Université Claude Bernard Lyon 1
Institut Camille Jordan (UMR CNRS 5208)

HABILITATION À DIRIGER DES RECHERCHES

Spécialité : Mathématiques

présentée par

Nicolás Matte Bon

Dynamical and geometric aspects of infinite groups and their non-free actions

Aspects dynamiques et géométriques des groupes infinis et leurs actions non-libres

Rapporteurs : Emmanuel Breuillard, Peter Haïssinsky et Andrés Navas.

Soutenue publiquement le 11 janvier 2024 devant le jury composé de :

Emmanuel Breuillard	University of Oxford
François Dahmani	Université Grenoble Alpes
Anna Erschler	CNRS et École Normale Supérieure de Paris
Damien Gaboriau	CNRS et École Normale Supérieure de Lyon
Peter Haïssinsky	Université d'Aix-Marseille
Andrés Navas	Universidad de Santiago de Chile
Todor Tsankov	Université Claude Bernard Lyon 1

Introduction

This text surveys part of my work after my Ph.D thesis. The results presented focus on dynamical aspects infinite groups, and related geometric and analytic aspects. The primary focus is on countable discrete groups, but some results also deal with larger groups (endowed or not with a topology). No formal proofs are given, but some ideas of proof are discussed. The text is divided into three chapters corresponding to three different themes; each chapter is (almost) self-contained.

Chapter 1 is devoted to the so called Chabauty dynamics of groups and its applications. Most of my work in this area is part of a long-term collaboration with Adrien Le Boudec, and also includes collaborations with him and Pierre-Emmanuel Caprace and with Todor Tsankov. For every discrete or locally compact group G, the set of subgroups of G, denoted $\mathrm{Sub}(G)$ is naturally a compact space when endowed with the Chabauty topology, on which the group G acts by conjugation. This action can be seen as a (topological) dynamical system intrinsically attached to any group, which carries information about it. For example its fixed points are just the normal subgroups of G. When looking at that action from the point of view of topological dynamics, natural generalisations of normal subgroups appear: *confined subgroups*, and the related *uniformly recurrent subgroups*. These arise as stabilisers of group actions on compact spaces. We have developed the study of these notions, providing various classes of groups for which they can be completely understood, and giving applications to problems of different nature. For example, we provide general conditions for discrete and locally compact groups acting faithfully by homeomorphisms on a topological space, that imply that the group acts freely on its Furstenberg boundary [1,4]: in the discrete case, by results of M. Kalantar and M. Kennedy, this property is equivalent to the simplicity of the reduced group C^*-algebra; our results elucidate these properties for some well-studied groups. More applications concern rigidity phenomena for various groups of dynamical origin, which give restrictions on the possible homomorphisms of such groups to other groups of homeomorphisms [2, 3]. Using confined subgroups as a tool, we study a new type of group property, that we call *Schreier growth gap*, that provides a quantitative obstruction for a finitely generated group to admit faithful actions with small growth of orbits, and establish various instances of this phenomenon [2, 3, 5]. Finally we establish a connection between the confined subgroups of a groups and its highly transitive actions; among applications we classify all highly transitive actions of a family of finitely generated groups (namely the Higman-Thompson groups), which is the first non-trivial example of such a classification result [6].

Chapter 2 is devoted to the study of group actions on the real line. The work presented there is part of collaborations with Joaquín Brum, Cristóbal Rivas and Michele Triestino. The main setting is the study of groups on manifolds of dimension 1 (namely the circle or a real interval), by homeomorphisms or by diffeomorphisms of a given class $C^r, r \geq 1$. Various restrictions are known for actions by C^r-diffeomorphisms for some $r > 1$; our main focus is on actions on the real line in very low regularity, namely C^0 or C^1, which are generally more flexible. In this setting a large part of the litterature is devoted to the problem of discussing the existence (or non-existence) of a C^0 or C^1 action for different families of countable groups: the existence of a faithful C^0 action on the line is well-known to be equivalent to left-orderability of the group. Results addressing the structure and classification of such actions are somewhat more rare. We introduce some tools and ideas that lead a satisfactory picture of actions on real intervals by C^0 and C^1-diffeomorphisms for some classes of groups. For example we show that every faithful minimal C^1 action of a finitely generated solvable group on a real interval is conjugate to an action by affine transformations on the line [9], and that for a vast class of groups arising via an action on an interval (including Thompson's group F), the natural defining action is essentially the unique C^1 action [8]. In fact these results follow from a comprehensive understanding of the possible C^0 actions of such groups, which turn out to be much richer. We shall put some emphasis on the role of an object called the *harmonic space* of G, which is a compact space with a flow $(\mathcal{D}, \Psi)$ associated to any finitely generated group G (whose construction depends on results of Deroin, Kleptsyn, Navas and Parwani [DKNP13]), unique up to flow equivalence, which plays a role of moduli space for the actions of G by homeomorphisms on the line up to semi-conjugacy. Reversing this point of view, we introduce a construction of groups acting on a class of flows on compact spaces, namely suspension flow of subshifts, which yields left-orderable groups satisfying various interesting properties (in particular which are simple and finitely generated, conceptualizing and improving the first construction of such groups, provided shortly before by J. Hyde and Y. Lodha) [7]. Considering a variant of this construction, we prove a realisation result for harmonic spaces of group actions on the line: every suspension flow of a subshift, satisfying mild additional conditions, is flow equivalent to the harmonic space $(\mathcal{D}, \Psi)$ of a finitely generated group [10]. This provides a source of finitely generated groups satisfying some new phenomena for actions on the line, related to their rigidity/flexibility properties and to the structure of (path-)connected components of the space of actions.

Chapter 3 is devoted to my recent work with Volodia Nekrashevych and Tianyi Zheng [11]. The general context is the study of amenability and the Liouville property for random walk in the realm of groups of dynamical origin; this was also the main subject of my Ph.D. The Liouville property is a strengthening of amenability; it means that any bounded harmonic function on the Cayley graph of the group is constant. We are interested in the class of *contracting self-similar groups*. These arise in connection to dynamics, namely as *iterated monodromy groups* of spaces endowed with an expanding (branched) self-covering (for example, post-critically finite rational functions on the Riemann sphere). The study of amenability of contracting self-similar groups has a long history: this class (or close relatives) contains the first examples non-elementary amenable groups (the Grigorchuk groups [Gri84]), and for a long time it remained essentially the only source of such examples. It is a well-known open question whether all contracting self-similar groups are amenable. We show amenability of many contracting groups by establishing a connection with the theory of *conformal dimension*, an invariant of self-similar metric spaces which is widely studied in geometric group theory and complex dynamics. Namely, we show that for any expanding partial self-covering, if the (Ahlfors-regular) conformal dimension of the space is less than 2, then the iterated monodromy group is Liouville (and hence amenable). This result recovers all previous amenability results for contracting groups, and applies to many new examples. In particular, it implies the amenability of the iterated monodromy group of any post-critically finite rational function whose Julia set is not the whole sphere.

The three chapters have common aspects, and there could have been other ways of grouping the results thematically. I have chosen to emphasize in the title one aspect which appears frequently in my research: the role played by group actions that are non-free in some strong sense, i.e. where non-trivial elements can admit a large set of fixed points. Conceiving groups as transformations groups, thus faithfully represented through their actions, the tension between actions that are free (or close to be free) and non-free (possibly in some strong sense) appears naturally, giving rise to situations with different taste and nature. Consider for example the case of finite simple groups, which (up to exceptions) essentially split in two families: those of Lie type, such as $\mathrm{PSL}(n, q)$, and the alternating groups $\mathrm{Alt}(n)$. Compare, say, the natural action of $\mathrm{PSL}(n, q)$ on the projective space $\mathbb{P}(\mathbb{F}_q^n)$ with the action of $\mathrm{Alt}(n)$ on $\{1, \ldots, n\}$. Neither action is free on the nose, but in the first case the set of fixed point of a non-trivial element is always a subspace of strictly smaller dimension, so that its size is at most a proportion $\frac{q^{n-1}-1}{q^n-1} \sim \frac{1}{q}$ of the space (in particular it is negligeable for q large); instead the groups $\mathrm{Alt}(n)$ contain non-trivial elements which fix an arbitrary proportion of points. In fact, by a result of R. Guralnick and K. Magaard [GM98], if a finite simple group admits a primitive action $G \curvearrowright \Omega$ with a non-trivial element fixing at least $\frac{4}{7}|\Omega|$ points, then G is isomorphic to an alternating group[1]. This is an example of a situation where a form of non-freeness gives rise to structure, a principle that will appear in various forms.

A large part of what is now called geometric group theory has been developed having in mind classical examples of groups from geometry and topology (discrete subgroups of Lie groups, fundamental groups of negatively curved manifolds..). Many of these examples admit natural actions that satisfy some form of freeness, in a geometric sense (proper isometric actions) or dynamical sense (topological freeness, essential freeness), and are suitably studied through those actions. In contrast groups defined by actions that are very far from being free actions have regularly appeared as source of examples considered somewhat surprising and "exotic" with respect to these classical examples. Two early examples of this philosophy are the Grigorchuk groups and Thompson's groups, and many more examples can be added.

I started to develop some taste for constructions based on non-free actions during my Ph.D, whose main results dealt with establishing the Liouville property and amenability of various groups. Many examples that are interesting in this context have dynamical origin, and share the feature to be given by an action on a compact space which is highly non-free. As a consequence the geometry of the group is not easily accessible from the action. The orbits of action, and the associated Schreier graphs, are tipically more manageable and play a crucial role in studying the group. Often the combinatorial and geometric structure of these graphs is determined by some underlying dynamical system, and the difficulty is to find a more direct relationship between the dynamical properties of the system and the geometry of the group. The results in Chapter 3 also belong to this line of research.

It was in this context that my collaboration with A. Le Boudec on the topic of Chapter 1 began when I was finishing my Ph.D. The content of our collaboration can be summarized as the systematic study of non-free actions in topological dynamics. I had just grew up in a context rich of examples and natural questions in this setting, he came to it from a different angle, namely the recent developements in C^*-simplicity (due to M. Kalantar, M. Kennedy, E. Breuillard, N. Ozawa, and himself), and we quickly learned from each other. Because of the nature of the topic, some of the results we developed turned out to be particularly suited to study groups of dynamical origin, although not limited to those.

[1] The use of this example in this context is borrowed from an inspiring talk by Pierre-Emmanuel Caprace.

Finally non-free actions also play a permeating role in Chapter 2. A fundamental reason is an old theorem of Hölder, which implies that any group acting freely on a manifold of dimension 1 is abelian and semiconjugate to a group of isometries. All the richness of the theory for arbitrary groups therefore comes from non-free actions. This is particularly true in regularity C^0, for which Hölder's theorem is *the* starting point of the theory: most other founding results (the classifications of actions according to their contraction properties, the theory of Conradian actions..) can be deduced by taking it as a blackbox. Additionally, the world of group actions on one-manifold is a rich source of examples of group actions which are very far from being free and arise naturally in other settings. The results in Chapter 2 are largely based on the use of non-freeness as a source of structure.

Acknowledgements

I am very grateful to Emmanuel Breuillard, Peter Haïssinsky and Andrés Navas for accepting being the referees of this memoir. I also thank François Dahmani, Anna Erschler, Damien Gaboriau, Todor Tsankov for accepting being part of the defense committee.

I thank all my collaborators and friends, and a particularly warm thank goes to Adrien Le Boudec, Joaquín Brum, Cristóbal Rivas, and Michele Triestino, for our long-term collaborations. I have been knowing and interacting with Volodia Nekrashevych and Tianyi Zheng for many years, and I'm very happy that this has lead to our recent collaboration.

Finally I want to thank Marzia for being supportive and (most often but not always) patient, but most importantly for just being there.

Contents

Chapter 1

Chabauty dynamics and applications

1.1 Introduction and preliminaires

For a locally compact group G, the space $\mathrm{Sub}(G)$ of subgroups of G is naturally a compact space endowed with the *Chabauty topology*. When G has the discrete topology (which for us will always be the case with the exception of Theorem 1.3.3) this is simply the topology induced from the product topology on the space of all subsets 2^G. The group G acts on $\mathrm{Sub}(G)$ by conjugation. This intrinsic dynamical system attached to every group G plays a role in the study of actions of the group G on other spaces, as it governs the structure of their stabilisers. In the last decade it was progressively realised that the a systematic study of the action of G on $\mathrm{Sub}(G)$ is fruitful and has various applications. In particular the study of G-invariant and stationnary probability measures on $\mathrm{Sub}(G)$ (named *invariant random subgroups* in [AGV14]) became a popular topic, see for instance the survey of Gelander [Gel18] for various applications to the study lattices in Lie groups and other locally compact groups.

In a long-term collaboration with Adrien Le Boudec, we have developed the study of the conjugation action of G on $\mathrm{Sub}(G)$ from the perspective of topological dynamics, with various applications to other aspects of infinite groups. Two notions play an important role in our study.

Definition 1.1.1. A subgroup $H \leq G$ is confined if the closure of its G-orbit in $\mathrm{Sub}(G)$ does not contain the trivial subgroup $\{1\}$.

When G is discrete, by definition of the topology a subgroup H is confined if and only if there exists a finite subset P of non-trivial elements of G such that $gHg^{-1} \cap P \neq \varnothing$ for every $g \in G$.

Definition 1.1.2 (Glasner and Weiss [GW15])**.** A uniformly recurrent subgroup (URS) is a closed minimal G-invariant subset of $\mathcal{H} \subset \mathrm{Sub}(G)$.

The two notions are related as follows: if a non-trivial subgroup H belongs to a URS, then it is confined, and conversely if H is a confined subgroup then the closure of its orbit contains a non-trivial URS.

The notion of URS was proposed by Glasner and Weiss [GW15] as a topological dynamical analogue of invariant random subgroups; it also appears implicitly before, e.g. in [GNS00, Gri11]. An early appearance of the notion of confined subgroups can be traced back to the study of ideals in group algebras of locally finite groups, notably in [SZ93, Zal95, HZ97]. In particular the terminology was introduced in [HZ97], in the special case of locally finite groups. While in our early work [1] the focus was more on uniformly recurrent subgroups, we progressively realised that the notion of confined subgroups is in a sense more fundamental, both for applications, and because results on uniformly recurrent subgroups usually pass from some understanding of confined subgroups.

Confined subgroups appear as stabilisers of group actions on compact spaces by homeomorphisms. Namely, suppose that a group G, for simplicity supposed to be countable discrete, acts on a compact space X by homeomorphisms (we shall also say that X is a compact G-space). For $x \in X$, we denote by $\mathrm{Stab}_G(x)$ its stabiliser subgroup, namely the set of group elements that fix x. In the topological dynamical setting, it is also important to consider another subgroup, that we call the *germ stabiliser* and denote $\mathrm{Stab}^0_G(x)$, which is the (normal) subgroup of $\mathrm{Stab}_G(x)$ consisting of elements that fix pointwise some neighbourhood of x. We thus have a pair of G-equivariant maps

$$\mathrm{Stab}_G \colon X \to \mathrm{Sub}(G), \qquad \mathrm{Stab}^0_G \colon X \to \mathrm{Sub}(G).$$

Neither of these maps is continuous in general (however, they are upper and lower semi-continuous respectively). It is not difficult to see that the continuity points of Stab_G are precisely those where $\mathrm{Stab}_G(x) = \mathrm{Stab}^0_G(x)$; an application of Baire's theorem shows that the latter is a dense G_δ subset of X. This discussion can be extended to second countable locally compact groups, with suitable changes [1].

Definition 1.1.3. Let X be a compact G-space.

(i) The action of G on X is said to be *topologically free* if $\mathrm{Stab}^0_G(x) = \{1\}$ for every $x \in X$ (equivalently, the set of fixed point $\mathrm{Fix}(g)$ of every non-trivial element g has empty interior).

(ii) We say that the action is *nowhere topologically free* if for every $x \in X$ we have $\mathrm{Stab}^0_G(x) \neq \{1\}$ (equivalently, every point is contained in the interior of $\mathrm{Fix}(g)$ for some non-trivial g).

The following observation is elementary.

Lemma 1.1.4. *if $G \curvearrowright X$ is a nowhere topologically free action on a compact space, then the germ stabiliser* $\mathrm{Stab}^0_G(x)$ *is a confined subgroup of G for every $x \in X$, and hence* $\mathrm{Stab}_G(x)$ *as well (since being confined passes to overgroups).*

If we further assume that the action of G on X is minimal, then Glasner and Weiss observed the following.

Proposition 1.1.5 ([GW15])**.** *For every minimal action $G \curvearrowright X$ of a locally compact group on a compact space, the closure of the image of the stabiliser map* $\mathrm{Stab}_G \colon X \to \mathrm{Sub}(G)$ *contains a unique URS, called the stabiliser URS of the action, that we shall denote by $\mathcal{S}_G(X)$. Moreover if G is second countable, then* $\mathrm{Stab}_G(x) \in \mathcal{S}_G(X)$ *for x in the (dense G_δ) subset of continuity points of the map* Stab_G*.*

[1]For a general locally compact group, $\mathrm{Stab}^0_G(x)$ should be defined as set of elements g with the property that for every neighbourhood U of g, there exists a neighbourhood V of x, such that every $y \in V$ is fixed by some element of U.

The stabiliser URS is trivial (that is, reduced to the trivial subgroup $\{1\}$) if and only if the action is topologically free. (If a minimal action is not topologically free, then it is nowhere topologically free).

A converse of these constructions also holds, as we observed with T. Tsankov [12]: if G is a locally compact group, then for every closed G-invariant subset $\mathcal{H} \subset \mathrm{Sub}(G)$, there exists a continuous action of G on a compact space X such that $\mathrm{Stab}_G \colon X \to \mathrm{Sub}(G)$ is everywhere continuous (in particular it is equal to the map Stab^0_G) and its image is equal to $\mathcal{H}$. It follows in particular that every URS of G is the stabiliser URS of some minimal action on a compact space. The question whether this is the case had been raised by Glasner and Weiss [GW15] (for finitely generated groups it was independently answered by Elek [Ele18]). Our proof is a close adaptation of the proof of the classical result of Veech, that every locally compact group acts freely on some compact space [Vee77] (this corresponds to the case $\mathcal{H} = \{\{1\}\}$).

When G is finitely generated, there is an equivalent interpretation of the action of G on $\mathrm{Sub}(G)$ in terms of Schreier graphs. Recall that for a subgroup H and a finite generating set S of G, the Schreier graph $\Gamma(G, G/H)$ is the graph whose vertex set is G/H and with edges (gH, sgH) for $s \in S$. It is naturally pointed at the trivial coset H, and its edges are labelled by elements of S. The correspondence $H \mapsto \Gamma(G, G/H)$ identifies the topology on $\mathrm{Sub}(G)$ with the topology induced from the space of marked graphs, where two graphs are close if large balls around the basepoints are isomorphic as labelled graphs. With this point of view, a subgroup H is confined if and only if the natural covering map from the Cayely graph of G to $\Gamma(G, G/H)$ has uniformly bounded injectivity radius; that is, $\Gamma(G, G/H)$ does not contain embedded copies of arbitrarily large balls in the Cayley graph of G. This characterisation will be useful in some situations, as it allows to use large-scale geometric invariants of graphs (such as volume growth and asymptotic dimension) to make confined subgroups appear.

The results discussed in this chapter are based on the study of confined subgroups or (URSs) in various families of groups, although this is not always apparent from the statements. They have been chosen so as to illustrate various types of applications to other aspects of groups, rather than focusing on the mere study of the conjugation action of G on $\mathrm{Sub}(G)$.

1.2 Confined subgroups and groups of homeomorphisms

This section surveys part of the results contained in [1, 2, 3]. A common denominator of these results is that they apply to groups naturally given by an action $G \curvearrowright X$ by homeomorphisms on some space, and the properties of this action are used to understand the confined subgroups and the URSs of G.

