ is irrational. Show that τ is continuous at f. (*Hint*: use Poincaré's theorem).

3. Suppose that $\tau(f) = 0$. Show that for every $\varepsilon > 0$ there exists $h_\varepsilon \in \widetilde{\mathrm{Homeo}}(\mathbb{R}/\mathbb{Z})$ such that $f_\varepsilon := h_\varepsilon f h_\varepsilon^{-1}$ satisfies $|f_\varepsilon(x) - x| < \varepsilon$ for every $x \in \mathbb{R}$ (*Hint*: use exercise 1).

4. Suppose that $\tau(f) = 0$. Show that τ is continuous at f.

5. Suppose that $\tau(f) \in \mathbb{Q}$. Show that τ is continuous at f.

6. Deduce that the rotation number $\rho\colon \mathrm{Homeo}_+(\mathbb{R}/\mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$ is also continuous for the topology of uniform convergence on $\mathrm{Homeo}_+(\mathbb{R}/\mathbb{Z})$.

Exercice 9. Let $f \in \mathrm{Diff}^2(\mathbb{R}/\mathbb{Z})$. For $\theta \in \mathbb{R}/\mathbb{Z}$, we let $f_\theta(x) = f(x) + \theta$. We want to study how the rotation number of f_θ varies when θ is small. We admit that the function $\theta \mapsto \rho(f_\theta)$ is continuous (previous exercise).

1. Suppose that $\rho(f)$ is irrational. Show that there exists θ_0 such that for all $\theta \in [-\theta_0, +\theta_0]$ we have $\rho(f_\theta) \neq \rho(f)$.

2. We say that f has a hyperbolic periodic point if there exists x such that $f^q(x) = x$ and $|Df^q(x)| < 1$. Show that if f has an hyperbolic periodic point, then there exists θ_0 such that f_θ has a hyperbolic periodic point for all $\theta \in [-\theta_0, \theta_0]$. Deduce that $\rho(f_\theta) = \rho(f)$ for every $\theta \in [-\theta_0, \theta_0]$.

2 Group actions on the circle by homeomorphisms

Exercice 10. Let $\Gamma \subset \mathrm{Homeo}_+(\mathbb{R}/\mathbb{Z})$ and assume that Γ has a finite orbit. Prove that all finite orbits of Γ have the same cardinality.

Exercice 11. Let R_α be an irrational rotation and $f \in \mathrm{Homeo}_+(\mathbb{R}/\mathbb{Z})$ be any element which is *not* a rotation. Prove that the group generated by R_α and f contains a ping-pong pair (*Hint*: use Exercise 5).

Exercice 12. Let $f, g \in \mathrm{Homeo}_+(\mathbb{R}/\mathbb{Z})$ be a ping pong pair with respect to sets U_-, U_+, V_-, V_+. Assume that $\overline{U_-} \cup \overline{U_+} \cup \overline{V_-} \cup \overline{V_+} \neq \mathbb{R}/\mathbb{Z}$. Show that the group generated by f and g has an exceptional minimal set.

Exercice 13. Show that if $f, g \in \mathrm{Homeo}_+(\mathbb{R}/\mathbb{Z})$ is a ping-pong pair, then there exists $x \in \mathbb{R}/\mathbb{Z}$ such that $f(x) = x$. Using this and Margulis' theorem, show that if $\Gamma \subset \mathrm{Homeo}_+(\mathbb{R}/\mathbb{Z})$ acts freely on $\mathbb{R}/\mathbb{Z}$, then Γ is abelian and semiconjugate to a group of rotations. (*Note*: this is a variant of Hölder's theorem).

Exercice 14. Show that for any two Cantor sets $\mathcal{C}, \mathcal{C}' \subset \mathbb{R}$, there exists $f \in \mathrm{Homeo}_+(\mathbb{R})$ such that $f(\mathcal{C}) = \mathcal{C}'$. Deduce the analogous results for two Cantor sets in the circle.

Exercice 15. Let Γ be a countable group and $\varphi\colon \Gamma \to \mathrm{Homeo}_+(\mathbb{R}/\mathbb{Z})$ a minimal action on the circle. Prove that there exists an action $\psi\colon \Gamma \to \mathrm{Homeo}_+(\mathbb{R}/\mathbb{Z})$ with an exceptional minimal set which is semi-conjugate to φ (*Hint*: argue in a similar way as in the Denjoy counterexample, see Exercise 7).