1.2.1 A general tool

The essential ingredient of our method is a general result that applies to groups $G \leq \mathrm{Homeo}(X)$ given by a faithful action by homeomorphisms of Hausdorff space X. Given a non-empty open set $U \subset X$ we denote by G_U the subgroup consisting of elements that are the identity on $X \setminus U$ (called the *rigid stabilizer* of U). We make the implicit assumption that the groups G_U are non-trivial for every non-empty open set: in this case say that G is *micro-supported* (if this is not the case, the result below has trivial conclusion).

Various countable groups of interest arise as micro-supported groups of homeomorphisms (some examples will appear later in this text). Among other things, the class of micro-supported groups has been a vast source of examples of groups that are simple, or more generally have few normal subgroups. The study of normal subgroups of such groups often starts with an elementary observation: for every $G \leq \mathrm{Homeo}(X)$, where X is any Hausdorff space, every non-trivial normal subgroup of G must contain $[G_U, G_U]$, the commutator subgroup of G_U, for some non-empty open set U. This is sometimes called the "double commutator lemma", as its proof is a simple trick involving two commutators; versions of this trick can be found e.g. in the proofs of old simplicity criteria of Higman [Hig54], or Epstein [Eps70]. In particular in order to study normal subgroups of G one is lead to understand the normal closures of the groups $[G_U, G_U]$; this can be done under additional assumptions on G and its action on X (typically versions of the so called *fragmentation property*, which require that the union of the groups G_U for sufficiently many open sets U generate a large subgroup of G).

Normal subgroups are (very special) examples of confined subgroups. It turns out that the statement mentioned above holds true for confined subgroups as well:

Theorem 1.2.1 ([3]). *Let X be a Hausdorff space and G be a discrete group acting faithfully by homeomorphisms on X. If $H \leq G$ is a confined subgroup of G (e.g. if H belongs to a non-trivial URS of G), then there exists a non-empty open subset $U \subset X$ such that H contains $[G_U, G_U]$.*

This results can be thought as a dichotomy for stabilisers of actions by homeomorphisms: if $G \leq \mathrm{Homeo}(X)$ is as above, and if $G \curvearrowright Y$ is an action of G by homeomorphisms on a compact space, then either some germ stabiliser $\mathrm{Stab}^0_G(y)$ is trivial, or all of them must be large enough and reminiscent of the action on X (each contains $[G_U, G_U]$ for some non-empty open set U).

The first versions of this result were obtained in [1] and [2], (which provide a weaker conclusion, or the same conclusion under stronger assumptions on the action of G on X); finally we removed all unnecessary assumption in [3]. It is convenient to phrase Theorem 1.2.1 for groups of homeomorphisms to make its statement cleaner, but it should be stressed that the role played by the topology on X is minor (given the little structure available): it is essentially is a combinatorial result about group actions on sets. The proof is elementary in nature, but substantially more involved than the simple trick mentioned above for normal subgroups.

Theorem 1.2.1 has a direct application to C^*-simplicity, postponed to §1.3 below. Beyond this, using Theorem 1.2.1 as main tool, and exploiting further assumptions on the action of G on X when available, we obtained a complete understanding of confined subgroups, or uniformly recurrent subgroups, of various classes of groups studied in the litterature, that appear via a micro-supported action on some topological space. These results provide restrictions on the possible actions of such groups on compact spaces. I discuss below some sample results in this spirit, without being exhaustive.

1.2.2 A first example: Thompson's groups

Famous examples of micro-supported groups are Thompson's groups $F \leq T \leq V$. Thompsons' group F is the group of all piecewise linear homeomorphisms of the unit interval, with finitely many discontinuity points for the derivative, all being dyadic rationals, and which are piecewise of the form $x \mapsto 2^n x + b$ for some $n \in \mathbb{Z}$ and $b \in \mathbb{Z}[\frac{1}{2}]$. A similar definition on the circle $\mathbb{R}/\mathbb{Z}$ yields Thompson's group T. Thompson's group V is defined as a group of homeomorphisms of the Cantor space of one-sided binary sequences $\{0,1\}^{\mathbb{N}}$, which are locally given by homeomorphisms of the form $v\xi \mapsto w\xi$ for some finite binary words $v, w \in \{0,1\}^*$ and $\xi \in \{0,1\}^{\mathbb{N}}$. The following result is proven in [1].

Theorem 1.2.2 ([1]). *The following hold for the Thompson's groups $F \leq T \leq V$.*

1. *The only non-trivial URSs of F are the singletons $\{N\}$, where N is a normal subgroup (these are the trivial subgroup and the subgroups that contain the commutator subgroup $[F, F]$).*

2. *The only non-trivial URSs of the group T are $\{\{1\}\}, \{T\}$ and the stabiliser URS associated to its action on $\mathbb{S}^1$.*

3. *The only non only non-trivial URSs of the group V are $\{\{1\}\}, \{V\}$ and the stabiliser URS associated to its action on the Cantor set.*

The last two items are a special case of a result from [1] which gives sufficient conditions on a minimal group action G on compact space X, under which the URS associated to the action is the only non-trivial URS of G (apart from $\{\{1\}\}$ and $\{G\}$). That criterion applies to other groups, including some variants of the Thompson's groups, and as groups acting on trees studied by A. Le Boudec in [LB16]. Prior to these results, there were essentially no groups whose uniformly recurrent subgroups were completely understood, leaving apart some rather degenerate cases (such as groups with only countably many subgroups). By now there are many more examples.

The scarcity of URSs in a group provides constraints on its minimal actions on compact spaces which are not topologically free. Indeed, from Theorem 1.2.2 and with some additional work, we get the following. Recall that a group action by homeomorphisms on a compact space $G \curvearrowright X$ *factors onto* another such action $G \curvearrowright Y$ if there exists a continuous, surjective G-equivariant map $p \colon X \to Y$.

Corollary 1.2.3.

1. *Every minimal faithful action of the group F on a compact space is topologically free.*

2. *A minimal action $T \curvearrowright X$ of Thompson's group T on a compact space which is not topologically free must factor onto its natural action on the circle $\mathbb{S}^1$.*

3. *A minimal action of the group V on a compact space set which is not topologically free must factor onto its natural action on the Cantor set.*

1.2.3 Rigidity of actions and embeddings: the case of topological full groups

Various interesting groups defined by a micro-supported action by homeomorphisms are obtained starting from a much simpler group or pseudogroup of transformations, and then considering transformations defined "piecewise" by those transformations. Thompson's groups above are an example of this principle. A general notion that systematizes this idea is the one of topological full group.

Given a group $\mathcal{G}$ of homeomorphisms of a topological space X, its topological full group is the group $F(\mathcal{G})$ of all homeomorphisms h of X that locally coincide with elements of $\mathcal{G}$, namely such that for every $x \in X$ there exists an open neighbourhood U of x and $g \in \mathcal{G}$ such that $h|_U = g|_U$. Since this definition is local, it makes sense more generally when $\mathcal{G}$ is a *pseudogroup* of homeomorphisms between opens subsets of X; this setting is more convenient to develop the theory but I shall avoid introducing it here for simplicity.

The notion of topological full group is mostly relevant when X is a Cantor space (i.e. homeomorphic to the Cantor set), as we assume below. It was first studied when $\mathcal{G}$ is the cyclic group generated by a single homeomorphism $\varphi\colon X \to X$ [GPS99, Mat06]. This construction attracted increasing attention after the result of Juschenko and Monod that the topological full group $F(\varphi)$ is amenable for every minimal homeomorphism $\varphi\colon X \to X$ of a Cantor space [JM13].

H. Matui showed that for various classes of minimal (pseudo-)groups $\mathcal{G}$ of homeomorphisms of the Cantor space, the commutator subgroup of $F(\mathcal{G})$ is simple and finitely generated [Mat06, Mat15]. For a general $\mathcal{G}$, Nekrashevych defined a subgroup $A(\mathcal{G}) \leq F(\mathcal{G})$ called the *alternating full group* of the action [Nek19]. It can be defined as the subgroup of $F(\mathcal{G})$ generated by elements h that have order 3 and whose support can be partitioned into 3 disjoint clopen sets which are permuted cyclically by h (such elements are analogues of 3-cycles in the symmetric groups). In many cases, the group $A(\mathcal{G})$ coincides with the commutator subgroup of $F(\mathcal{G})$; it is not known whether this is true for every minimal action $\mathcal{G} \curvearrowright X$. Nekrashevych showed that if $\mathcal{G} \curvearrowright X$ is minimal, then the group $A(\mathcal{G})$ is simple and contained in every non-trivial normal subgroup of $F(\mathcal{G})$. In addition, if $\mathcal{G}$ is finitely generated the action $\mathcal{G} \curvearrowright X$ is expansive, then the group $A(\mathcal{G})$ is finitely generated. A finitely generated group action on the Cantor space $\mathcal{G} \curvearrowright X$ is expansive if and only if it is conjugate to a subshift, that is, a closed $\mathcal{G}$-invariant subset of $A^{\mathcal{G}}$, where A is a finite alphabet.

In [2] I obtain a classification of the confined subgroups of the groups $F(\mathcal{G})$ and $A(\mathcal{G})$, which holds for any minimal group action $\mathcal{G} \curvearrowright X$ on the Cantor set: essentially, the only confined subgroups are the stabilisers and germ stabilisers of finite collections of points $\{x_1, \ldots, x_r\} \subset X$ (and subgroups nested between the two). Using that classification as main tool, I obtained a rigidity result for actions of full groups a rigidity result for actions on compact spaces which are nowhere topologically free. I shall state here only a simplified version of that result. Given a space X, the *configuration space* of at most r points in X is defined as the space of non-empty open subsets of X of cardinality at most r, and is denoted $X^{[r]}$.

Theorem 1.2.4 (Rigidity of non-free actions of topological full groups [2])**.** *Let $\mathcal{G} \curvearrowright X$ be a minimal action on the Cantor set, and G be a group such that $A(\mathcal{G}) \leq G \leq F(\mathcal{G})$. Assume that $G \curvearrowright Y$ is a faithful action on a compact space which is nowhere topologically free. Then there exists a partition $Y = Y_0 \sqcup Y_1$ into disjoint G-invariant clopen subsets such that:*

- *the action of $A(\mathcal{G})$ on Y_0 is trivial (and hence the action $G \curvearrowright Y_0$ descends to an action of the quotient $G/A(\mathcal{G})$).*
- *The action $G \curvearrowright Y_1$ factors onto the action on a configuration space $G \curvearrowright X^{[r]}$ for some $r \geq 1$.*

Remark 1.2.5. The actual main result in [2] has a stronger conclusion, which uses the language of pseudo-groups (which is relevant in its statement even if one is interested only in the special case of group actions). Roughly speaking, it says that any nowhere topologically free action of the group $A(\mathcal{G})$ is induced by a homomorphisms between pseudogroups in a suitable sense; this formulation implies the existence of a factor map as above, but also gives information on the fibers of the factor map, and relates well with various invariants of pseudogroups (such as homology and geometric invariants), which are useful in applications

[2]. This has led me to develop properly the study of a notion of morphisms between pseudo-groups of homeomorphisms and the associated groupoids of germs, which was missing in the litterature and occupies a large part of [2].

Note that contrary to Corollary 1.2.3, Theorem 1.2.4 applies to actions $G \curvearrowright Y$ that are not necessarily minimal. One advantage of a result holding for all topologically nowhere free actions of a group G (without any minimality assumption), is that such a result implies restrictions on the possible embeddings of G into other groups defined via an action by homeomorphisms. Indeed if $H \leq \mathrm{Homeo}(Y)$ is a group acting on a compact space, there is no reason that a group embedding $G \hookrightarrow H$ should give rise to a minimal action (its minimal sets could be just fixed points). However, the properties of the action $H \curvearrowright Y$ might suffice to enforce that all embeddings $\varphi\colon G \hookrightarrow H$ induce an action which is nowhere topologically free. One way to ensure this is to analyse the geometry of the *graph of germs* of the action of the target group $H \curvearrowright Y$, which are defined as the Schreier graphs associated to the germ-stabilisers $\mathrm{Stab}^0_H(y)$. It is enough for this to show that there is "no room" in these graphs to contain copies of G embedded in a Lipschitz way (which would be the case if the G-action is somewhere topologically free). A large-scale invariant which is adapted to this purpose is asymptotic dimension, introduced by Gromov [Gro93a]. In the setting of topological full groups, this is particularly natural because the groups $A(\mathcal{G})$ or $F(\mathcal{G})$ always have infinite asymptotic dimension, but their graph of germs are quasi-isometric to those of the original action $\mathcal{G} \curvearrowright X$, and in many interesting examples their asymptotic dimension is finite. Thus Theorem 1.2.4 has the following corollary which allows to understand embeddings between the topological full groups of a vast class of group actions.

Corollary 1.2.6 (Rigidity of embeddings between topological full groups [2])**.** *Let $\mathcal{G} \curvearrowright X$ be a minimal action on a Cantor space of a finitely generated group, and G be a group such that $A(\mathcal{G}) \leq G \leq F(\mathcal{G})$. Let $\mathcal{H} \curvearrowright Y$ be an action of a finitely generated group on a compact space, whose graphs of germs have finite asymptotic dimension (for instance, if the action is topologically free and the group $\mathcal{H}$ has finite asymptotic dimension).*

Then for any group embedding $G \hookrightarrow F(\mathcal{H})$, the induced action of G on Y satisfies the conclusion of Theorem 1.2.4.

Previously it had long been known that if $\mathcal{G} \curvearrowright X, \mathcal{H} \curvearrowright Y$ are minimal group actions on the Cantor set, then for every group *isomorphism* $\varphi\colon F(\mathcal{G}) \to F(\mathcal{H})$, there exists a homeomorphism $q\colon X \to Y$ that $\varphi(g) = qgq^{-1}$ for every $g \in G$. This result follows from a more general result of M. Rubin [Rub89] (it was rediscovered in this setting by Matui [Mat15]); it is an example of a so called "reconstruction result" for groups of homeomorphisms. Such results give conditions on a group action by homeomorphism $G \curvearrowright X$ under which the space X and the action can be abstractly reconstructed from X. Early examples of reconstructions results were proven for the groups $\mathrm{Homeo}(M)$ and $\mathrm{Diff}(M)$ of homeomorphisms and diffeomorphisms of manifolds (Whittaker [Whi63], Filipkiewicz [Fil82]). M. Rubin [Rub89, Rub96] later provided a vast array of such results applicable to very general micro-supported group of homeomorphism $G \leq \mathrm{Homeo}(X)$ satisfying some mild additional condition (tipically expressed in terms of the action of the groups G_U on U). Rubin was mostly motivated by the group of all homeomorphisms of suitable spaces, but his results are also applicable to countable subgroups and have been widely used (or rediscovered in special cases) to distinguish such groups up to isomorphisms. A well-known limit of those classical reconstruction methods is that they can't be applied to study embeddings between groups; this is because the conditions on the action necessary to apply them are too strong and need not be satisfied by the image of an arbitrary embedding. This very general problem was raised by Rubin, who asked (with a carefully vague formulation): "are any reasonable assumptions under which an embedding of the group of homeomorphisms of a space X into another group of homeomorphism of a space Y implies that X is some kind of continuous image of Y?" [Rub89]. Corollary 1.2.6 provides a satisfactory answer in the setting of topological full groups. Other results in this spirit have been proven for homeomorphisms and diffeomorphisms groups of manifolds, for which the problem of studying embeddings was independently raised by É. Ghys and and has been studied by various authors (see e.g. Hurtado [Hur15], Chen and Mann [CM23]). That setting is quite different, as those groups admit a natural Polish topology which is crucial to the arguments, while Corollary 1.2.6 deals with countable groups. Nevertheless, there is a striking formal analogy between Theorem 1.2.4 and the results of L. Chen and K. Mann [CM23].

I shall conclude this part by illustrating Corollary 1.2.6 in a couple of concrete special cases. First consider the case of topological full group $F(\varphi)$ of a $\mathbb{Z}$ action, generated by a homeomorphism $\varphi\colon X \to X$. It is not difficult to see that if $\psi\colon Y \to Y$ is another homeomorphism that factors onto (X, φ), then there is a natural embedding of the group $F(\varphi)$ into $F(\psi)$. In addition for every $\psi_0 \in F(\psi)$ the group $F(\psi_0)$ is naturally a subgroup of $F(\psi)$. Using Corollary 1.2.6, one can show that all embeddings between the groups $F(\varphi)$ arise from these two observations:

Theorem 1.2.7 ([2])**.** *Let $(X, \varphi), (Y, \psi)$ be homeomorphisms of Cantor spaces, with (X, φ) minimal. Then there exists an embedding of the group $F(\varphi)$ into $F(\psi)$ if and only if there exists some element $\psi_0 \in F(Y, \psi)$ supported on some clopen subset $Y_0 \subset Y$ and such that (Y_0, ψ_0) factors onto (X, φ).*

One of my personal motivations to prove Theorem 1.2.4 was the fact that the subgroups structure of topological full groups is largely mysterious. In other words we know very little about how the dynamical properties of an action $\mathcal{G} \curvearrowright X$ limit the possible subgroups of its topological full group. From this perspective it is natural to ask when they can embed into each other. Corollary 1.2.6 allows to show that many natural dynamical invariants of actions provide obstructions to such embeddings, confirming in some situations the natural intuition that "simpler" group actions have "smaller" topological full group. A basic example is the following consequence of Theorem 1.2.7, where h_{top} denotes topological entropy.

Corollary 1.2.8. *Let $(X, \varphi), (Y, \psi)$ be a homeomorphism of the Cantor set, and assume that $h_{top}(X, \varphi) > 0$ and $h_{top}(Y, \psi) = 0$. Then the group $F(\varphi)$ does not admit any embedding into $F(\psi)$.*

Remark 1.2.9. The numerical value of entropy is not an obstruction to group embeddings, as can be seen by the fact that $h_{top}(\varphi^2) = 2h_{top}(\varphi)$ and $F(\varphi^2) \leq F(\varphi)$.

As another example, let us consider subgroups of the group IET of interval exchange transformations. The group IET is the group of all piecewise continuous bijection of the circle $\mathbb{R}/\mathbb{Z}$, with finitely many discontinuities, which in restriction to each interval of continuity coincide with a translation $x \mapsto x + \alpha$. Given a finitely generated subgroup $\Lambda \leq \mathbb{R}/\mathbb{Z}$, denote by IET($\Lambda$) the subgroup of IET consisting of all elements whose discontinuity points belong to Λ and that are piecewise of the form $x \mapsto x + \lambda$ with $\lambda \in \Lambda$. The group IET(Λ) is finitely generated. As observed in [Cor14] the group IET(Λ) can be identified with the topological full group of a minimal action of the finitely generated abelian group Λ on the Cantor set, obtained from its action by translations on the circle $\mathbb{R}/\mathbb{Z}$ by "blowing up" the orbit of 0 as in the classical Denjoy counterexample [Cor14]. The subgroup structure of the group IET is quite mysterious. Since every finitely generated subgroup of IET is contained in $\mathrm{IET}(\Lambda)$ for some finitely generated $\Lambda \leq \mathbb{R}/\mathbb{Z}$, it is natural to ask how the possible subgroups of $\mathrm{IET}(\Lambda)$ depend on the nature of Λ. There are very few results in this direction. The following result, also based on Corollary 1.2.6, goes in this direction by classifying when the groups $\mathrm{IET}(\Lambda)$ embed into each other.

Theorem 1.2.10 ([2]). *Let $\Lambda, \Delta \leq \mathbb{R}/\mathbb{Z}$ be subgroups, and denote $\tilde{\Lambda}, \tilde{\Delta} \leq \mathbb{R}$ their preimages. Then the following are equivalent*

(i) exists an (abstract) group embedding of $\mathrm{IET}(\Lambda) \hookrightarrow \mathrm{IET}(\Delta)$*;*

(ii) there exists some $0 < \alpha \leq 1$ such that the affine map $x \mapsto \alpha x$ sends $\tilde{\Lambda}$ inside $\tilde{\Delta}$.

In [3], we prove also several results in a similar spirit within the class of (weakly) branch groups acting on rooted trees, which I shall not discuss here.

1.3 Furstenberg boundary and C^*-simplicity

In topological dynamics, a *boundary action* of a group G is a continuous action by homeomorphisms on a compact space X which is minimal and strongly proximal, that is, every probability measure on X can be accumulated by G on some Dirac mass (the space X is also called a G-boundary). Every group has a universal boundary $G \curvearrowright \partial_F G$, called the Furstenberg boundary, which factors onto every boundary of G.[2] For a discrete non-amenable group G, the space $\partial_F G$ is never metrisable, and is highly non-explicit.

Boundary actions and their stabilisers have turned out to be intimately connected to the study of unitary representations and the associated C^*-algebras. Recall that a group G is C^*-simple if its reduced C^*-algebra (namely the C^*-algebra generated by the left-regular representation $G \curvearrowright \ell^2(G)$) is simple; equivalently if every unitary representation of G which is weakly contained in the left-regular representation is weakly equivalent to it. There has been an extensive literature on the problem of determining which groups have this property. A series of breakthrough papers of by Kalantar and Kennedy [KK17], Breuillard, Kalantar, Kennedy, Ozawa [BKKO17] and Kennedy [Ken15] lead to the following characterisation.

Theorem 1.3.1 ([KK17, BKKO17, Ken15]). *For a discrete group G, the following conditions are equivalent.*

(i) The G-action on $\partial_F G$ is free.

(ii) There is a G-boundary on which the G-action is topologically free.

(iii) The group G does not have any non-trivial URS consisting of amenable subgroups.

(iv) G does not have any amenable confined subgroup.

(v) The group G is C^*-simple.

The combination of Theorem 1.2.1 with Theorem 1.3.1 yields imediatly the following.

Corollary 1.3.2 (of Theorems 1.2.1 and 1.3.1, [1]). *Let G be a discrete group acting faithfully by homeomorphisms on a Hausdorff space. Assume that for every non-empty open set $U \subset X$, the group G_U is non-amenable. Then G has no non-trivial amenable confined subgroup. In particular G is C^*-simple.*

[2]The space $\partial_F G$ should not be confused with the Poisson–Furstenberg boundary, which is a measure space associated to the choice of a random walk on G

This criterion is suitably applied to various classes of groups of homeomorphisms, as described in [1]. For example it implies that Thompson's group V is C^*-simple, and also applies to various generalisations of it appearing in the litterature. More examples to which Corollary 1.3.2 applies arise as subgroups of the groups of piecewise projective homeomorphisms of the real line studied by Monod [Mon13], which are non-amenable and have no free subgroups, non-amenable branch groups acting on rooted trees, the topological full group of any topologically free action $\mathcal{G} \curvearrowright X$ of a non-amenable group.

Haagerup and Olesen discovered a connection between the C^*-simplicity problem for the Thompsons' groups $F < T$ and the long-stading question whether Thompsons' group F is amenable: if the group T is C^*-simple, then the group F is non-amenable [HO17]. From Corollary 1.3.2, it follows that the non-amenability of F is actually equivalent to the C^*-simplicity of T, as well as to the C^*-simplicity of F itself.

In the more recent work with P.-E. Caprace and A. Le Boudec [4], we provide a condition for the freeness of the action on the Furstenberg boundary for a class of locally compact, non-discrete groups. (Note that Theorem 1.2.1 is not true in this setting). Following Caprace and Monod, a subgroup H of a locally compact group G is called *relatively amenable* if every continuous G-action on a compact space has an H-invariant probability measure; for discrete groups this condition is equivalent to the amenability of H, but for general locally compact groups this is not known [CM14]. For a locally compact group G the equivalence between (i)–(iv) in Theorem 1.3.1 remains intact upon replacing "amenable" with "relatively amenable" (all implications are elementary except (ii)⇒(iii), which follows from recent work of A. Le Boudec and T. Tsankov [LBT23]). The equivalence of these conditions with the C^*-simplicity (v) is still an open problem.

While for non-amenable discrete groups the property to act freely on the Furstenberg boundary is quite common, for non-discrete groups the situation is very different. Indeed many natural non-discrete groups admit amenable confined subgroups: this is the case in particular for all connected Lie groups (which admit cocompact amenable subgroups). As a consequence of this and of [BM02, Theorem 3.3.3] any locally compact group acting freely on its Furstenberg boundary must be totally disconnected. Even in this case, many natural examples fail to have this property, for instance semi-simple algebraic groups over local fields, or the group $\mathrm{Aut}(T)$ of automorphisms of a tree. We show the following.

Theorem 1.3.3 ([4])**.** *Let G be a totally disconnected locally compact group. Suppose that G admits a faithful action on a compact totally disconnected space X such that for every non-empty clopen subset $U \subset X$, the action of G_U on U is minimal and strongly proximal. Then G does not have any relatively amenable confined subgroup; equivalently, G acts freely on its Furstenberg boundary.*

Examples of groups to which this result applies are the Neretin groups $\mathcal{N}_d$. The group $\mathcal{N}_d$ can be defined as the topological full group of the action of the group of automorphism $\mathrm{Aut}(T_d)$ of a d-regular rooted tree on its boundary ∂T_d. It is endowed with a locally compact topology defined by the condition that the inclusion of $\mathrm{Aut}(T_d)$ is continuous and open.

Corollary 1.3.4. *for every $d \geq 3$, the Neretin group $\mathcal{N}_d$ acts freely on its Furstenberg boundary.*

Part of the recent research on totally disconnected locally compact groups is focused on the class of compactly generated, non-discrete simple groups (see e.g. the ICM address by P.E. Caprace and G. Willis [CW22]). A guiding paradigm is to understand analogies and differences with the classical cases of Lie groups and simple algebraic groups over local fields. To our knowledge, the groups $\mathcal{N}_d$ provide the first examples of groups in this class which act freely on their Furstenberg boundary.

It is an open question to find a complete characterisation of the confined subgroups (or of the uniformly recurrent subgroups) of the groups $\mathcal{N}_d$.

1.4 Growth gaps for group actions

In this section we discuss a different type of application of the study of the conjugation action $G \curvearrowright \mathrm{Sub}(G)$, namely to the geometry of the graphs of faithful actions of a group G. Some first results in this direction appear in [2, 3], and this topic is the primary object of [5].

1.4.1 Framework and context

Let G be a finitely generated group endowed with a symmetric, finite generating set S. Assume that G acts on a set X (just called a G-set for short). We say that a G-set is *faithful* if the action of G on it is faithful. Given a G-set, the graph of the action (or Schreier graph) is the graph $\Gamma(G, X)$ with vertex set X and where any two points are connected by an edge if one is the image of the other under some generator $s \in S$. We drop S from the notation as we are interested in the large scale geometric properties of this graph, which do not depend on S (up to bi-Lipschitz equivalence). We do not require here that the action is transitive, so that the graph $\Gamma(G, X)$ need not be connected: its connected components are precisely the G-orbits in X.

Graph of actions are much more flexible than Cayley graphs, and there are not many general results that relate directly the properties of a group G with the properties of the graphs of the possible faithful actions of G. The importance to study this general question has gradually emerged in recent years, motivated in part from the developments in the study of various groups of dynamical origin arising via natural actions with very explicit graphs, which turn out to be a fundamental tool to study them (see for example [BG02, JM13, Nek18].)

Our main focus here is on the volume growth of balls. Denote by $B_{\Gamma(G,X)}(x, n)$ the ball of radius n around a point x, and by $B_G(n)$ the ball of radius n in G. We define the growth of the action of G on X is the function $\mathrm{vol}_{G,X}(n)$ measuring the size of the largest ball, namely

$$\mathrm{vol}_{G,X}(n) := \max_{x \in X} |B_{\Gamma(G,X)}(x, n)| = \max_{x \in X} |B_G(n) \cdot x|.$$

Given functions $f, g \colon \mathbb{N} \to \mathbb{N}$, we write $f(n) \preceq g(n)$ if there exists $C > 0$ such that $f(n) \leq Cg(Cn)$ for every $n \geq 1$, and $f(n) \simeq g(n)$ if $f(n) \preceq g(n) \preceq f(n)$. Then $\mathrm{vol}_{G,X}(n)$ does not depend on the choice of generating set up to the equivalence $\simeq$.

It should be noted that the function $\mathrm{vol}_{G,X}(n)$ can be unbounded also if all individual G-orbits are finite (so that $\Gamma(G, X)$ is a disjoint union of finite graphs), and this situation is included in our setting.

For the action of G on itself, this invariant coincides with the usual word growth of G, denoted $\mathrm{vol}_G(n) := |B_G(n)|$. It is clear that the growth of every G-set is bounded above by the growth of G, but can be much smaller. In fact many finitely generated groups admit faithful G-sets X whose growth is linear, i.e. $\mathrm{vol}_{G,X}(n) \simeq n$ (this is the smallest possible growth for a faithful action of an infinite group); for instance it is not difficult to construct such actions for finitely generated free groups. More generally, all right-angled

Artin groups (RAAGs) admit such actions, as observed by Salo [Sal21]; as a consequence many groups that are known to embed into RAAGs also do (including surface groups and many hyperbolic groups). More examples include Grigorchuk groups, see [BG02], topological full groups of $\mathbb{Z}$-actions, etc.

Conversely it is natural to wonder what are obstructions, in terms of G, to admit actions of small growth. To formalise this question we state the following definition.

Definition 1.4.1. Let G be a finitely generated group, and $f\colon \mathbb{N} \to \mathbb{N}$ a function. We say that G has *Schreier growth gap* $f(n)$ if for every faithful G-set X we have $\mathrm{vol}_{G,X}(n) \succcurlyeq f(n)$. The gap is said to be *non-trivial* if f is not asymptotically linear, i.e. $\limsup f(n)/n = \infty$.

Some familiar group properties imply the existence of a Schreier growth gap. For instance it is well known that a group with Kazhdan's property (T) has a Schreier growth gap $\exp(n)$ (for transitive actions, this observation is pointed out by Gromov in [Gro93b], where it is attributed to Kazhdan; for general actions see [JdlS15]). As another example, it is not too difficult to see that if a group G contains a cyclically distorted element, then it satisfies a non-trivial Schreier growth gap (which can be estimated from quantitative control on the distorsion). It is natural to seek for more instances of this phenomenon.

It should also be observed that the knowledge of a Schreier growth gap for a class of groups $\mathcal{C}$ can be used to rule out that other groups G admit subgroups in $\mathcal{C}$, in view of the fact that if X is a G-set, the action of every finitely generated subgroup H of G satisfies $\mathrm{vol}_{H,X}(n) \preccurlyeq \mathrm{vol}_{G,X}(n)$. This approach is natural in the setting of topological full groups and other groups of dynamical origin, whose subgroup structure is quite mysterious. This was part of the context that lead us to identify this notion, making it natural to study it in its own right.

1.4.2 Some gaps arising from classification of confined subgroups

There is a direct connection between growth function $\mathrm{vol}_{G,X}(n)$ of G-sets and confinement of subgroups. Indeed suppose that a G-set X, the collection of stabilisers $\{\mathrm{Stab}_x(G)\colon x \in X\}$ accumulates on the trivial subgroup $\{1\}$. Then the graph $\Gamma(G,X)$ will contain arbitrarily large balls isomorphic to balls in the Cayley graphs of G, and it follows that $\mathrm{vol}_{G,X}(n) \simeq \mathrm{vol}_G(n)$. In particular the inequality $\mathrm{vol}_{G,X}(n) \preceq \mathrm{vol}_G(n)$ can only be strict if all stabilisers $\mathrm{Stab}_x(G)$ are (uniformly) confined.

This observation can be used to establish Schreier growth gaps when one has a complete classification of the confined subgroups of a group G (sometimes a classification of URSs suffices). This is the case for various groups given by an action by homoemomorphisms thanks to the results discussed in §1.2. For example, as an application of the classification of confined subgroups of topological full groups in [2], I showed the following.

Theorem 1.4.2 ([2]). *Let $\mathcal{G} \curvearrowright X$ be an expansive, minimal action by homeomorphisms of a finitely generated group $\mathcal{G}$ on a Cantor space, and $A(\mathcal{G})$ be the alternating full group of the action (recall that it is finitely generated by Nekrashevych* [Nek19]*). Then $A(\mathcal{G})$ has a Schreier growth gap* $\mathrm{vol}_{\mathcal{G},X}(n)$.

It is not difficult to see that the natural action of $A(\mathcal{G})$ on X has the same growth as the original action of $\mathcal{G}$, so that this result is sharp. Thus Theorem 1.4.2 can be seen as a source of examples of groups satisfying a sharp Schreier growth gap $f(n)$ for a vast class of functions (indeed the growth function of a minimal expansive group action on the Cantor set can have a wide range of behaviours, although a complete characterisation seems to be missing from the literature).

In [3], we show a similar phenomenon for branch groups. Recall that a *profinite* action of a group G is an action on a Cantor space X which is an inverse limit of finite actions. This is equivalent to the fact that X can be identified with the boundary ∂T of a locally finite rooted tree, on which T acts by isometries. Following Grigorchuk [Gri00], a *branch group* is a group $G \leq \mathrm{Homeo}(X)$ given by a faithful minimal profinite action on a Cantor space, with the property that for every open cover $U_1, \cdots, U_k$ of X, the subgroup of G generated by the rigid stabilisers $G_{U_1}, \ldots, G_{U_k}$ has finite index in G. The class of branch groups includes many well-studied examples of groups acting on rooted trees, including the Grigorchuk groups [Gri84]. This setting a source of group actions whose graphs can have a variety of possible growths and geometric properties. In [3] we provide a structure theorem on confined subgroups and URSs for (weakly) branch groups. As an application we show the following

Theorem 1.4.3 ([3])**.** *Let $G \leq \mathrm{Homeo}(X)$ be a finitely generated branch group. Then G has a Schreier growth gap $\mathrm{vol}_{G,X}(n)$.*

1.4.3 Non-foldable subsets

The two results in the previous subsection rely on a precise characterisation of the confined subgroups or URS of the groups under consideration. Although this method to establish Schreier growth gaps is viable in some cases, it has rather limited scope, indeed most infinite groups admit a wild pool of confined subgroups, which are too many to be classified (even classifying normal subgroups is often hopeless).

Here I explain a method to study Schreier growth gaps which requires only a partial understanding of confined subgroups. This method comes from [5], were we used it to study Schreier growth gaps in the setting of finitely generated solvable groups (the main results in that setting will be explained in §1.4.4). It is based on the following notion, which has some independent interest.

Definition 1.4.4. Let G be a group. A subset $\mathcal{L} \subset G$ is *non-foldable* if for every faithful G-set X and every finte subset $\Sigma \subset \mathcal{L}$, there exists a point $x \in X$ such that the orbital map $g \mapsto gx$ is injective on Σ.

In other words if a subset $\mathcal{L}$ of a group is non-foldable, then for every fatifhful G-set, every finite subset of $\mathcal{L}$ appears "embedded" in the graph $\Gamma(G, X)$ through a Lipschitz map (with respect to the distance on $\mathcal{L}$ induced by G). In terms of growth this implies that G must satisfy a Schreier growth gap at least equal to the relative growth of $\mathcal{L}$ in G. However the data of a non-foldable subset provides more explicit geometric information on the graphs of faithful actions of G; for instance it also provides lower bounds on their asymptotic dimension.

The fact that a given subset of $\mathcal{L}$ of a group G is non-foldable has an elementary reformulation in terms of a suitable family of confined subgroups of G. To explain this connection, recall that a subgroup H of G is confined if and only if there exists a finite subset $P \subset G \setminus \{1\}$ which intersects non-trivally every conjugate of H, we call P a *confining set* for H. For a finite set P, denote by $S_G(P)$ the collection of confined subgroups of G for which P is a confining set P. The set $S_G(P)$ is a closed G-invariant subset of $\mathrm{Sub}(G)$ avoiding $\{1\}$.

Lemma 1.4.5 (Non-foldable subsets and confined subgroups)**.** *Let G be a group and $\mathcal{L}$ be a subset of G. The following are equivalent:*

(i) $\mathcal{L}$ *is a non-foldable subset of* G*;*

(ii) for every finite subset $\Sigma \subset \mathcal{L}$ *we have*

$$\bigcap_{H \in S_G(P)} H \neq \{1\},$$

where $P = \{g^{-1}h \colon g, h \in \Sigma, g \neq h\}$.

The intersection in the statement is a normal subgroup, its non-triviality means that all confined subgroups in $S_G(P)$ must be induced from some proper quotient of G.

In practice, Lemma 1.4.5 is used by first guessing an explicit subset $\mathcal{L} \subset G$ that one hopes to be confined, and then proving so by verifying condition (ii) is satisfied. A good guess of the set $\mathcal{L}$ is essential, as it narrows down the set of confined subgroups of G that one needs to study. (A trivial, but perhaps helpful example is when $\mathcal{L}$ is an infinite cyclic subgroup, which is always non-foldable).

1.4.4 Growth gaps for actions of solvable groups

The main results form [5] establish Schreier growth gaps for some classes of finitely generated solvable groups. One motivation to look at this setting is that the nature of the problem suggests to look at rather "small" groups first. In fact solvable groups also played a role in the early developments of the theory of growth of groups. Recall that Wolf [Wol68] and Milnor [Mil68] proved if G is a finitely generated solvable groups, then the growth of G is either polynomial or exponential, and the first case arises exactly when G is virtually nilpotent. Moreover virtually nilpotent groups satisfy $\mathrm{vol}_G(n) \simeq n^d$ for some $d \in \mathbb{N}$ given by the Bass – Guivarch formula [Bas72, Gui73].

Another motivation is that the class of solvable groups does contain some very explicit examples of actions of small growth, which serve as test case for our study. For example, consider the wreath product $G = A \wr B$ of two finitely generated abelian groups, namely the semi-direct product $A^{(B)} \rtimes B$, where $A^{(B)}$ is the set of functions $f \colon B \to A$ with finite support, on which B acts by translation. The group G admits a natural action on the set $X = B \times A$, given by the formula

$$(f, b) \cdot (b_1, a_1) = (bb_1, f(bb_1)a) \qquad (f, b) \in A^{(B)} \rtimes B,\ (b_1, a_1) \in B \times A.$$

This action is transitive and faithful, and its Schreier graph is quasi-isometric to the graph obtained by taking a copy of the Cayley graph of A, and gluing at each vertex a copy of the Cayley graph of B (see Figure 1.1). In particular $\mathrm{vol}_{G,X}(n) \simeq \mathrm{vol}_A(n) \times \mathrm{vol}_B(n)$, which is polynomial. For example, if $G = C_p \wr \mathbb{Z}^d$ we have $\mathrm{vol}_{G,X} \simeq n^d$, and if $G = \mathbb{Z} \wr \mathbb{Z}^d$ we have $\mathrm{vol}_{G,X}(n) \simeq n^{d+1}$. Similar constructions cam be made for other metabelian groups, for instance by embedding them into (unrestricted) wreath product using versions of the Magnus embedding. This way one can show, for instance, that the free metabelian group $\mathbb{FM}_d$ on d generators has a faithful action of growth n^{d+1}.

Finally we mention that solvable groups are generally quite different from the examples considered in §1.4.2 in regard of the fact that they tipically have a more complicated structure of confined (or even normal) subgroups.

Before stating the main results in [5], we shall point out the elementary fact that if G is a finitely generated group which is virtually abelian, then G always admits a faithful G-set X with $\mathrm{vol}_{G,X}(n) \simeq n$. Thus such groups are excluded when studying Schreier growth gap.

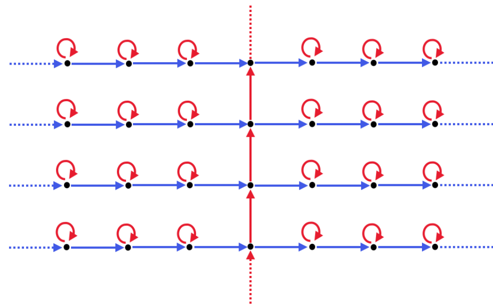

Figure 1.1: Schreier graph of the standard wreath product action of $\mathbb{Z} \wr \mathbb{Z}$: the "comb".