Exercice 16. Let $\Gamma \leq \mathrm{Homeo}_+(\mathbb{R}/\mathbb{Z})$ and let $\Delta \leq \Gamma$ be a subgroup of finite index. Prove that Γ acts minimally on the circle if and only if Δ acts minimally on the circle.

Exercice 17. Let Γ be the subgroup of $\mathrm{Homeo}_+(\mathbb{R})$ generated by the transformations $f(x) = 2x$ and $g(x) = x+1$. (*Note*: this group is known as the *Baumslag-Solitar group*, and is usually denoted $\mathrm{BS}(2,1)$).

1. Prove that Γ consists exactly of all homeomorphisms of $\mathbb{R}$ of the form $x \mapsto ax+b$ such that $a = 2^n$ for some $n \in \mathbb{Z}$, and $b \in \mathbb{Z}[\frac{1}{2}]$. Here $\mathbb{Z}[\frac{1}{2}]$ denotes the additive group of *dyadic rationals* $\mathbb{Z}[\frac{1}{2}] = \{\frac{p}{2^q} : p, q \in \mathbb{Z}\}$.

2. Show that Γ is isomorphic to a semi-direct product of the form $\mathbb{Z}[\frac{1}{2}] \rtimes \mathbb{Z}$. Deduce that it is solvable.

3. By identifying the circle with the Alexandroff compactification $\mathbb{R} \cup \{\infty\}$, the group Γ can also be seen as a group of homeomorphisms of the circle, whose action fixes ∞. Show that for every action $\varphi\colon \Gamma \to \mathrm{Homeo}_+(\mathbb{S}^1)$ the element $\varphi(g)$ must have a fixed point (*Hint*: observe first that $fgf^{-1} = g^2$).

4. Construct an embedding $\varphi\colon \Gamma \hookrightarrow \mathrm{Homeo}_+(\mathbb{S}^1)$ for which $\varphi(f)$ does not have fixed points.

Exercice 18. Show that an amenable subgroup $\Gamma \subset \mathrm{Homeo}_+(\mathbb{R}/\mathbb{Z})$ which acts minimally is necessarily abelian. Give examples of non-abelian amenable subgroups of $\mathrm{Homeo}_+(\mathbb{R}/\mathbb{Z})$ (*Hint*: previous exercise). Give such examples that do not have a finite orbit.

Exercice 19. Show that the projective action of $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{P}^1(\mathbb{R}) \simeq S^1$ is minimal and contracting.

Exercice 20. The purpose of this exercise is to show that every countable subgroup $\Gamma \subset \mathrm{Homeo}_+(\mathbb{R}/\mathbb{Z})$ is conjugate to a group of bi-Lipschitz homeomorphisms. Here $f \in \mathrm{Homeo}_+(\mathbb{R}/\mathbb{Z})$ is said to be bi-Lipschitz if both f and f^{-1} are Lipschitz with respect to the distance on $\mathbb{R}/\mathbb{Z}$.

Choose $\mu\colon \Gamma \to \mathbb{R}$ such that $\sum_{g\in\Gamma} \mu(g) = 1$ (that is, μ defines a probability measure on Γ). We assume that $\mu(g) > 0$ for every $g \in \Gamma$ and that $\mu(g) = \mu(g^{-1})$. For every probability measure $\nu \in \mathrm{Prob}(\mathbb{R}/\mathbb{Z})$, we let $\mu * \nu = \sum_{g\in\Gamma} \mu(g) g_*\nu$ be the *convolution* of μ and ν, which is again a probability measure on $\mathbb{R}/\mathbb{Z}$.

1. Prove that there exists $\nu \in \mathrm{Prob}(\mathbb{R}/\mathbb{Z})$ such that $\mu * \nu = \nu$. (*Hint*: remember the proof of the Kakutani fixed point theorem).

2. Prove that if Γ has no finite orbit, then ν is atomless. (*Hint*: assuming the contrary, choose $x \in \mathbb{R}/\mathbb{Z}$ such that $\nu(\{x\})$ is maximal to derive a contradiction).

3. Prove that $\mathrm{supp}(\nu)$ is a closed Γ-invariant subset.

4. Assume that Γ is minimal. Prove that there exists $h \in \mathrm{Homeo}_+(\mathbb{R}/\mathbb{Z})$ such that h_* is the Lebesgue measure. Deduce that $h\Gamma h^{-1}$ is a group of bi-Lipschitz homeomorphisms.