It is natural to address first the case of polycyclic groups (which include in particular all virtually nilpotent groups). This class is somewhat degenerate for the problem considered here, due to the fact that a polycyclic group has only countably many subgroups, which implies its URSs are all finite (i.e. consist of subgroups which are normalised by some finite index subgroup of G). Thanks to this, an elementary compactness argument in the space $\mathrm{Sub}(G)$ shows that the study of growth of actions of polycyclic groups reduces to the classical results on growth of such groups:

Proposition 1.4.6. *Let G be a polycyclic group.*

1. *If G is virtually nilpotent, then for every faithful G-set there exists $d \in \mathbb{N}$ such that $\mathrm{vol}_{G,X}(n) \simeq n^d$.*
2. *If G is virtually nilpotent, then G has a sharp Schreier growth gap n^{α_G}, where α_G can be computed explicitly and $\alpha_G \geq 4$ unless G is virtually abelian.*
3. *If G is not virtually nilpotent, then G has a Schreier growth gap $\exp(n)$.*

The situation is quite different when we consider more general solvable groups, which might have a much richer subgroup structure. The first case to look at is metabelian groups. Even in this class, there are groups of exponential growth which admit faifthul actions of linear growth. An example of such a group is the lamplighter group $C_2 \wr \mathbb{Z}$, whose natural wreath product action (described above) has linear growth. Actually, the group $C_2 \wr \mathbb{Z}$ admits faithful actions whose growth is equivalent to an arbitrarily prescribed function $f(n)$ between linear and exponential and satisfying some mild conditions, see [5, Proposition 2.3]. So unlike for polycyclic groups, strictly *nothing* can be said on the possible growth of actions for *all* metabelian groups. Nevertheless the lamplighter group can be considered a somewhat degenerate example, for instance it is not finitely presented, and contains a very large subgroup consisting of torsion elements. The first main result in [5] says that a uniform Schreier growth gap does hold for two natural classes of metabelian groups:

Theorem 1.4.7. *Let G be a finitely generated metabelian group which is not virtually abelian. Assume that G satisfies any of the following conditions:*

- *G is finitely presented,*
- *G is is torsion-free.*

Then G has a Schreier growth gap n^2. If moreover both conditions hold, then G has a Schreier growth gap n^3.

This result is sharp, in the sense that there are examples of metabelian groups admitting faithful actions of quadratic growth, both in the finitely presented and in the torsion-free case.

After such a uniform bound for a whole class of groups is established, the next natural question is to seek for quantitative criteria that depend on finer invariants of the group G. A natural invariant associated to a metabelian group is its Krull dimension. This is a well-known invariants of rings, whose relevance in the setting of metabelian groups was pointed out by Cornulier [Cor11] and further exploited by Jacoboni [Jac19], who showed that it provides bounds on the return probability of the random walk. We show the following.

Theorem 1.4.8. *Let G be a finitely generated metabelian group which is not virtually abelian and has Krull dimension d. Then G has a Schreier growth gap n^d.*

A relevant special case of application of Theorem 1.4.8 is given by the wreath products $(\mathbb{Z}/p\mathbb{Z})\wr\mathbb{Z}^d$ and $\mathbb{Z}\wr\mathbb{Z}^d$, whose Krull dimension is d and $d+1$ respectively. For these groups the Schreier growth gap established by Theorem 1.4.8 is sharp, as it is matched by the standard wreath product action mentioned above. Another case in which the theorem is sharp is the free metabelian group $\mathbb{FM}_d$ on d generators, whose Krull dimension is $d+1$. Both Theorems 1.4.7 and 1.4.8 are based on a criterion that allows to find explicit non-foldable subsets in metabelian groups. For example, in the case of wreath products $G = C_p \wr \mathbb{Z}$ and $\mathbb{Z} \wr \mathbb{Z}^d$, that criterion allows to construct a non-foldable subset $\mathcal{L}$ which is quasi-isometric to the graph of their standard wreath product action. That is, that graph is the "smallest" in the sense that all graphs of faithful actions of such groups must contain large balls of it.

For arbitrary solvable groups of higher solvability length the situation appears to be much more complicated in full generality. Nevertheless the following result provides a class of solvable groups which satisfy the strongest possible Schreier growth gap. Recall that a slovable group G has finite Prüfer rank if there exists $k > 0$ such that every finitely generated subgroup of G can be generated by at most k elements. This class contains all solvable groups which are linear over the field $\mathbb{Q}$ (but also examples that are not linear over any field).

Theorem 1.4.9. *Let G be a finitely generated solvable group of finite Prüfer rank. If G is not virtually nilpotent, then it has a Schreier growth gap $\exp(n)$.*

Polycyclic groups have finite Prüfer rank. Thus this result is a generalisation of the third item in Proposition 1.4.6. However the class of solvable groups with finite Prüfer rank is quite larger (in particular such groups can admit an uncountable set of subgroups), and the proof of Theorem 1.4.9 requires substantially more involved arguments compared to the polycyclic case, based on Malcev's theory. Theorem 1.4.9 is based on a criterion that allows to find explicit non-foldable subsets in groups that admit a nilpotent normal subgroup $N \trianglelefteq G$ of finite Prüfer rank, provided the conjugation action of G on N satisfies a certain irreducibility condition [5, Theorem 4.9]. Theorem 1.4.9 cannot be extended even to solvable groups which are linear over fields more general than $\mathbb{Q}$: for example if $\mathbb{K}$ is any extension of $\mathbb{Q}$ containing a transcendental element, then there there are solvable groups which are linear over $\mathbb{K}$ which are not virtually nilpotent, but admit faithful actions of polynomial growth (for instance $\mathbb{Z} \wr \mathbb{Z}$).

Another problem discussed in [5] is whether the Schreier growth gap n^2 for torsion-free metabelian groups from Theorem 1.4.7 extends to torsion-free solvable groups of arbitrary length. We conjecture an affirmative answer

Conjecture 1.4.10. *Let G be a finitely generated solvable group which is virtually torsion-free and not virtually abelian. Then G has a Schreier growth gap n^2.*

We provide various partial answers of the conjecture which cover many familiar classes of torsion-free solvable groups (including e.g. all linear ones), but could not solve the conjecture in general.

1.5 Highly transitive actions and confined subgroups

We conclude this chapter by explaining yet another application of confined subgroups, to the study of highly transitive actions. An action of a group G on an infinite set set Ω is highly transitive if the induced action on the set of ordered $n$tuples of distinct points is transitive for every n. Equivalently, an action is highly transitive if the induced homomorphism from G to the group $\mathrm{Sym}(\Omega)$ of all permutations of Ω has dense image, where $\mathrm{Sym}(\Omega)$ is endowed with the pointwise convergence topology. An example of a highly transitive action is for instance given by the action of the group $\mathrm{Sym}_f(\Omega)$ of finitary permutations of Ω, or even its subgroup $\mathrm{Alt}_f(\Omega)$ of alternating permutations. Here a permutation is termed finitary if it moves only finitely many points. Recall that $\mathrm{Alt}_f(\Omega)$ and $\mathrm{Sym}_f(\Omega)$ are normal subgroups of $\mathrm{Sym}(\Omega)$, and every non-trivial normal subgroup of $\mathrm{Sym}(\Omega)$ contains $\mathrm{Alt}_f(\Omega)$ [Ono29, SU33]. For a group G acting faithfully and highly transitively on Ω, one easily verifies that the image of G in $\mathrm{Sym}(\Omega)$ contains a non-trivial finitary permutation if and only if it contains $\mathrm{Alt}_f(\Omega)$. If a group G admits an action with this property, we say that G is **partially finitary**. This can be reformulated as an intrinsic property of the group, namely G is partially finitary if and only if it has a normal subgroup N isomorphic to $\mathrm{Alt}_f(\Omega)$, with trivial centraliser in G [6, §2.3]. This is a somewhat trivial class of examples, and we are mostly interested in highly transitive actions of groups that are not partially finitary.

In [6] we establish the following connection between confined subgroup and highly transitive actions.

Theorem 1.5.1. *Suppose that a group G admits a faithful and highly transitive action on Ω and that G is not partially finitary. If H is a confined subgroup H of G, then the union $\Omega_{H,f}$ of finite H-orbits is a finite subset of Ω, and the action of H on $\Omega_{H,\infty} := \Omega \setminus \Omega_{H,f}$ is highly transitive.*

Theorem 1.5.1 is an application of a characterisation of the confined subgroups of $\mathrm{Sym}(\Omega)$ (with respect to the discrete topology on $\mathrm{Sym}(\Omega)$), that we provide in [6, Theorem 3.7]. That result relies on the proof mechanism of Theorem 1.2.1 from [3]. Theorem 1.5.1 is then deduced using by exploiting the fact that if H is a confined subgroup of G, then the closure of H under a dense embedding $G \hookrightarrow \mathrm{Sym}(\Omega)$ (with respect to the Polish topology on $\mathrm{Sym}(\Omega)$) is confined in the whole group $\mathrm{Sym}(\Omega)$ (with respect to its discrete topology!).

Recently there has been a growing interest for a better understanding of which (say countable) groups admit faithful highly transitive actions. Actually the question is already rich for primitive actions, and was investigated in depth by Gelander and Glasner [GG08], partly motivated by earlier results of Margulis and Soifer [MS81]. See [GGS22] for a recent survey. A series of consecutive works shows that various groups admitting a rich geometric or dynamical action also admit a highly transitive action. This is for instance the case for free groups [McD77, Dix90, Ols15, EG16, LM18], surface groups [Kit12], hyperbolic groups [Cha12], outer automorphism groups of free groups [GG13a], acylindrically hyperbolic groups [HO16], unbounded Zariski dense subgroups of SL_2 over a local field

[GGS22], and many groups acting on trees [FLMMS22]. More examples arise as groups of homeomorphisms. For example, the action of the Higman-Thompson's group V_d on each of its orbits in the Cantor set is highly transitive on each orbit. More generally, for every group action $\mathcal{G} \curvearrowright X$ on the Cantor set, the action of the topological full group $F(\mathcal{G})$ (or even the alternating full group $A(\mathcal{G})$) on each infinite orbit is highly transitive (these were defined in §1.2.3).

While there is by now an abundant litterature establishing the existence of highly transitive actions for various classes of groups, there are relatively few general obstructions that prevent a group from having such actions, mostly of algebraic nature (see [6, §2.4] for a more detailed discussion around this). The following consequence of Theorem 1.5.1 provides a new criterion to rule out the existence of highly transitive actions.

Corollary 1.5.2. *Let G be a group that is not partially finitary, and let H be a confined subgroup of G. If H does not admit any faithful highly transitive action, then neither does G.*

As an example of application of this criterion, we provide a family of groups that do not admit any faithful highly transitive action, without satisfying any of the main previously known obstructions to this property. These groups belong to the family of groups acting on suspension flows of subshifts from [7], that will be extensively discussed in §2.4 (for a different purpose).

Beyond the problem of deciding which groups admit highly transitive actions, it is natural to try to classify all highly transitive actions of a given group. In the case of primitive actions, similar questions have been considered by Gelander–Glasner in [GG08, GG13b] for countable linear groups. It follows from an old result of Jordan-Wielandt that the group $\mathrm{Alt}_f(\Omega)$ admits a unique highly transitive action up to conjugacy, and more generally the same is true for every partially finitary group (as we observe in [6, Proposition 2.4]). Apart from this observation, there was before [6] no example of a finitely generated highly transitive group whose highly transitive actions were completely understood. Using Theorem 1.5.1 as main tool we obtain an example of such a classification result, for the family of the Higman–Thompson's group V_d acting on the Cantor set:

Theorem 1.5.3 ([6])**.** *Every faithful highly transitive action of the Higman–Thompson group V_d is conjugate to its natural action on some orbit in the Cantor set.*

The proof of Theorem 1.5.3 can be adapted to other groups acting on compact spaces with suitable dynamical properties. It is a natural question whether the same result holds true for the topological full group of every minimal group action on the Cantor set.

Chapter 2

Group actions on the real line

2.1 Introduction

This chapter presents the papers [7, 8, 9, 10], in collaboration with J. Brum, C. Rivas and M. Triestino. The general context of these results is the study of dynamics of groups actions by orientation pereserving homeomorphisms, or diffeomorphisms, on connected manifolds of dimension 1 (that is, the circle and real intervals, possibly with boundary points). The study of actions of the group of integers $\mathbb{Z}$ on the circle is a classical topic in dynamics going back to the fundamental work of Poincaré and Denjoy. The study of one-dimensional actions of more general groups has its origin in the study of co-dimension one foliations and independently in the theory of (left-)orderable groups. This gave rise to a theory that considerably expanded in the last decades, after the results of Ghys, Matsumoto, Witte Morris, Margulis, Navas, and many others.

In a nutshell, one of the main goals in the field is to be able to address the following questions for some groups G of interest:

1. to determine whether G can act (non-trivially or faithfully) on a one-manifold;

2. if this is the case, to describe its possible actions.

In the study of these questions, the regularity of the action plays a crucial role. Namely one can consider actions by homeomorphisms or by diffeomorphisms of a given class C^r. In regularity C^r with $r > 1$, several restrictions on actions by diffeomorphisms are based on the possibility of controlling the affine distorsion (namely how much the ratio between length of intervals is modified): a prototipical example is Denjoy's theorem for circle diffeomorphisms of class C^2. Our main focus will be on actions in very low regularity, namely C^0 and C^1.

In regularity C^0 it is enough to consider the circle and the real line (boundary points of intervals play a role only when derivability at them is assumed). Question 2 is better understood in the case of the circle, at least for actions *without finite orbits*. É. Ghys showed that such actions are classified up to semi-conjugacy by a complete invariant, the bounded Euler class, taking values in the second bounded cohomology of the group G [Ghy87a]; a more elementary interpretation was given by Matsumoto in terms of rotation numbers [Mat86]. These invariants have been largely used to get a global understanding of the space of actions on the circle in various cases (an important example are surface groups [Mat87, Man15, MW17]). The case of actions with a finite orbit reduces, up to passing to a finite index subgroup, to study actions on the real line, for which no such tools are available.

For actions on the real line by homeomorphisms, a vast part of the literature is devoted to question 1 above. For a countable group G, the existence of a faithful action on $\mathbb{R}$ is equivalent to the existence of a left-invariant total order on G, and various results investigate the structure of the space of invariant orders. The pool of left-orderable groups for which question 2 has been addressed, or can be answered using results on left-orders, is somewhat limited (almost always groups that admit very few actions on the line up to semi-conjugacy).

If many rigidity and classification results are known for actions on intervals in regularity $C^r, r > 1$, these are also more rare in regularity C^1. Here distorsion arguments are not applicable, and essentially the only additional tools available over the C^0 case come from the analysis of elements with hyperbolic fixed points, which exist in great generality by a result of Deroin, Kleptsyn and Navas [DKN07].

The result presented here aim at filling in part this gap, by proving structure and rigidity theorems for C^0 and C^1 actions on the real line for some classes of groups, proposing a framework with new type of questions, and exhibiting new examples and phenomena in this setting.

2.2 The space of representations to $\mathrm{Homeo}_+(\mathbb{R})$ up to semi-conjugacy

This section is mostly introductory and contains almost no original results, with the exception of Theorem 2.2.3 (proved in [8]), which revisits and complements results of Deroin, Kleptsyn, Navas and Parwani [DKNP13]. However, the point of view presented on the study of actions on the line, motivated in part by that result and the question it suggests, has perhaps an original component.

2.2.1 Minimal sets and semi-conjugacy

Let G be a finitely generated group. We denote by $\mathrm{Rep}_{\mathrm{irr}}(G, \mathbb{R})$ the set of representations $\rho\colon G \to \mathrm{Homeo}_+(\mathbb{R})$ whose image has no global fixed point in $\mathbb{R}$; such representations will be called *irreducible* (in this context, we will use the term "representation" and "action" interchangeably). Restricting to irreducible representations is not a loss of generality (upon removing the fixed points, the induced action on every connected component is irreducible). The space $\mathrm{Rep}_{\mathrm{irr}}(G, \mathbb{R})$ is naturally a topological (Polish) space, with the topology induced from pointwise convergence of representations in the compact open topology on $\mathrm{Homeo}_+(\mathbb{R})$.

Given $\rho \in \mathrm{Rep}_{\mathrm{irr}}(G, \mathbb{R})$, a non-empty closed, minimal G-invariant subset $\Lambda \subset \mathbb{R}$ will be called for short *minimal set* for ρ. It is well-known (see [Nav11, §2.1]) that if G is finitely generated, every irreducible representation $\rho \in \mathrm{Rep}_{\mathrm{irr}}(G, \mathbb{R})$ admits a minimal set. Every minimal set Λ satisfies one of the following possibilities.

- Either $\Lambda = \mathbb{R}$, that is, the representation ρ is minimal.
- Or Λ has no isolated point and empty interior (also called an *exceptional minimal set*); in this case Λ is the unique minimal set of ρ.
- Or Λ is discrete and consists of a single orbit, in this case all minimal sets have this property, and the action of G on any of them factors through an epimorphism to $\mathbb{Z}$.

If a representation ρ is not minimal, its study naturally splits into the analysis of the action of G on a minimal set Λ, and the action of the stabilisers of each connected component of $\mathbb{R} \setminus \Lambda$. One can associate to ρ a new representation $\bar{\rho}$ that models how ρ acts on its minimal set (this process is sometimes called the *minimalisation*). Namely if ρ admits an exceptional minimal set Λ, we can associate to it a minimal representation $\bar{\rho}$ by collapsing all connected components of $\mathbb{R} \setminus \Lambda$ to obtain a new copy of the real line, together with a continuous, non-decreasing map $h\colon \mathbb{R} \to \mathbb{R}$ which intertwines ρ and $\bar{\rho}$. The case of a discrete minimal set Λ is somewhat degenerate and collapsing components of $\mathbb{R} \setminus \Lambda$ does not give back a copy of the real line; in this case we proceed in two steps by first blowing up every point in Λ to a small interval (extending the action by interpolation), and then collapse the connected components of the complement: the resulting action is conjugate to an action $\bar{\rho}$ which takes values in the group of integer translations $\mathbb{Z}$. An action taking values in the group of translations will be called *cyclic*.

This discussion naturally leads to the notion of *semi-conjugacy*.

Definition 2.2.1. Two representations $\rho_1, \rho_2 \in \mathrm{Rep}_{\mathrm{irr}}(G, \mathbb{R})$ are semi-conjugate if there exists a non-decreasing map $h\colon \mathbb{R} \to \mathbb{R}$ which intertwines ρ_1 and ρ_2.

Semi-conjugacy is an equivalence relation. The map h is not required to be continuous to make sure that the relation reflexive; but semi-conjugacy coincides with the equivalence relation *generated by* the pairs (ρ_1, ρ_2) such that there exist a *continuous*, non-decreasing map which intertwines ρ_1 and ρ_2, as follows from the previous discussion. It also follows that every representation $\rho \in \mathrm{Rep}_{\mathrm{irr}}(G, \mathbb{R})$ is semiconjugate to a representation $\bar{\rho}$ which is either minimal or cyclic. Moreover such a minimal or cyclic representative is unique up to topological conjugacy. In fact, a non-decreasing equivariant map between minimal actions on $\mathbb{R}$ is automatically a homeomorphism.

2.2.2 The harmonic space of a group acting on the line

Given a finitely generated group, we will be interested on the topological and dynamical properties of representations in $\mathrm{Rep}_{\mathrm{irr}}(G; \mathbb{R})$ up to semi-conjugacy. One might be tempted to consider the quotient of $\mathrm{Rep}_{\mathrm{irr}}(G; \mathbb{R})$ by the semi-conjugacy relation as a moduli space for actions of G on the line; unfortunately this quotient is usually quite ill-behaved. In fact for many groups G the semi-conjugacy relation is not *smooth* in the Borel sense, namely there is no Borel map from $\mathrm{Rep}_{\mathrm{irr}}(G; \mathbb{R})$ to any standard Borel space which is constant on semi-conjugacy classes and separates them. This can be seen as an obstruction to find explicit invariants that classify actions up to semi-conjugacy. This contrasts with the case of actions on the circle (by the results of Ghys [Ghy87b] and Matsumoto [Mat86] described above) for which the quotient by the semi-conjugacy relation is actually a compact metrisable space (see Mann and Wolff [MW17]).

For finitely generated groups, a partial solution to this problem one can however consider a slightly larger quotient space, called the *harmonic space* of G, which is a compact space endowed with a real flow $(\mathcal{D}, \Psi)$ and a G-action preserving each Ψ-orbits, such that the Ψ-orbits correspond to semi-conjugacy classes. The construction of this space depends on results of Deroin, Kleptsyn, Navas and Parwani [DKNP13], and elaborates on a previous construction of Deroin [Der13]. The starting point of this construction is to observe that the space $\mathrm{Rep}_{\mathrm{irr}}(G, \mathbb{R})$ is naturally endowed with a flow induced by the conjugation action of the group of translations, namely

$$\Psi\colon \mathbb{R} \times \mathrm{Rep}_{\mathrm{irr}}(G, \mathbb{R}) \to \mathrm{Rep}_{\mathrm{irr}}(G, \mathbb{R}), \quad \Psi^t(\rho)(g) = T_t \rho(g) T_{-t}$$

where T_t denotes the translation $x \mapsto x + t$. The flow Ψ will be called the *translation flow*. The group G acts on $\mathrm{Rep}_{\mathrm{irr}}(G, \mathbb{R})$ by preserving Ψ-orbits via the formula

$$g \cdot \rho = \Psi^{\rho(g)(0)}(\rho).$$

Deroin showed in [Der13] that every representation $\rho \in \mathrm{Rep}_{\mathrm{irr}}(G, \mathbb{R})$ can be conjugated to a representation contained in some compact Ψ-invariant subset of $\mathrm{Rep}_{\mathrm{irr}}(G, \mathbb{R})$. A more "universal" version of his construction can be obtained from subsequent results of Deroin, Kleptsyn, Navas and Parwani [DKNP13] on random walk on groups acting on the line. Namely the choice on G of a symmetric, finitely supported probability measure μ allows to consider a distinguished set of repesentatives of representations in $\mathrm{Rep}_{\mathrm{irr}}(G, \mathbb{R})$, the *$\mu$-harmonic representations* (those for which the Lebesgue measure is μ stationnary). The results in [DKNP13] imply that every representation ρ is semi-conjugate to a μ-harmonic one, and that the space of μ-harmonic representations, normalised in a suitable sense, is compact and Ψ-invariant. We resume below the properties of this construction that will be important for our purposes.

Theorem 2.2.2 (Deroin, Kleptsyn, Navas, Parwani [DKNP13], see §3 in [8])**.** *Let G be a finitely generated group. Then there exists a subset $\mathcal{D} \subset \mathrm{Rep}_{irr}(G, \mathbb{R})$ satisfying the following.*

1. *$\mathcal{D}$ is compact invariant under the translation flow Ψ.*
2. *Every $\rho \in \mathcal{D}$ is either minimal or cyclic.*
3. *For every $\rho \in \mathrm{Rep}_{irr}(G, \mathbb{R})$ there exists a unique $\rho_* \in \mathcal{D}$ and a semi-conjugacy $h \colon \mathbb{R} \to \mathbb{R}$ between ρ and ρ_* such that $h(0) = 0$ (the map h is called a* pointed *semi-conjugacy).*

By definition, a space $\mathcal{D}$ as in Theorem 2.2.2 comes with a map

$$r_{\mathcal{D}} \colon \mathrm{Rep}_{\mathrm{irr}}(G, \mathbb{R}) \to \mathcal{D},$$

which associates to every representation ρ its unique representative ρ_* up to pointed semi-conjugacy. In [8, §3] we provide a proof of the following (this formulation appears in [10]).

Theorem 2.2.3 (see §3 in [8])**.** *Let $\mathcal{D}$ be any space as in Theorem 2.2.2. Then the map*

$$r_{\mathcal{D}} \colon \mathrm{Rep}_{irr}(G, \mathbb{R}) \to \mathcal{D}$$

is a G-equivariant continuous retraction, with the property that two representations ρ_1, ρ_2 are semi-conjugate if and only if $r(\rho_1)$ and $r(\rho_2)$ belong to the same Ψ-orbit.

The non-trivial statement in Theorem 2.2.3 is the continuity of the map $r_{\mathcal{D}}$ (the second assertion follows from the properties of $\mathcal{D}$ in Theorem 2.2.2). This is not apparent from the results in [DKNP13], as the semi-conjugacy of an action ρ to a harmonic action is obtained through a probabilistic argument to find a stationary measure. Our proof proceeds through an intermediate step, namely identifying of the space $\mathcal{D}$ with a suitable quotient of the space of left-invariant pre-orders on G.

A consequence of Theorem 2.2.3 is that the space with flow $(\mathcal{D}, \Psi)$ is uniquely determined by G in the following sense.

Definition 2.2.4. A flow equivalence between spaces with flows $(Y_1, \Psi_1), (Y_2, \Psi_2)$ is a homeomorphism $\mathfrak{h} \colon Y_1 \to Y_2$ such that for every $y \in Y_2$, there exists $h \in \mathrm{Homeo}_+(\mathbb{R})$ with $\mathfrak{h}(\Psi_1^t(y)) = \Psi^{h(t)}(\mathfrak{h}(y))$.

Corollary 2.2.5 (Uniqueness up to flow equivalence)**.** *Let $\mathcal{D}_1, \mathcal{D}_2$ be spaces as in Theorem 2.2.2. Then the map $\mathfrak{h} = r_{\mathcal{D}_2}|_{\mathcal{D}_1}$ is a G-equivariant flow equivalence $\mathfrak{h}\colon (\mathcal{D}_1, \Psi|_{\mathcal{D}_1}) \to (\mathcal{D}_2, \Psi|_{\mathcal{D}_2})$, which by construction satisfies $r_{\mathcal{D}_2} = \mathfrak{h} \circ r_{\mathcal{D}_1}$.*

Definition 2.2.6. Let G be a finitely generated group. Given a subset $\mathcal{D} \subset \mathrm{Rep}_{\mathrm{irr}}(G, \mathbb{R})$ satisfying the conclusion of Theorem 2.2.2, we will refer to the space with flow $(\mathcal{D}, \Psi)$, considered up to G-equivariant flow equivalence, as the *harmonic space* of G. It is endowed with the associated retraction $r_{\mathcal{D}}\colon \mathrm{Rep}_{\mathrm{irr}}(G, \mathbb{R}) \to \mathcal{D}$ as in Theorem 2.2.3. [1]

We shall specify that we consider a *representative* of $(\mathcal{D}, \Psi)$ it when we choose $\mathcal{D}$ as in Theorem 2.2.2, and let Ψ be the restriction of the translation flow. By [DKNP13] such a representative can always be chosen consisting of μ-harmonic representations (and this is actually the only general way to find one known so far). This choice has additional useful properties (see in particular Deroin and Hurtado [DH20]), however we shall not need this.

Remark 2.2.7. The terminology has nothing to do with the notion of abstract harmonic space in analysis (which is a locally compact space equipped with a sheaf of functions satisfying a series of axioms); hopefully the context leaves no room for confusion.

Let us go back to our initial discussion. Even though the space of semi-conjugacy classes in $\mathrm{Rep}_{\mathrm{irr}}(G, \mathbb{R})$, defined naively as the quotient by the semi-conjugacy relation, is badly behaved (as a topological, or even just Borel space), Theorem 2.2.3 clarifies exactly how “bad” it can be: it identifies with the space of orbits $\mathcal{D}/\Psi$ of the flow $(\mathcal{D}, \Psi)$. In other words, the dynamics of the flow $(\mathcal{D}, \Psi)$ is responsible for all the complexity of the semi-conjugacy relation in $\mathrm{Rep}_{\mathrm{irr}}(G, \mathbb{R})$, and retains topological information on how the semi-conjugacy classes sit inside $\mathrm{Rep}_{\mathrm{irr}}(G, \mathbb{R})$. This has a few immediate consequences, and this point of view suggests various qualitative questions that should be addressed on the harmonic space $(\mathcal{D}, \Psi)$ of a group G, which provide viable goals even when a completely explicit description of $(\mathcal{D}, \Psi)$ is not.

As perhaps the most basic consequence, Theorem 2.2.3 clarifies completely the possible nature of the semi-conjugacy relation on $\mathrm{Rep}_{\mathrm{irr}}(G, \mathbb{R})$ in the sense of Borel bireducibility. This theory provides a framework to tell how difficult a classification problem is (in our case, classifiying actions of a given group G up to semi-conjugacy); see the book of Kechris [Kec21]. Most classification problems can be interpreted as a (Borel) equivalence relation $\mathcal{R}$ on a standard Borel space X. The simplest type are the *smooth* equivalence relations, which are those for which there exists a Borel map $f\colon X \to \mathbb{R}$, such that $(x, y) \in \mathcal{R}$ if and only if $f(x) = f(y)$ (they correspond to classification problems that can be solved by exhibiting a complete invariant). More generally one says that an equivalence relation $(X_1, \mathcal{R}_1)$ is *Borel reducible* to $(X_2, \mathcal{R}_2)$ if there exists a Borel map $f\colon X_1 \to X_2$ such that x, y are equivalent if and only if $f(x), f(y)$ are (this means that the first classification problem can be reduced to the second), and that the relations are *bireducible* if the symmetric statement also holds. From Theorem 2.2.3 it follows that the semi-conjugacy relation on $\mathrm{Rep}_{\mathrm{irr}}(G, \mathbb{R})$ is Borel bireducible to the orbit-equivalence relation of the flow $(\mathcal{D}, \Psi)$. Equivalence relations induced by flows are well-understood and can have only two bireducibility types, see [Kec21, Theorem 8.32], leading to the following:

[1] In [8, 10], we call $(\mathcal{D}, \Psi)$ the *Deroin space* for G as it is, indeed, a universal version of the construction in Deroin [Der13], where the idea of looking at spaces with flows in this context was first introduced (this idea is not explicitly mentioned in [DKNP13]). However, after reflection and discussions with colleagues, I now believe that this choice does not give enough credit to the fundamental results in [DKNP13], and decided to change it.

Corollary 2.2.8. *The semi-conjugacy equivalence relation on* $\mathrm{Rep}_{irr}(G, \mathbb{R})$ *is either smooth, or it is Borel bireducible to the orbit-equivalence relation of the Bernoulli shift* $\mathbb{Z} \curvearrowright \{0, 1\}^{\mathbb{Z}}$*. It is smooth if and only if the harmonic space* $(\mathcal{D}, \Psi)$ *has a smooth orbit equivalence relation; equivalently if it admits a Borel transversal (a subset intersecting each* Ψ*-orbit in exactly one point).*

Which case in Corollary 2.2.8 holds depends on the group G under consideration, tracing a natural dividing line among groups acting on the line according to the complexity of their actions up to semi-conjugacy. The following question then can be asked when addressing the structure of actions on the line of any given group.

Question 2.2.9. For which groups G the semi-conjugacy relation on $\mathrm{Rep}_{\mathrm{irr}}(G, \mathbb{R})$ (equivalently the orbit-equivalence relation on $(\mathcal{D}, \Psi)$) is smooth in the Borel sense?

Another direction of application of Theorem 2.2.3 is related to the study of rigidity and flexibility of group actions on the line. In a broad sense, this topic aims at understanding, given a group G, how its representations can be modified (up to semi-conjugacy) by small perturbations, which boils down to understand how semi-conjugacy classes accumulate onto each other. This leads to several tightly related notions of rigidity and flexibility for representations, appearing in the literature on group actions on one-manifolds, see e.g. [Man18, MR18, MW17, KKM19]. We will use here the following terminology, analogous to the one used by Mann and Wolff [MW17] for actions on the circle.

Definition 2.2.10. A representation $\rho \in \mathrm{Rep}_{\mathrm{irr}}(G; \mathbb{R})$ is *locally* C^0*-rigid* if it is an interior point in its semi-conjugacy class. It is C^0*-rigid* if its whole semi-conjugacy class is open. A representation which is not locally C^0-rigid is called *flexible*.

As a consequence of Theorem 2.2.3, the C^0-rigidity of a representation can be read in the harmonic space.

Corollary 2.2.11 ([8])**.** *Let* G *be a finitely generated group and* $(\mathcal{D}, \Psi)$ *be its harmonic space. Then a representation* $\rho \in \mathrm{Rep}_{irr}(G, \mathbb{R})$ *is* C^0*-rigid if and only if the orbit of* $r_{\mathcal{D}}(\rho)$ *is an open subset of* $\mathcal{D}$*, equivalently if* $r_{\mathcal{D}}(\rho)$ *is an isolated point in any local cross-section of* Ψ*.*

Moreover local C^0*-rigidity and* C^0*-rigidity are equivalent concept for a* minimal *representation.*

Question 2.2.12. For given groups G, study which representations are (locally) C^0-rigid representations. More generally find interesting Ψ-invariant open subset of the harmonic space $(\mathcal{D}, \Psi)$ (these correspond to rigid families of representations, in the sense that any small perturbation will remain semi-conjugate to a representation in the same family).

Another general theme in the study of representation spaces is the classification of their connected and path-connected components. See for instance the works of Matsumoto [Mat86], Mann [Man15] and Mann and Wolff [MW17] for the case of surface group actions on the circle (for which this problem remains open in general). It is not difficult to see that for a finitely generated group G, the semi-conjugacy classes in $\mathrm{Rep}_{\mathrm{irr}}(G; \mathbb{R})$ are always path connected. Again in analogy with the terminology of [MW17], one may call a representation $\rho \in \mathrm{Rep}_{\mathrm{irr}}(G; \mathbb{R})$ *path-rigid* if its path-component coincides with its semi-conjugacy class (that is, ρ cannot be deformed *along a continuous path* into any non-semi-conjugate representation).

Since every ρ is connected by a path to $r_{\mathcal{D}}(\rho)$, Theorem 2.2.3 implies the following.

Corollary 2.2.13. *For every finitely generated group, the map $r_{\mathcal{D}}$ puts the (path-)connected components of* $\mathrm{Rep}_{irr}(G,\mathbb{R})$ *in one-to-one correspondence with those of the harmonic space $\mathcal{D}$ of G.*

Question 2.2.14. Given a finitely generated group G, study the (path-)components of its harmonic space. In particular determine which representations of G are path-rigid.

To conclude this part with a simple observation interpreting the fixed points and periodic orbits of the flow $(\mathcal{D},\Psi)$, which will be useful in what follows.

Proposition 2.2.15 (see Proposition 3.10 in [10])**.** *Let $(\mathcal{D},\Psi)$ be the Deroin space of a finitely generated group G.*

1. *The set of fixed points of Ψ correspond to the the semi-conjugacy classes of actions of G by translations, i.e non-zero homomorphisms of G to $(\mathbb{R},+)$ up to a positive scalar. In particular it is either empty or homeomorphic to a sphere $\mathbb{S}^d, d\geq 0$.*

2. *The non-trivial periodic orbits of $(\mathcal{D},\Psi)$ correspond to semi-conjugacy classes of minimal actions of G on $\mathbb{R}$ which commute with the group of integer translations, and descend to an action on the circle $\mathbb{R}/\mathbb{Z}$ (which is moreover minimal and proximal).*

2.2.3 A short survey of some previous results

I conclude this section with a (non-exhaustive) survey of some classes of groups for which C^0 and C^1 actions on the line were well-understood, and give some first examples of description of their harmonic spaces. These examples add some context to the results that will come later.

A classical argument, which is based essentially on Hölder's theorem [Höl01], implies the following.

Theorem 2.2.16 (See Plante [Pla75], Navas [Nav10])**.** *Let G be a finitely generated group which does not contain a free semi-group on two generators (e.g. if G hassubexponential growth). Then every $\rho\in\mathrm{Rep}_{irr}(G,\mathbb{R})$ is semi-conjugate to a representation taking values in the group of translations $(\mathbb{R},+)$.*

It follows from Proposition 2.2.15 that the harmonic space of G is either empty (if G has no non-trivial action on $\mathbb{R}$) or homeomorphic to a sphere $\mathbb{S}^n, n\geq 0$, with trivial flow.

An important class of groups falling in the setting of Theorem 2.2.16 are finitely generated abelian and nilpotent groups. In this case there is also considerable literature on actions on regularity $C^r, r\geq 1$. A combination of the classical results of Hölder's [Höl01], Denjoy [Den32] and Kopell [Kop70], implies that if G is an abelian group, then every action $\rho\colon G\to\mathrm{Diff}^2([0,1])$ without fixed point in the interior is topologically conjugate (on $(0,1)$) to an action by translations. These results fail to hold in regularity C^1, as one can see from the examples of Tsuboi [Tsu95], generalising the classical constructions of Denjoy [Den32] and Pixton [Pix77]. The situation for nilpotent groups is completely analougous to the abelian case. Plante and Thurston showed (building on the case of abelian groups) that every nilpotent subgroup of $\mathrm{Diff}^2_0([0,1])$ is abelian [PT76]; in particular it is conjugate to a group of translations provided it is irreducible on $(0,1)$. This is far from being true for C^1 actions, as shown by Farb and Franks [FF03].

Beyond Theorem 2.2.16, there were some more groups for which actions by homeomorphisms (or even C^1-diffeomorphisms) on $\mathbb{R}$ were completely understood up to semi-conjugacy, in particular many groups that admit only a finite number of such actions.

An well-studied family of examples are the solvable Baumslag-Solitar groups $\mathsf{BS}(1,n) = \langle a, b \colon aba^{-1} = b^n \rangle$. The group $\mathsf{BS}(1,n)$ has a natural (faithful) action on $\mathbb{R}$ generated by the affine transformations $a\colon x \mapsto nx$ and $b\colon x \mapsto x+1$, as well as a (non-faithful) cyclic action coming from its abelianisation $G^{ab} \simeq \mathbb{Z}$. Every representation $\rho \in \mathrm{Rep}_{\mathrm{irr}}(\mathsf{BS}(1,n), \mathbb{R})$ is semi-conjugate to one of these two examples, up to the orientation (see Rivas [Riv10]). From this, one can easily describe the harmonic space of $BS(1,n)$, which is homeomorphic to a circle, with flow having two parabolic fixed points and two non-degenerate orbits, see Figure 2.1. For C^1-actions, work of Bonatti, Monteverde, Navas and Rivas [BMNR17] shows that every action $\varphi\colon BS(1,n) \to \mathrm{Diff}^1([0,1])$ without fixed point in the interior is conjugate on $(0,1)$ to the affine action, and moreover the conjugacy (which need not be C^1) preserves the values of derivatives of fixed points of homotheties (this very useful phenomenon is called rigidity of multipliers).

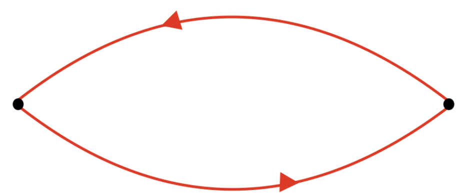

Figure 2.1: The harmonic space of the Baumslag-Solitar group $BS(1,2)$.

Another particular source of examples are central extension of some groups acting on the circle. From every group $G \leq \mathrm{Homeo}_+(\mathbb{S}^1)$, one can pass to a central extension $\tilde{G}$ with infinite cyclic center that acts on the line. Actions of such groups can often be understood by reducing the problem to the circle. Various groups are known to admit only one action on the circle up to semi-conjugacy and to the orientation, such as Thompson's group T or the Fuchsian group $\Delta(2,3,7)$; the same holds true for the corresponding central extension acting on the real line [7, MT23]. The harmonic space of such a group will consist of two copies of $\mathbb{S}^1$, which are periodic orbits for the flow.