5. Get rid of the assumption that Γ is minimal (*Hint*: it is enough to find $\Gamma' \supset \Gamma$ which is minimal).

3 Orderable groups, group actions on the real line

Exercice 21. Let $\preceq$ be a left-invariant order on a group Γ. Assume that for every $\gamma, \delta \in \Gamma$ such that $\gamma \preceq \delta$ we have $\delta^{-1} \preceq \gamma^{-1}$. Prove that $\preceq$ is bi-invariant.

Exercice 22. Prove that a left-invariant order $\preceq$ on a countable group Γ is Archimedean if and only if every dynamical realisation $\Gamma \hookrightarrow \mathrm{Homeo}_+(\mathbb{R})$ of $\preceq$ acts freely on $\mathbb{R}$. Conversely, show that every subgroup of $\mathrm{Homeo}_+(\mathbb{R})$ which acts freely on $\mathbb{R}$ admits an Archimedean left-order, and deduce that it is abelian.

Exercice 23. Let Γ be a countable group endowed with a bi-invariant order $\preceq$, and $\Gamma \hookrightarrow \mathrm{Homeo}_+(\mathbb{R})$ be a dynamical realisation. Prove that for every $\gamma \in \Gamma$ we have either $\gamma(x) \geq x$ for every $x \in \mathbb{R}$, or $\gamma(x) \leq x$ for every $x \in \mathbb{R}$. Conversely, prove that a subgroup of $\mathrm{Homeo}_+(\mathbb{R})$ with this property is bi-orderable.

Exercice 24 (Uniqueness of the dynamical realisation up to conjugacy)**.** Let $(\Gamma, \preceq)$ be a countable left-ordered group. Recall that the dynamical realisation $\Gamma \hookrightarrow \mathrm{Homeo}_+(\mathbb{R})$ of $\preceq$ depends on the choice of a numeration $\Gamma = (\gamma_n)_{n\in\mathbb{N}}$. Show that a different choice of numeration gives rise to a conjugate subgroup of $\mathrm{Homeo}_+(\mathbb{R})$.

Exercice 25. Prove that in a group endowed with an Archimedean order, every proper convex subgroup is trivial.

Exercice 26. We analyse some orders on the group $\mathbb{Z}^2$.

1. The *lexicografic order* on $\mathbb{Z}^2$ is defined by setting $(n_1, m_1) \leq (n_2, m_2)$ if and only if $n_1 \leq n_2$ or if $n_1 = n_2$ and $m_1 \leq m_2$. Prove that the lexicographic order is a non-Archimedean left-invariant order on $\mathbb{Z}^2$.

2. Let $\ell \subset \mathbb{R}^2$ be a line through the origin with irrational slope. Let $P \subset \mathbb{Z}^2$ be the set of elements which lie in one of the half-planes delimited by ℓ. Prove that P is the positive cone of an order on $\mathbb{Z}^2$. Prove that this order is Archimedean, and that every Archimedean left-order on $\mathbb{Z}^2$ arises in this way for some line ℓ with irrational slope.

Exercice 27. Let $\mathrm{Aff}(\mathbb{R})$ be the subgroup of $\mathrm{Homeo}_+(\mathbb{R})$ consisting of the transformations $x \mapsto ax + b$, with $a > 0, b \in \mathbb{R}$. We further let $\Gamma \subset \mathrm{Aff}(\mathbb{R})$ be the countable subgroup of maps $x \mapsto ax + b$ such that $a = 2^n$ for some $n \in \mathbb{Z}$, and $b \in \mathbb{Z}[\frac{1}{2}]$ (cf. Exercise 17).

1. The *lexicographic* order on $\mathrm{Aff}(\mathbb{R})$ is defined as follows: the map $x \mapsto a_1x + b_1$ is not greater than $x \mapsto a_2x + b_2$ if $a_1 \leq a_2$ or if $a_1 = a_2$ and $b_1 \leq b_2$. Show that the lexicographic order is a left-invariant order.

2. Let $\iota \colon \Gamma \hookrightarrow \mathrm{Homeo}_+(\mathbb{R})$ the dynamical realisation of the lexicographic order restricted to Γ. Is it true that $\iota(\Gamma)$ is conjugate to the standard inclusion $\Gamma \subset \mathrm{Homeo}_+(\mathbb{R})$?