2.3 Structure theorems for actions on the line via laminations

This section describes some results in our monograph [8] and the subsequent paper [9]. These works introduce some new ideas that allow to understand C^0 and C^1 actions on the line of various groups. The common ingredient in both papers is the role played by actions which have an invariant lamination. For expository reason [9] is presented first, although this paper follows and builds on [8].

2.3.1 Solvable groups and affine actions

In [9] we consider the class of finitely generated solvable groups. Looking at this case is natural in view of the classical results on abelian and nilpotent groups described above, and because this class contains perhaps the smallest of finitely generated groups not covered by Theorem 2.2.16. The study of this case was initiated in by J.F. Plante [Pla83]. A fundamental source of examples of solvable (in fact metabelian) groups of homemorphisms of the real line is given by subgroups of the real affine group

$$\mathrm{Aff}(\mathbb{R}) = \{x \mapsto ax + b, \quad a \in \mathbb{R}_{>0}, b \in \mathbb{R}\}.$$

Motivated by the analogy with Theorem 2.2.16, Plante initiated the study of the following question: if G is a finitely generated solvable group, under which conditions an action $\varphi \in \mathrm{Rep}_{\mathrm{irr}}(G, \mathbb{R})$ is semi-conjugate to an affine action $\psi \colon G \to \mathrm{Aff}(\mathbb{R})$? He gave an example showing that this is not always the case. More precisely, he constructed a faithful minimal action of the group $G = \mathbb{Z} \wr \mathbb{Z}$ on $\mathbb{R}$ which is not semi-conjugate to an affine action (described in Example 2.3.7 below). However he provided some sufficient conditions on an action which imply that the answer is affirmative: he showed in particular that this is always the case if the group G is polycyclic, or if the action is by analytic diffeomorphisms. Since Plante's work, a number of papers have addressed the study of actions of various classes of solvable groups on the line, with affine actions often playing a central role [Pla84, Nav04a, Nav04b, BMNR17, GR18] . In particular Navas showed that actions by diffeomorphisms of class at least C^2 are always semi-conjugate to affine action, and obtained an algebraic description of solvable groups acting faithfully by C^2 diffeomorphisms on one-manifolds [Nav04a]. These results is crucially based on the restrictions on abelian groups of C^2-diffeomorphisms, in particular Koppel's lemma, which fail in class C^1.

The following result from [9] shows that Plante's question has nevertheless an affirmative answer for actions by C^1-diffeomorphisms on compact intervals.

Theorem 2.3.1 (C^1-actions of solvable groups on intervals)**.** *Let G be a finitely generated virtually solvable group. Then for every C^1 action $\varphi \colon G \to \mathrm{Diff}^1_+([0,1])$ without global fixed points in the interior, the action of G on $(0,1)$ is semi-conjugate to an affine action $\psi \colon G \to \mathrm{Aff}(\mathbb{R})$. Moreover, the semi-conjugacy is automatically a topological conjugacy provided ψ has non-abelian image.*

The main new content of the result is the existence of the semi-conjugacy; it had been previously shown in [BMNR17] that any C^1-action on $[0,1]$ which is semi-conjugate (on the interior) to a non-abelian affine action is automatically minimal, and hence the semi-conjugacy is a conjugacy.

Even if the answer to Plante's question is negative for actions that are merely C^0, we show that it is still always positive in a *local* sense, at the level of germs near any given fixed point. A precise statement is the following.

Theorem 2.3.2 (Semi-conjugacy "at infinity" to an affine action)**.** *Let G be a finitely generated virtually solvable group. Then for every $\varphi \in \mathrm{Rep}_{irr}(G, \mathbb{R})$ there exists an affine action $\psi \colon G \to \mathrm{Aff}(\mathbb{R})$ without fixed points, and a non-decreasing map $h \colon (a, +\infty) \to \mathbb{R}$ with $\lim_{x \to +\infty} h(x) = +\infty$, such that for every $g \in G$ we have $\psi(g) \circ h(x) = h \circ \varphi(g)(x)$ for all x large enough. Such an action ψ is unique up to affine conjugacy.*

This result can be compared with a result of Chiswell and Kropholler [CK93], stating that a finitely generated solvable group admits a non-trivial homomorphisms to $\mathrm{Homeo}_+(\mathbb{R})$ if and only if it admits a non-trivial homomorphism to the group $(\mathbb{R}, +)$ of translations (for any group, the existence of a non-trivial homomorphism to $\mathrm{Aff}(\mathbb{R})$ is easily shown to be equivalent to the existence of a non-trivial homomorphism to $(\mathbb{R}, +)$). This result was famously generalised to all amenable groups by Witte Morris [Mor06]. In those results the construction of a homomorphism to $(\mathbb{R}, +)$ is quite indirect and does not say much on the original action one started with; instead Theorem 2.3.2 allows to associate to an any given action on the line of a solvable group an affine action with a clear dynamical interpretation. This stronger phenomenon appears so far to be specific to solvable groups.

Finally for a class of finitely generated solvable group, we show that any small perturbation of a non-abelian affine action in the space of the representations $\mathrm{Rep}_{\mathrm{irr}}(G, \mathbb{R})$ must remain semi-conjugate to an affine action, which can also be seen as positive answer to Plante's question in a local (in the space $\mathrm{Rep}_{\mathrm{irr}}(G, \mathbb{R})$) sense.

Theorem 2.3.3 (C^0-rigidity of the family of affine actions)**.** *Let G be a finitely generated group which is virtually nilpotent-by-nilpotent (e.g. any virtually solvable group which is linear over a field). Then the set of representations $\varphi \in \mathrm{Rep}_{irr}(G, \mathbb{R})$ which are semi-conjugate to a non-abelian affine action is open in $\mathrm{Rep}_{irr}(G, \mathbb{R})$.*

This result is false for arbitrary solvable groups: we give an example of a 3-step solvable group failing to have this property.

Even though affine actions appear to be the main character in the results above, proving them requires some understanding actions of finitely generated solvable group that are *not* semi-conjugate to affine actions. The main content of [9] is a qualitative description of the dynamics of such actions, summarized in Theorem 2.3.11 below. That result is based on the study of a class of actions with an invariant lamination, developed in [8]. The discussion of that result is postponed after a digression on this.

2.3.2 Invariant laminations and $\mathbb{R}$-focal actions

We denote by $\mathbb{R}^{(2)} = \{(x, y) \in \mathbb{R}^2 \colon x < y\}$,identified with the set of bounded open intervals of $\mathbb{R}$, with the topology induced from $\mathbb{R}^2$. We say that two intervals I, J are *linked* if they overlap non-trivially, but neither one is contained in the other.

Definition 2.3.4. A lamination of the real line $\mathbb{R}$ is a closed subset of $\mathcal{L} \subset \mathbb{R}^{(2)}$ which does not contain any linked pair of intervals.

Definition 2.3.5. Given a group G, an action $\rho \in \mathrm{Rep}_{\mathrm{irr}}(G, \mathbb{R})$ is $\mathbb{R}$-focal if it preserves a lamination $\mathcal{L}$ such that for every $I \in \mathcal{L}$, there exists a sequence of (g_n) of such that $g_n(I)$ is an exhaustion of $\mathbb{R}$.

This notion is invariant under semi-conjugacy, so that for our purposes it is enough to study *minimal* $\mathbb{R}$-focal actions.

The terminology comes from the study of actions on trees (and hyperbolic spaces). Recall that an isometric group action on a simplicial tree $\mathbb{T}$ is focal if it fixes a unique end $\omega \in \partial\mathbb{T}$ and contains hyperbolic elements. In fact we develop a formalism in [8] which allows to effectively visualise $\mathbb{R}$-focal actions in terms of trees, that I only briefly outline here. If $\mathcal{L}$ is a a lamination, the partially ordered set $(\mathcal{L}, \subset)$ can be always completed through a standard procedure (described in [8]) to obtain a G-action on a partially ordered set $(\mathbb{T}, \triangleleft)$ whose totally ordered subsets are isomorphic to $(\mathbb{R}, \leq)$, and where any two point admit a smallest common upper bound. We call such a set a *directed tree* in [8]. In other words $(\mathbb{T}, \triangleleft)$ can be identified with a real tree (more precisely a *non-metric real tree* in the terminology of [FJ04]), with the partial order $\triangleleft$ induced by the choice of an end $\omega_{\mathbb{T}} \in \partial\mathbb{T}$, called the *focus*, which is a maximal point at infinity. The $\mathbb{R}$-focal condition on the G-action on the lamination implies that that G-orbit of every point $x \in \mathcal{T}$ is cofinal in $(\mathbb{T}, \triangleleft)$; we call such actions *focal* in this non-metric context. The lamination $\mathcal{L}$ also carries a second partial order $<$, where $I < J$ denotes the fact that $\sup I \leq \inf J$. The two partial orders $\subset, <$ on $\mathcal{L}$ are *orthogonal*: a pair of elements is comparable for one if and only if it is not comparable for the other. The order $<$ also extends to a G-invariant partial order $\prec$ on $\mathbb{T}$ which is orthogonal to the tree order $\triangleleft$. A triple $(\mathbb{T}, \triangleleft, \prec)$ of a directed tree with an additional orthogonal order is called a *planar directed tree* (as $\prec$ can be thought of an order on branches determined by the choice of an embedding of $\mathbb{T}$ in the plane). In particular $\preceq$ induces a linear order on the set of ends $\partial\mathbb{T} \setminus \{\omega_{\mathbb{T}}\}$ with the focus removed. From the G-action on the ordered set $(\partial\mathbb{T} \setminus \{\omega_{\mathbb{T}}\}, \prec)$ one can recover the original action on $\mathbb{R}$ through a standard dynamical realisation procedure (an intuitively correct picture is to imagine the tree $\mathbb{T}$ as "hanging over the line" in the upper half plane, with the focus ω at ∞ and the remaining ends on the line). This construction implies the following (more precise definitions and details can be found in [8]).

Proposition 2.3.6 ([8])**.** *An action $\varphi \in \mathrm{Rep}_{irr}(G, \mathbb{R})$ is $\mathbb{R}$-focal if and only if there exists a focal action of G on a separable planar directed tree $(\mathbb{T}, \triangleleft, \prec)$ such that φ is semi-conjugate to the dynamical realisation of the action of G on $(\partial\mathbb{T} \setminus \{\omega_{\mathbb{T}}\}, \prec)$.*

The point of view of trees is extensively used in [8, 9]. Here it won't be needed to understand the statements of the results, although it can be kept in mind to help intuition.

Example 2.3.7. A fundamental example of $\mathbb{R}$-focal action is the action of $\mathbb{Z} \wr \mathbb{Z}$ constructed by Plante [Pla83], which is not semi-conjugate to an affine action. To define it, it is convenient to think of $\mathbb{Z} \wr \mathbb{Z}$ as the semi-direct product $\mathbb{Z}[X, X^{-1}] \rtimes \mathbb{Z}$, where $\mathbb{Z}$ acts on the ring of Laurent polynomial $\mathbb{Z}[X, X^{-1}]$ by powers of X. The group $\mathbb{Z} \wr \mathbb{Z}$ then acts affinely on $\mathbb{Z}[X, X^{-1}]$. This action preserves the natural lexicographic order $\prec$ on $\mathbb{Z}[X, X^{-1}]$, obtained by declaring a polynomial $p \succ 0$ if its coefficient of smallest degree is positive. By taking a completion of the ordered set $(\mathbb{Z}[X, X^{-1}], \succ)$ one obtains an action on $\mathbb{R}$. Figure 2.2 shows a portion of the graph of some elements of $\mathbb{Z} \wr \mathbb{Z}$ in this action. One can show that this action is minimal and $\mathbb{R}$-focal, see [8, Example 7.11]. The natural invariant lamination is discrete, and the corresponding tree coincides with the simplical Bass-Serre tree associated to a natural splitting of $\mathbb{Z} \wr \mathbb{Z}$ as an ascending HNN-extension. (Of course we could have also defined the partial order $\prec$ by looking at the coefficient with highest degree. Considering also the flip of orientations, this construction gives four non-semiconjugate actions in total.)

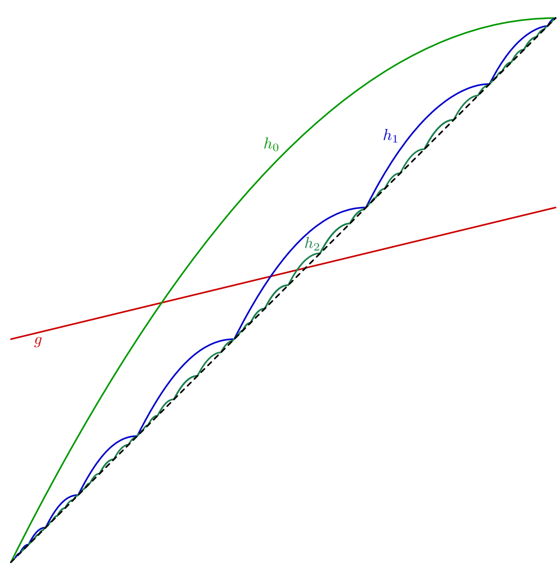

Figure 2.2: The Plante action. Here $\mathbb{Z} \wr \mathbb{Z} = \langle g, h | [g^n h g^{-n}, h] = 1 \forall n \rangle$, and $h_i = g^i h g^{-1}$.

An important consequence of $\mathbb{R}$-focality of an action is that individual elements $g \in G$ satisfy a dichotomy analogous to the classification of isometries of trees into hyperbolic and elliptic. Let us fix the following terminology. We will say that a homeomorphism of the real line $f \in \mathrm{Homeo}_+(\mathbb{R})$ is:

- a *pseudohomothety* if it has a non-empty compact set of fixed point $K \subset \mathbb{R}$ and for every $x \notin [\min K, \max K]$ we have $|f^n(x)| \to \infty$ for $n \to +\infty$ (in which case we say that f is an *expanding* pseudohomothety) or the same holds for $n \to -\infty$ (in which case we say that f is *contracting*).
- *totally bounded* if its orbits are all bounded (equivalently its set of fixed points accumulate on $\pm\infty$).

Proposition 2.3.8 ([8])**.** *If G is a group and $\rho \in \mathrm{Rep}_{irr}(G, \mathbb{R})$ is an $\mathbb{R}$-focal action, then every element of G is either a pseudohomothety or it is totally bounded.*

The fact that an action is $\mathbb{R}$-focal, alone, gives limited information: such actions can be quite wild (think of the subgroup of all homeomorphisms of $\mathbb{R}$ that preserve a lamination). It becomes much more useful if in the presence of additional structure preserved by the G action on an invariant lamination $\mathcal{L}$. The best possible case would be that the G-action on a tree associated to a lamination preserves an invariant $\mathbb{R}$-tree metric. While this is often too much to ask, in many situation it is still possible to find a notion of "horospheres" on $\mathcal{L}$ (or on the associated tree), so that the G-action on the set of horospheres is governed by another G-action on $\mathbb{R}$. This is the content of the following definition, which will be crucial in what follows.

Definition 2.3.9. A *horograding* of a lamination $\mathcal{L}$ is a map $h\colon \mathcal{L} \to \mathbb{R}$ such that $h(I) \leq h(J)$ whenever $I \subseteq J$.

Let $\rho \in \mathrm{Rep}_{\mathrm{irr}}(G, \mathbb{R})$ be an $\mathbb{R}$-focal action, and $j \in \mathrm{Rep}_{\mathrm{irr}}(G, \mathbb{R})$ be another (irreducible) action. We say that ρ can be horograded by j if there exists an invariant lamination $\mathcal{L}$ and an equivariant horograding $h\colon \mathcal{L} \to \mathbb{R}$ which intertwines ρ and j.

For example, the the case where the horograding action j takes values in the group of translations $(\mathbb{R}, +)$ corresponds exactly to the case where the group action of an associated tree $(\mathbb{T}, \triangleleft)$ preserves an $\mathbb{R}$-tree metric on $\mathbb{T}$. The Plante actions of $\mathbb{Z} \wr \mathbb{Z}$ described above is horograded by the standard cyclic action by integer translation of the natural $\mathbb{Z}$-quotient.

The horograding action retains large-scale information on the dynamics of the original action. For instance we have the following.

Proposition 2.3.10. *Let $\rho \in \mathrm{Rep}_{irr}(G, \mathbb{R})$ be an $\mathbb{R}$-focal action, and assume that ρ is positively horograded by j. Then for every $g \in G$ the following holds*

1. *if $j(g)$ has no fixed point accumulating to $+\infty$, then $\rho(g)$ is a pseudo-homothety, which is expanding if $j(g)(x) > 0$ for all x large enough, and contracting otherwise. If moreover $j(g)$ has no fixed point on $\mathbb{R}$, then $\rho(g)$ has a unique fixed point (we say that it is a* homothety*).*
2. *Else, $\rho(g)$ is totally bounded.*

Although these notions might seem technical, a fact that we realized from [8] is that $\mathbb{R}$-focal actions appear more often than one might think. In particular this class contains many examples of actions that can be considered "exotic" in some way (including many isolated examples that had been studied before, such as the Plante action of $\mathbb{Z} \wr \mathbb{Z}$). The notion of horograding turns out to be crucial, as it it gives a way to relate them to more "familiar" actions, in a way which is weaker than semi-conjugacy, but still useful. The results below will provide examples of this principle.

2.3.3 General actions of solvable groups

With the language of $\mathbb{R}$-focal actions (and their horogradings) in hand, we can now state a result that elucidates the behaviour of arbitrary C^0-actions of finitely generated solvable groups in the line, which is the actual main result in [9]. Recall that the Fitting subgroup $\mathrm{Fit}(G)$ of a group G is the union of all nilpotent normal subgroups of G (it is a subgroup by Fitting's theorem).

Theorem 2.3.11 ([9])**.** *Let G be a finitely generated virtually solvable group. Then every action $\rho \in \mathrm{Rep}_{irr}(G, \mathbb{R})$ satisfies one of the following two cases;*

1. *either ρ is semi-conjugate to an affine action;*
2. *or ρ is semi-conjugate to a minimal $\mathbb{R}$-focal action, and in this case ρ can be horograded by an action ρ_* which factors through $G/\mathrm{Fit}(G)$.*

A crucial feature of Theorem 2.3.11 is that it can be applied inductively until finding an affine action. Indeed note that $G/\mathrm{Fit}(G)$ has smaller solvable length than G. If ρ is $\mathbb{R}$-focal, and if the horograding action ρ_* is not semi-conjugate to an affine action, then it must be again $\mathbb{R}$-focal and horograded by an action $(\rho_*)_*$ of a quotient of even smaller solvable length. In finitely many steps, we must reach an affine action $\psi\colon G \to \mathrm{Aff}(\mathbb{R})$ (this is in fact the affine action appearing in Theorem 2.3.2, which is a direct consequence of Theorem 2.3.11). This shows that actions on the line of a solvable groups on the line are naturally organized in a tower of finitely many levels of complexity, where affine actions are the simplest.

Although it might not be apparent from its statement, Theorem 2.3.11 is yet another form of the dychotomy between freeness and non-freeness, where the second item captures the structure imposed by non-freeness. The idea of its proof is the following: let $\rho \in \mathrm{Rep}_{\mathrm{irr}}(G, \mathbb{R})$; up to taking a semi-conjugate action and passing to a quotient, we can suppose without loss of generality that ρ is minimal and faithful. Consider then a non-trivial abelian normal subgroup $N \trianglelefteq G$. If N acts freely, by Hölder's theorem it is semi-conjugate to a subgroup of $(\mathbb{R}, +)$. In that case, one can conclude that ρ is semi-conjugate to an affine action (essentially because the normaliser of a dense group of translations is contained in $\mathrm{Aff}(\mathbb{R})$). If N does not act freely, by inspecting the combinatorics of the fixed points of its elements (and the maximal intervals of their complement) one obtains the existence of an invariant lamination. To find a horograding one needs to extract a minimal lamination, and then study the action of G on the associated tree $\mathbb{T}$: roughly speaking, the action of any nilpotent normal subgroup on it must be of elliptic type, in the sense that it cannot move any point towards the focus. This allows to quotient $\mathbb{T}$ by $\mathrm{Fit}(G)$ to obtain a new directed tree, and conclude by induction.

Example 2.3.12 (Classifying all actions of $\mathbb{Z} \wr \mathbb{Z}$.). Let us illustrate Theorem 2.3.11 with the example of $G = \mathbb{Z} \wr \mathbb{Z} = \mathbb{Z}[X, X^{-1}] \rtimes \mathbb{Z}$. We have $\mathrm{Fit}(G) = \mathbb{Z}[X, X^{-1}]$. Hence by Theorem 2.3.11 any $\mathbb{R}$-focal action is necessarily horograded by an action of the standard cyclic quotient of G. Out of this and of Proposition 2.3.10 it is not difficult to see that the only $\mathbb{R}$-focal actions of G are the four Plante actions described in Example 2.3.7 [9, Proposition 6.3]. The affine actions of G are easily described by a separate argument [9, Lemma 6.1].

Recalling that G has the presentation $G = \langle t, h | [t, h^n t h^n] = 1 \forall n \rangle$, then the affine actions of G are, up to semi-conjugacy:

- actions by translations obtained by setting

$$\rho(t)\colon x \mapsto x + \alpha, \quad \rho(h)\colon x + \beta \qquad (\alpha^2 + \beta^2 = 1);$$

- non-abelian affine actions obtained by setting

$$\rho(t)\colon x \mapsto x \pm 1, \quad \varphi(h)\colon x \mapsto \lambda x \qquad (\lambda > 0, \lambda \neq 1).$$

The non-abelian affine actions are obtained from the action on the ring $\mathbb{Z}[X, X^{-1}]$ by endowing it with the pre-order induced by evaluation of polynomials on λ. For $\lambda \to \infty$ and for $\lambda \to 0$ this resembles a lexicographic order: from this one can show that the corresponding representative in the harmonic space $(\mathcal{D}, \Psi)$ must necessarily converge to a Plante action. With slightly more care, it is not too difficult to obtain a full the description of the harmonic space of $\mathbb{Z} \wr \mathbb{Z}$, shown in Figure 2.3.

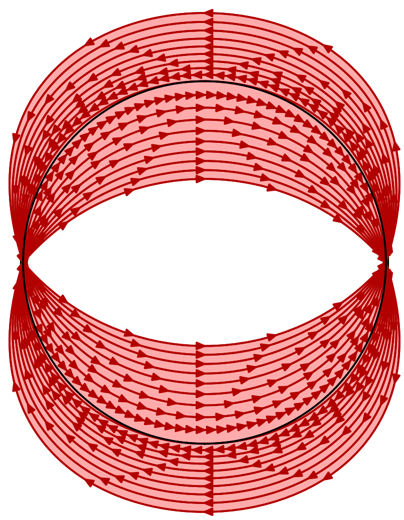

Figure 2.3: The harmonic space $(\mathcal{D}, \Psi)$ of the group $\mathbb{Z} \wr \mathbb{Z}$. The space $\mathcal{D}$ is homoeomorphic to the connected sum of two closed disks at two points on the boundary. The black circle is fixed by the flow, and parametrises actions by translations of $(\mathbb{Z} \wr \mathbb{Z})^{ab} \simeq \mathbb{Z}^2$ (Proposition 2.2.15). The rest of the space is foliated by Ψ-orbits as shown: the two limit points are the cyclic action of the standard $\mathbb{Z}$-quotient, and its conjugate by the reflection (this is because in all affine actions and in the Plante action, the displacement of the homothety $\rho(h)$ is infinitely larger than the one of $\rho(t)$ near ∞). The four boundary orbits are the Plante actions, the remaining orbits are the non-abelian affine actions, divided in four regions according to the choices $\lambda \in (0, 1), \lambda \in (1 + \infty)$, and $\rho(t)\colon x \mapsto x \pm 1$. When λ approaches 1, the orbits approach the circle of fixed points: the flow slows down, and inverts its direction when λ crosses 1. I thank M. Triestino for the picture.

The proof of Theorem 2.3.1 uses the example of $\mathbb{Z} \wr \mathbb{Z}$. As another consequence of Theorem 2.3.11, we show every minimal $\mathbb{R}$-focal action of a solvable group G on the line, there is a copy of $\mathbb{Z} \wr \mathbb{Z}$ in G whose action has no global fixed point and is semi-conjugate to a Plante action. This reduces the problem to see that the latter cannot be C^1 (up to semi-conjugacy). We do so by combining the intuition given by Figure 2.3, that shows that the latter is a limit of affine actions, together with the method used in [BMNR17] to show that a topological conjugacy of an affine action to any C^1 action preserve the derivative at hyperbolic fixed points (which explode in the limit).

Finally Theorem 2.3.3 is a manifestation of the fact that, if the group G is metanilpotent (e.g. linear), then $G/\operatorname{Fit}(G)$ is nilpotent and can only act by translations up to semi-conjugacy (Theorem 2.2.16). In this case Theorem 2.3.11 gives stronger information.

It is apparent from Figure 2.3 that the harmonic space $(\mathcal{D}, \Psi)$ of $\mathbb{Z} \wr \mathbb{Z}$ has very restricted dynamics: it admits a locally closed section intersecting each orbit exactly once (in particular its space of orbits is smooth in the Borel sense), and all orbits converge to a limit point. In view of Theorem 2.2.3, this corresponds to the fact that the structure of semi-conjugacy classes in $\operatorname{Rep}_{\mathrm{irr}}(\mathbb{Z} \wr \mathbb{Z}, \mathbb{R})$ is very gentle. Using Theorem 2.3.11, one can in fact prove analogous conclusions for the harmonic space of an arbitrary finitely generated solvable group. This is the object of an ongoing project with the same collaborators; the results are still evolving but I would like to announce here the following special case, which could be deduced quite directly from Theorem 2.3.11.

Theorem 2.3.13 (Brum–MB–Rivas–Triestino, in preparation)**.** *Let G be a finitely generated solvable group with $\operatorname{Rep}_{irr}(G, \mathbb{R}) \neq \varnothing$, and $(\mathcal{D}, \Psi)$ be any representative of the harmonic space of G. Then there exists Ψ-invariant closed subsets $\mathcal{D} = \mathcal{D}_r \supset \mathcal{D}_{r-1} \supset \cdots \supset \mathcal{D}_1$ such that*

1. *The set $\mathcal{D}_1$ consists of the fixed points of Ψ and is homeomorphic to a sphere $\mathbb{S}^{d-1}$, for $d = \dim \mathsf{Hom}(G, \mathbb{R})$ (Proposition 2.2.15).*
2. *For every $\rho \in \mathcal{D} \setminus \mathcal{D}_1$, the limits $\lim_{t\to+\infty} \Psi^t(\rho)$ and $\lim_{t\to-\infty} \Psi^t(\rho)$ exist (and belong to $\mathcal{D}_1$); they are opposite points on character sphere.*
3. *The convergence to the limit is uniform over compact subsets of $\mathcal{D}_{i+1} \setminus \mathcal{D}_i$ for every $i \geq 1$. As a consequence the restriction of Ψ to $\mathcal{D}_{i+1} \setminus \mathcal{D}_i$ is proper, and the quotient $(\mathcal{D}_{i+1} \setminus \mathcal{D}_i)/\Psi$ is a locally compact metrisable space.*

By the discussion after Theorem 2.3.11, one can deduce the following:

Corollary 2.3.14. *For a finitely generated solvable group G with $\operatorname{Rep}_{irr}(G, \mathbb{R}) \neq \varnothing$, the following hold:*

- *the semi-conjugacy relation on the space $\operatorname{Rep}_{irr}(G, \mathbb{R})$ is smooth (in the Borel sense);*
- *the space $\operatorname{Rep}_{irr}(G, \mathbb{R})$ is path-connected, unless it contains only two semi-conjugacy classes corresponding to a cyclic action $G \to \mathbb{Z}$ and its conjugate by the reflection.*

This naturally raises the following questions.

Question 2.3.15. Let $(\mathcal{D}, \Psi)$ be the harmonic space of a finitely generated amenable group.

1. Is it true that all Ψ-orbits converge to a limit?
2. Is it true that the orbit-equivalence relation of the flow $(\mathcal{D}, \Psi)$ is smooth?

It follows from [10, Proposition 3.10] that if G is amenable, then all orbits in $(\mathcal{D}, \Psi)$ must admit a fixed point in their closure; this is related to Witte-Morris' theorem [Mor06]. The convergence to a fixed point is however a substantially more delicate property. While a positive answer for elementary amenable group is plausible and, hopefully, reachable, a generalisation to all amenable groups is connected the amenability problem for Thompson's group F (I'll come back to this in Remark 2.3.30).

2.3.4 Locally moving groups and Thompson's group F

The goal of the main results in the monograph [8] is to describe the possible actions on the real line of a class of subgroups $G \leq \mathrm{Homeo}_+(\mathbb{R})$. If $G \leq \mathrm{Homeo}_+(\mathbb{R})$, we will refer to the inclusion of $G \hookrightarrow \mathrm{Homeo}_+(\mathbb{R})$ as the *standard* action of G. For a group given in this way, and under suitable assumptions on the standard action, we will be interested in analysing the other possible actions of G on the line, in particular to determine whether these must be semi-conjugate to the standard action (up to the orientation).

Taking the same notation and terminology from §1.2, for a non-empty open interval $I \subset \mathbb{R}$ we denote by G_I the subgroups of G that fixes all points in $\mathbb{R} \setminus I$. The main assumption on G will be the following.

Definition 2.3.16. A subgroup $G \leq \mathrm{Homeo}_+(\mathbb{R})$ is locally moving if for every non-empty open interval I, the rigid stabiliser G_I acts on I without fixed points. (It then follows that every G_I acts minimally on I).

One evidence that this assumption should imply some form of rigidity is the result of M. Rubin that if $G \leq \mathrm{Homeo}(Z)$ where Z is a perfect locally compact space, and if $G_U \curvearrowright U$ is minimal for U in a basis of the topology, then the space Z and the action $G \curvearrowright Z$ can be reconstructed from G [Rub89][2]. Thus a locally moving group $G \leq \mathrm{Homeo}_+(\mathbb{R})$ has a unique locally moving action up to conjugacy; this says a-priori nothing, however, on other actions of G that might fail to have this property.

A famous example of locally moving subgroup of $\mathrm{Homeo}_+(\mathbb{R})$ is Thompson's group F (which was defined in §1.2.2). Here the standard action is the action by PL-homeomorphisms on the interval $(0, 1)$ (identified with $\mathbb{R}$). We also mention that F is actually the "smallest" locally moving group acting on the line: every other such group contains a copy of F, by an argument going back to Brin [Bri99] and largely generalised by Kim, Koberda and Lodha [KKL19].

Thompson's group F is one of the single most studied examples of groups acting on the line, yet very little was known about its possible actions. Besides its standard action, another source of examples of actions comes from the fact that the group F has an infinite abelianisation $F^{ab} \simeq \mathbb{Z}^2$. This implies that it admits actions on the line by translations, which correspond to homomorphisms of $\mathbb{Z}^2$ into the additive group $(\mathbb{R}, +)$ of the reals. There are also (non-minimal) faithful actions which are semi-conjugate to actions by translations (for instance this is always the case for actions arising from bi-invariant orders, which exist on F and were classified by Navas and Rivas [NR10]). However it did not appear to be known whether these are the only actions of F up to semi-conjugacy, equivalently whether every *minimal* and *faithful* action of the group F by homeomorphisms on the line is conjugate its standard action. The project that lead to [8] started by investigating this question. One reason to expect an affirmative answer was that the analogous result for Thompson's group T acting on the circle holds true, as was first observed by É. Ghys using bounded cohomology.

Finally one of my personal motivations that lead me to investigate this type of question was the analogy with some of the results in §1.2, in particular Corollary 1.2.3 and Theorem 1.2.4.

[2]The reader should be warned that the terminology "locally moving" is used in some of the papers of M. Rubin as a synonym of what we called here "micro-supported" (§1.2). The condition that we call here "locally moving" has various names in the litterature. Since the terminologies for these properties are not stable even in Rubin's papers, we decided to chose our own.

Existence of exotic actions

Definition 2.3.17. In what follows, for a group given by some, minimal faithful action $G \leq \mathrm{Homeo}_+(\mathbb{R})$, we will term an action $\rho \in \mathrm{Rep}_{\mathrm{irr}}(G, \mathbb{R})$ *exotic* if it is minimal, faithful, and not semi-conjugate to its standard action up to the orientation.

The evidence described above turned quickly out, with some initial surprise, to be completely wrong: it turns out that many classes of locally moving subgroups of $\mathrm{Homeo}_+(\mathbb{R})$ admit exotic actions. We provide various sufficient conditions for the existence of such an action; a special case is the following.

Proposition 2.3.18 ([8])**.** *Le $G \leq \mathrm{Homeo}_+(\mathbb{R})$ be a finitely generated group whose standard action is minimal and conjugate to an action by piecewise linear or piecewise projective homemomorphisms (with finitely many pieces) on some open interval $X \subseteq \mathbb{R}$. Assume that G is not metabelian. Then there exists an exotic action $\rho \in \mathrm{Rep}_{irr}(G, \mathbb{R})$.*

Example 2.3.19 (Idea of proof in the PL case). Assume that G is a finitely generated group of PL homeomorphisms of $X = (a, b)$ as above, and let us sketch a construction that produces an exotic action of G on the line (at least for many natural examples of G). For this, denote by $D_-g(x), D_+g(x)$ the right-derivative of an element $g \in G$ at a point $x \in X$, which we extend to the extreme point b in the natural way. Consider the map $c\colon G \times X \to \mathbb{R}$ defined by

$$c(g, x) = \log(D_+g(x)/D_-g(b)).$$

This map is a cocycle, i.e. it satisfies $c(gh, x) = c(g, hx) + c(h, x)$. By construction, it has the property that for every $g \in G$, the function $c(g, \cdot)$ vanishes on a neighbourhood of b. It follows that for $g \in G$, the point $x_g := \sup\{x \colon c(g, x) \neq 0\}$ belongs to the interior of X (if $c(g, \cdot)$ is identically 0, we set $x_g = b$). Define $g \succeq 1$ if $c(g, x_g) > 0$, and $g \succeq h$ if $h^{-1}g \succeq 1$. An elementary (but somewhat tedious) verification shows that this defines a left-invariant pre-order on G (like an order except that $g \preceq h$ and $h \preceq g$ does not imply $g = h$). A pre-order descends a left-invariant order on a coset space G/H, where H is the subgroup of elements $\preceq$-equivalent to 1. By completing the ordered set $(G/H, \prec)$, one obtains an action of G on the line. It turns out this construction yields a minimal exotic action. (This action is actually $\mathbb{R}$-focal). This construction can be compared with the Plante action of $\mathbb{Z} \wr \mathbb{Z}$ (Example 2.3.7). Of course, we can also reverse the orientation of X and carry out the same construction, the resulting action is not conjugate (even up to the orientation).

In particular, the group F admits exotic actions. The previous construction has the advantage to work for very general PL groups, but provides only two such actions. Moreover this construction yields honest exotic actions for the group F, but it is not satisfactory for its derived subgroup $[F, F]$ for which it restricts to an action which is not minimal (and actually coincides with an action arising from a well-known bi-invariant order [NR10]). However in the case of the group F, we provide in [8] many quite different constructions, some which are more specific to it and do not work in the generality of all PL groups. In particular a more delicate construction gives the following.

Theorem 2.3.20. *Thompson's group F admits uncountably many pairwise non semi-conjugate faithful minimal actions $\rho\colon F \to \mathrm{Homeo}_+(\mathbb{R})$, which remain minimal and pairwise non-semiconjugate in restriction to $[F, F]$.*

Main results

The three results below could be considered the main results of [8]; they show that nevertheless, for various classes of locally moving groups (and for Thompson's group F in particular), actions on the line do satisfy various rigidity phenomena. First, exotic actions are never C^1.

Theorem 2.3.21 (C^1-actions of locally moving groups)**.** *Let* $G \leq \mathrm{Homeo}_+(\mathbb{R})$ *be a locally moving group. Then every action* $\rho\colon G \to \mathrm{Diff}^1_+(\mathbb{R})$ *is semi-conjugate, up to the orientation, to one of the following:*

(i) the standard action.

(ii) a non-faithful action.

In particular every faithful minimal action $\rho\colon G \to \mathrm{Diff}^1_+(\mathbb{R})$ *is topologically conjugate (up to the orientation) to the standard action.*

Remark 2.3.22. In the case of Thompson's group F, the standard action is indeed topologically conjugate to a C^∞ action by a well-known result of É. Ghys and V. Sergiescu [GS87]. I general it can happen that the standard action nor any non-faithful action can are semi-conjugate to a C^1 action. In that case, the result simply says that G does not admit non-trivial C^1-actions on $\mathbb{R}$.

Remark 2.3.23. The non-faithful actions of F are precisely the actions by translations of $F^{ab} \simeq \mathbb{Z}^2$, since every non-trivial normal subgroup of F contains $[F, F]$. For a general locally moving group, denote by G_c the subgroup of G consisting of elements with compact support (in the standard action). Then the commutator subgroup $[G_c, G_c]$ is simple and contained in every non-trivial normal subgroup of G [8, Proposition 4.4]. Therefore the non-faithful actions of G are necessarily induced from the largest quotient $G/[G_c, G_c]$. If G is finitely generated, then the latter is non-trivial and always has non-trivial actions on the line. In many interesting concrete cases, the quotient $G/[G_c, G_c]$ can be described and its actions understood, but not much can be said about it in full generality.

Our second main result is a qualitative description of all C^0 actions of a vast class of locally moving groups. For this result we need to add a technical finite generation assumption on some subgroups of G, stated here in a non-optimal form for simplicity.

Definition 2.3.24. We denote by $\mathcal{C}$ the class of all subgroups $G \leq \mathrm{Homeo}_+(\mathbb{R})$ that satisfy the following conditions

(i) G is locally moving.

(ii) There exist $x, y \in \mathbb{R}$ such that the subgroups $G_{(-\infty,x)}$ and $G_{(y,+\infty)}$ are finitely generated.

The group F, and many other examples, belong to the class $\mathcal{C}$. In fact, every countable subgroup of $\mathrm{Homeo}_+(\mathbb{R})$ is contained in a finitely generated group in $\mathcal{C}$ (with a slightly more general formulation of (ii))

Theorem 2.3.25 (Structure theorem for C^0-actions)**.** *Let* G *be a subgroup of* $\mathrm{Homeo}_+(\mathbb{R})$ *in the class* $\mathcal{C}$*. Then every* $\rho \in \mathrm{Rep}_{irr}(G, \mathbb{R})$ *is semi-conjugate, up to the orientation, to one of the following.*

(i) A non-faithful action.

(ii) The standard action.

(iii) A minimal $\mathbb{R}$-focal action, which can be horograded by the standard action or by its conjugate by the reflection.

The main point of this theorem is that, although the group G may admit many non-semiconjugate exotic actions on the line, these all have a similar qualitative dynamics and, most importantly, the standard action of G governs the behaviour of all of them at large scale. In particular, the knowledge of how elements act in the standard action gives information on how they can act in other actions via Proposition 2.3.10.

Among its main applications, Theorem 2.3.25 can be used to gain insight on the the space or representations $\mathrm{Rep}_{\mathrm{irr}}(G, \mathbb{R})$. In particular under a mild additional assumption on G, we show that its $\mathbb{R}$-focal representations cannot accumulate on the standard one, thus obtaining the following.