3. Choose $t \in \mathbb{R} \setminus \mathbb{Z}[\frac{1}{2}]$. Define an order $\preceq_t$ on Γ by letting $\gamma_1 \preceq_t \gamma_2$ if and only if $\gamma_1(t) \leq \gamma_2(t)$. Show that $\preceq_t$ is a left-invariant order whose dynamical realisation is conjugate to the standard inclusion $\Gamma \subset \mathrm{Homeo}_+(\mathbb{R})$.

4. Show that the maximal convex subgroup of Γ with respect to the lexicographic order is the group of translations $\{x \mapsto x + b \colon b \in \mathbb{Z}[\frac{1}{2}]\}$. In contrast, show that the maximal convex subgroup with respect to the orders $\preceq_t$ is trivial.

Exercice 28. Given a set I and a group Γ, the *direct sum* $\oplus_I \Gamma$ is the group of all of collections $(\gamma_i) \colon i \in I$ of elements of Γ such that $\gamma_i = \mathrm{Id}_\Gamma$ for all but finitely many $i \in I$. The *wreath product* of two groups Γ and Δ is the group

$$\Gamma \wr \Delta = \oplus_\Delta \Gamma \rtimes \Delta$$

where the semi-direct product is taken with respect to the translation action of Δ on $\oplus_\Delta \Gamma$ given by $\delta \cdot (\gamma_i)_{i \in \Delta} = (\gamma_{\delta^{-1} i})_{i \in \Delta}$.

1. Prove that the wreath product of two left-orderable groups is left-orderable.

2. Let $f \in \mathrm{Homeo}_+(\mathbb{R})$ be any non-trivial homomorphism. Choose $x \in \mathbb{R}$ such that $f(x) \neq x$, and let $g \in \mathrm{Homeo}_+(\mathbb{R})$ be any homeomorphisms supported in the interval between x and $f(x)$. Prove that the group generated by f and g is isomorphic to $\mathbb{Z} \wr \mathbb{Z}$.

Exercice 29. The purpose of this exercise is to establish establish an analogue for actions on the real line of the "fundamental trichotomy" for actions on the circle. Let $\Gamma \leq \mathrm{Homeo}_+(\mathbb{R})$.

1. Assume that Γ has no discrete orbit (we say that an orbit is discrete if the topology induced from $\mathbb{R}$ on it is the discrete topology). Prove that whenever $C, C' \subset \mathrm{Homeo}_+(\mathbb{R})$ are non-empty, closed and Γ-invariant, the intersection $C \cap C'$ is non-empty.

2. Assume Γ finitely generated. Prove that there exists a compact interval $I \subset \mathbb{R}$ such that for every closed, non-empty Γ-invariant subset C we have $C \cap I \neq \varnothing$.

3. Deduce the following result: if $\Gamma \subset \mathrm{Homeo}_+(\mathbb{R})$ is finitely generated, then either it has a discrete orbit, or it has a unique non-empty closed minimal invariant subset $M \subset \mathbb{R}$. Furthermore, in the latter case we either have $M = \mathbb{R}$, or M has empty interior and no isolated point.

4. Give a counterexample to the above statement when Γ is not finitely generated.

Exercice 30. Let Γ be a finitely generated group. Assume that Γ is amenable and has trivial abelianisation. Prove that every homomorphism $\Gamma \to \mathrm{Homeo}_+(\mathbb{R}/\mathbb{Z})$ is trivial.

3.1 Actions by diffeomorphisms

Exercice 31. Let Γ be a finitely generated group. Assume that Γ has no free subgroup and trivial abelianisation. Prove that every homomorphism $\Gamma \to \mathrm{Diff}^1_+(\mathbb{R}/\mathbb{Z})$ is trivial. (*Note*: Compare with exercise 30)

Exercice 32. Let Γ be a subgroup of $\mathrm{Diff}^2_+([0,1])$, and $\Delta \leq \Gamma$ be a normal subgroup of finite index. Assume that Δ is abelian. Show that Γ is abelian.

Exercice 33. Let Γ_1, Γ_2 be commuting subgroups of $\mathrm{Diff}^2_+([0,1])$ (meaning that every element of Γ_1 commutes with every element of Γ_2). Prove that the commutator subgroups Γ_1', Γ_2' have disjoint supports. Here the *support* of Γ_i is the set of point in $[0,1]$ which are moved by at least an element of Γ_i.