Theorem 2.3.26 (C^0-rigidity of the standard action)**.** *Let G be a finitely generated subgroup of $\mathrm{Homeo}_+(\mathbb{R})$ in the class $\mathcal{C}$, and assume further that G contains an element $f \in G$ without fixed point in $\mathbb{R}$. Then the semi-conjugacy class of the standard action $G \hookrightarrow \mathrm{Homeo}_+(\mathbb{R})$ is open in $\mathrm{Rep}_{irr}(G, \mathbb{R})$ (namely the standard action is C^0-rigid according to Definition 2.2.10).*

This result provides a new source of examples of C^0-rigid actions of finitely generated groups on the line. All previously known examples of (locally) C^0-rigid actions that I am aware of seem to be groups having finitely many actions up to semi-conjugacy, and the many groups known to admit an isolated order (see for instance [MR18]): every such order gives rise to a locally C^0-rigid action which is, however, never minimal (and tipically not C^0-rigid). Instead, there are many more examples of C^0-rigid actions on the circle (an important one being Fuchsian actions of surface groups [Mat86, Man15]). The lack of compactness of $\mathbb{R}$ is a relevant difference, as a neighbourhood of a representation $\rho \in \mathrm{Rep}_{\mathrm{irr}}(G, \mathbb{R})$ only sees how ρ behaves on some compact subset of $\mathbb{R}$.

The harmonic space of Thompson's group F

Theorem 2.3.26 is proven by using Theorem 2.3.25 to analyse the qualitative behaviour of the flow on the harmonic space $(\mathcal{D}, \Psi)$ of a group in the class $\mathcal{C}$, which turns out to have very restricted dynamics. Figure 2.4 provides an illustration of this for Thompson's group F. The goal of this paragraph is to explain that picture.

According to Theorem 2.3.25, the harmonic space $\mathcal{D}$ of F can be naturally partitioned into three Ψ-invariant subsets $\mathcal{D} = \mathcal{A} \cup \mathcal{S} \cup \mathcal{F}$, where:

1. The set $\mathcal{A}$ consists of non-faithful actions, which in this case are actions by translations of $\mathbb{Z}^2 = F^{ab}$. Thus it is homeomorphic to $\mathbb{S}^1$, and it is fixed by the flow Ψ (Proposition 2.2.15). There are 4 distinguished points on it, corresponding to the cyclic actions $\tau_0, \tau_1 \colon F \to \mathbb{Z}$ determined by the logarithm (in base 2) of the derivatives at the extreme points of $[0, 1]$, up to sign.

2. The set $\mathcal{S}$ contains exactly two flow orbits $\mathcal{S}_+$ and $\mathcal{S}_-$, corresponding to the standard action of F and its conjugate by the reflection.

3. The set $\mathcal{F}$ consists of actions which are $\mathbb{R}$-focal, it further splits into two subsets $\mathcal{F} = \mathcal{F}_+ \sqcup \mathcal{F}_-$, namely those that are horograded by the standard action on $(0, 1)$ and by its conjugate by the reflection.

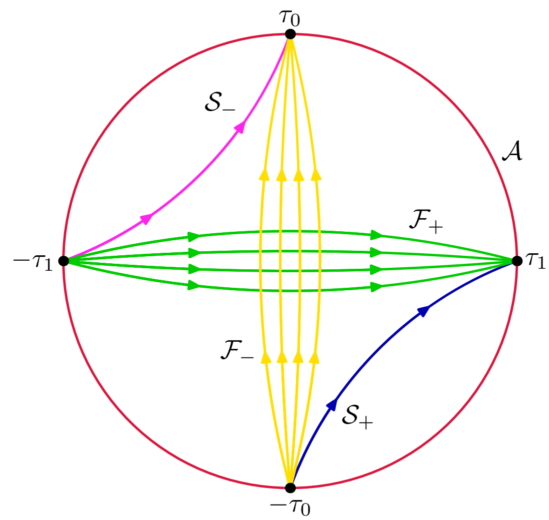

Figure 2.4: The harmonic space of Thompson's group F.

We have the following.

Theorem 2.3.27. *Let $(\mathcal{D}, \Psi)$ be a representative of the harmonic space of Thompson's group F, and retain all notations above. Then*

- *The orbits $\mathcal{S}_+, \mathcal{S}_-$ of the standard actions converge (in the past and the future) to the limits shown in Figure 2.4.*
- *Each set $\mathcal{F}_\pm$ is open and Ψ-invariant, and admits a compact transversal $\mathcal{F}_\pm^0$ intersecting each Ψ-orbit in exactly one point. Moreover all orbits converge to the limits shown in Figure 2.4, and the convergence is uniform on the compact transversals $\mathcal{F}_\pm^0$.*

In particular $\overline{\mathcal{F}_+} = \mathcal{F}_+ \cup \{\tau_1, -\tau_1\}$, and $\overline{\mathcal{F}_-} = \mathcal{F}_- \cup \{\tau_0, -\tau_0\}$, and thus the sets $\mathcal{S}_+, \mathcal{S}_-$ are open.

Note that the first claim, on the convergence of standard orbits $\mathcal{S}_\pm$, is straightforward; the content of the theorem is the conclusion concerning the set $\mathcal{F}$ of exotic actions, which is based on Theorem 2.3.25. This result can be used to answer various general questions about the space of representations $\mathrm{Rep}_{\mathrm{irr}}(F, \mathbb{R})$, by Theorem 2.2.3 and the subsequent discussion. In particular it has the following corollaries.

Corollary 2.3.28. *(i) The standard action of Thompson's group F on $\mathbb{R}$ is C^0-rigid (Theorem 2.3.26 in this case).*

(ii) The semi-conjugacy equivalence relation on $\mathrm{Rep}_{irr}(F, \mathbb{R})$ is smooth in the Borel sense. In fact it admits a closed transversal; moreover if we remove from $\mathrm{Rep}_{irr}(F, \mathbb{R})$ the semi-conjugacy classes of the four actions $\pm\tau_0, \pm\tau_1$, the quotient of the resulting space by the semi-conjugacy relation is locally compact and Polish.

(iii) The space $\mathrm{Rep}_{irr}(F, \mathbb{R})$ is path connected. However, every path that connects two actions having different type in Theorem 2.3.25 must meet the semi-conjugacy class of one of the four actions $\pm\tau_0, \pm\tau_1$.

While Theorem 2.3.27 gives a good qualitative picture of the dynamics of the flow $(\mathcal{D}, \Psi)$, it does not describe completely the topology of the space $\mathcal{D}$: the main question left open is to understand the topology of the compact transversals $\mathcal{F}^0_\pm$ (equivalently, the quotient spaces $\mathcal{F}_\pm/\Psi$). We know little about them, beyond the fact that they are compact and uncountable. In particular we do not know whether they have isolated points, which is equivalent to the following question.

Question 2.3.29. Are the standard action and its conjugate by the reflection the unique C^0-rigid representations in $\mathrm{Rep}_{\mathrm{irr}}(F, \mathbb{R})$?

Understanding the topology of the transversals $\mathcal{F}^0_\pm$ seems to require a very explicit and fine classification of all the $\mathbb{R}$-focal actions of the group F up to semi-conjugacy, that we do not have at the moment. Although these all have very similar qualitative behavior, and can be in principle be explicitly classified (by Corollary 2.3.28 item (ii) in Corollary 2.3.28), in practice there is a wild plethora of such actions (essentially because there are a complicated amount of focal actions of F on trees and they often preserve many planar orders). Many of the constructions we found tolerate an arborescent set of modifications, which keep giving new non semi-conjugate actions, see [8]. At the moment we are not even able to guess a reasonably clean recipe that would cover all the examples we are aware of.

Remark 2.3.30. By Theorem 2.3.27 the harmonic space of the group F satisfies the conclusion of Question 2.3.15. Thus it might seem that an affirmative answer would not have any implications on the amenability problem for F. However we can show that if F is amenable, then there is another amenable group, constructed from F via elementary amenable operations, for which the first item (at least) in Question 2.3.15 fails.

Locally moving groups with very few actions

Within the class $\mathcal{C}$ of groups covered by Theorem 2.3.25, there are examples satisfying very different behaviours. This is already visible among finitely generated groups of piecewise linear homeomorphisms. Even though all such groups admit exotic actions (by Proposition 2.3.18), the amount precise nature of such actions depend subtly on the group. To illustrate this, given an open interval $X \subset \mathbb{R}$ and $\lambda \geq 1$ let us denote by $G(\lambda, X) \leq \mathrm{Homeo}_+(\mathbb{R})$ the group of all PL homeomorphisms of X with finitely many discontinuity points for the derivative, all in $\mathbb{Z}[\frac{1}{\lambda}]$, that are piecewise of the form $x \mapsto \lambda^n x + b$ for some $b \in \mathbb{Z}[\frac{1}{\lambda}]$. This group belongs to the family of Bieri-Strebel groups, studied in [BS16a]. The group $G(2, (0, 1))$ is Thompson's group F. We saw above that it admits a large amount of minimal $\mathbb{R}$-focal actions. Consider instead groups of the form $G(\lambda, \mathbb{R})$. For λ algebraic, these are finitely generated and belong to the class $\mathcal{C}$ (using the results of [BS16b]). Using Theorem 2.3.25 as main tool we show the following.

Theorem 2.3.31 (PL groups with 3 faithful minimal actions)**.** *Fix $\lambda \geq 1$ algebraic, and set $G = G(\lambda; \mathbb{R})$. Then every faithful minimal action $\rho \in \mathrm{Rep}_{irr}(G, \mathbb{R})$ is conjugate, up to the orientation, either to the standard action, or to one of the two exotic actions obtained through the construction in Example 2.3.19.*

Modifying this example (and leaving the PL-world, so as to avoid Proposition 2.3.18) we are able to show the following.

Theorem 2.3.32 (Locally moving groups with no exotic actions)**.** *There exist finitely generated locally moving group $G \leq \mathrm{Homeo}_+(\mathbb{R})$ (in the class $\mathcal{C}$), such that every faithful minimal action $\rho\colon G \to \mathrm{Homeo}_+(\mathbb{R})$ is topologically conjugate to the standard one (up to the orientation).*

2.4 Groups acting on flows and a realisation result for spaces of actions

This section describes [7, 10]. The construction of Deroin [Der13] (or the construction of the harmonic space from [DKNP13]) shows that every finitely generated group acting on the line also acts on a compact space with a flow, by preserving each orbit of the flow. Here, we shall reverse this point of view and consider family of groups defined by an action on a class of flows, namely suspension flows of subshifts. In particular we use this idea to prove a realisation result for harmonic spaces of a groups acting on the line, showing that every flow in a suitable class arises as the harmonic space $(\mathcal{D}, \Psi)$ of some finitely generated group, up to flow equivalence. This produces examples of finitely generated groups whose space of actions $\mathrm{Rep}_{\mathrm{irr}}(G, \mathbb{R})$ has various prescribed properties.

2.4.1 The group construction and its first properties

The main input for our construction is a pair (X, φ) consisting of a totally disconnected, compact metrisable space and a homeomorphism φ of X. The *suspension* (or *mapping torus*) of (X, φ) is the space $Y^\varphi = X \times \mathbb{R}/(x,t) \simeq (\varphi(x), t-1)$, this space is locally homeomorphic to a totally disconnected set $\times$ interval, and is naturally endowed with a system of local coordinates. Indeed, for every clopen subset $C \subset X$ and every interval I of length $|I| \leq 1$, the restriction of the projection maps homeomorphically $C \times I$ to a subset $Y_{C,I}$, that we shall call a *chart*. Note that changes of charts are locally of the form $\varphi^n \times (t \mapsto t-n)$. The space Y^φ is naturally endowed with a flow $\Phi \colon \mathbb{R} \times Y^\varphi \to Y^\varphi$, the suspension flow of φ.

The definition below also depends on the choice of a pseudogroup $\mathcal{G}$ of homemorphisms between intervals. Here this simply means that $\mathcal{G}$ is a collection of homeomorphisms $f \colon I \to J$ between (open) intervals, which is closed under composition (whenever defined), inversion, and restriction to an open sub-interval. We further assume that $\mathcal{G}$ contains all integer translations $x \mapsto x+n$ (in restriction to any open interval).

Definition 2.4.1. Let (X, φ) and $\mathcal{G}$ be as above. We denote by $\mathsf{T}_{\mathcal{G}}(\varphi)$ the group of all homeomorphisms g of Y^φ such that every $y \in Y^\varphi$ is contained in some chart where g coincides (in chart coordinates) with a transformation of the form $\varphi^n \times f$, for some $n \in \mathbb{Z}$ and some $f \in \mathcal{G}$.

The fact that $\mathsf{T}_{\mathcal{G}}(\varphi)$ is a group follows easily from the form of the changes of charts, together with the assumption that $\mathcal{G}$ contains all integer translations. Since the group $\mathsf{T}_{\mathcal{G}}(\varphi)$ preserves orbits of the flow Φ, every non-periodic orbit provides an action of it on the real line, leading to the following.

Proposition 2.4.2. *If the set of points with infinite φ-orbit is dense in X, then the group $\mathsf{T}_{\mathcal{G}}(\varphi)$ is left-orderable.*

In [12], we consider only the case where $\mathcal{G}$ is the pseudogroup of all PL dyadic homeomorphisms, namely all maps $f \colon I \to J$ between open intervals which are piecewise linear, with a discrete set of discontinuity point for the derivative all being in $\mathbb{Z}[\frac{1}{2}] \cap I$, and such that the restriction of f on every piece is of the form $x \mapsto 2^n x + b$ for some $n \in \mathbb{Z}$ and $b \in \mathbb{Z}[\frac{1}{2}]$. For this choice of $\mathcal{G}$, the group $\mathsf{T}_{\mathcal{G}}(\varphi)$ is denoted simply $\mathsf{T}(\varphi)$. The following results are proven in [12].

Theorem 2.4.3. *If the system (X, φ) is minimal, then the group $\mathsf{T}(\varphi)$ is simple.*

Recall that subshift is a system (X, φ) given by a closed shift-invariant subset $X \subset A^{\mathbb{Z}}$, where A is a finite alphabet, and the restriction φ of the shift. When X is totally disconnected, a system (X, φ) is conjugate to a subshift if and only if it is expansive.

Theorem 2.4.4. *If (X, φ) is conjugate to a subshift, then $\mathsf{T}(\varphi)$ is finitely generated.*

These statements are largely inspired by the analogous results of H. Matui [Mat06] for the commutator subgroup of the topological full group of φ (they are in fact even a bit simpler to prove). Our initial motivation in [7] was to provide a conceptualisation of the construction of J. Hyde and Y. Lodha of the first examples of finitely generated simple groups of homeomorphisms of the real line [HL19]. Their original construction was based on defining carefully some subgroups of $\mathrm{Homeo}_+(\mathbb{R})$ given by some explicit set of generators, which act by PL dyadic homeomorphisms with infinitely many pieces in a quasi-periodic manner. These groups are easily seen to embed into $\mathsf{T}(\varphi)$ for some special choice of (X, φ). The main point is that the more intrinsic definition of $\mathsf{T}(\varphi)$, and its action on the *compact* space Y^φ, allows to prove simplicity by a very short and standard argument for micro-supported groups of homeomorphisms. By analysing the action on the real line directly, the argument is necessarily more tedious and involved (a finitely generated simple group can't have a micro-supported action on a non-compact space).

We also established the following [7].

Theorem 2.4.5. *If (X, φ) is minimal, then every action of the group $\mathsf{T}(\varphi)$ on the circle by homeomorphisms has a fixed point.*

This provided the first example of finitely generated subgroups of $\mathrm{Homeo}_+(\mathbb{R})$ with this property. The existence of such examples helps clarifying the difference between groups acting on the line and on the circle and the limitations of some methods to study them, answering a question of B. Deroin, A. Navas and C. Rivas [DNR16, §3.5] (discussed more in details in [Nav18, Question 4]).

Several additional properties of the groups $\mathsf{T}(\varphi)$ are proven in [7], that I shall not discuss here.

Finally we mention that even though the only case considered in [7] is for $\mathcal{G}$ consisting of PL-dyadic maps, the proofs the previous results are robust and only depend on some formal properties of groups of PL-dyadic maps (fragmentation property, cutting and pasting conditions, etc.). Although we did not try to write an optimal statement in terms of an abstract $\mathcal{G}$, these proofs can be adapted to other choices of $\mathcal{G}$. A different choice will be important in the next subsection.

2.4.2 A realisation result for harmonic spaces

In the previous sections we have seen a few examples of groups whose harmonic space $(\mathcal{D}, \Psi)$ can be understood. In all these examples it has very restricted dynamical properties, in the sense that the flow Ψ is not cahotic at all (all orbits converge to a limit, there are nice transversals to the orbits, etc.). In view of Theorem 2.2.3, this corresponds to the fact that the semi-conjugacy equivalence relation on $\mathrm{Rep}_{\mathrm{irr}}(G, \mathbb{R})$ is particularly well-behaved. The goal of [10] is to use the groups construction discussed in the previous section to show that the suspension flow of any subshift, satisfying some mild assumptions, arises the harmonic space $(\mathcal{D}, \Psi)$ of some explicit finitely generated group (up to flow equivalence). Before explaining the details of this construction let me state this realisation result in an intrinsic (but non-explicit) form. We say that a flow (Y, Ψ) is *freely reversible* if there exists a continuous involution $\hat{\sigma} \colon Y \to Y$ which establishes a flow equivalence between (Y, Φ)

and its time-reverse (Y, Φ^{-1}). The harmonic space $(\mathcal{D}, \Psi)$ of a finitely generated group always has this property: the involution is induced by conjugation of representations in $\mathrm{Rep}_{\mathrm{irr}}(G, \mathbb{R})$ by the reflection $x \mapsto -x$, and it commutes with the group action.

Theorem 2.4.6 (Realisation result for harmonic spaces [10])**.** *Let (Y, Φ) be an expansive flow without fixed points on a compact space of covering dimension 1. Suppose that (Y, Φ) is freely reversible and topologically free. Then there exists a finitely generated group G whose harmonic space is flow-equivalent to (Y, Φ).*

This provides many explicit examples of harmonic spaces which have truly chaotic flows.

Let me now discuss the group construction used to prove Theorem 2.4.6. The groups $\mathsf{T}_{\mathcal{G}}(\varphi)$ are natural candidates to prove such a result, indeed it is natural to wonder whether their harmonic space identifies with (Y^{φ}, Φ), the suspension flow of (X, φ); this essentially amounts to show that all actions of $\mathsf{T}(\varphi)$ on the line arise from one Φ-orbit (a question we asked already in [7], with this application in mind). There is however an obvious obstruction for that statement to be literally true, namely the reversibility of the harmonic space. In general the harmonic space of $\mathsf{T}_{\mathcal{G}}(\varphi)$ must contain at disjoint a copy (Y^{φ}, Φ^{-1}) of the time reverse of (Y^{φ}, Φ), which encodes actions with the opposite orientations. So, at best, the family $\mathsf{T}_{\mathcal{G}}(\varphi)$ would allow to realise the disjoint union of (Y^{φ}, Φ) and (Y^{φ}, Φ^{-1}) as the harmonic space of some group. This would also be interesting but is too restrictive for some of the applications we are looking for (for instance a flow of this type is never minimal).

To take this into account, instead of starting with a subshift (X, φ), we take as input a triple (X, φ, σ), where (X, φ) is a subshift and σ is a fixed point-free involution of X that conjugates φ to φ^{-1}. The map σ naturally induces a map $\hat{\sigma}\colon Y^{\varphi} \to Y^{\varphi}$ on the suspension space, which conjugates the flow Φ to its inverse; this map has no fixed points provided σ does not preserve any φ-orbit. We shall call such a triple (X, φ, σ) a *freely reversible subshift*.

Definition 2.4.7. Assume that (X, φ, σ) is a freely reversible susbhift. We denote by $\mathsf{T}_{\mathcal{G}}(\varphi, \sigma)$ the centraliser of $\hat{\sigma}$ in $\mathsf{T}_{\mathcal{G}}(\varphi)$.

For instance, choosing $\mathcal{G}$ to be the pseudogroup of PL-dyadic transformations, we obtain a family of groups denoted $\mathsf{T}(\varphi, \sigma)$. This family was defined in this terms in [6] (for an application to high transitivity), and the groups originally defined by Hyde and Lodha [HL19] turn out to be particular instances of this family (by an alternative description that they gave later with Navas and Rivas [HLNR21].) However, in [10] we also choose a different pseudogroup $\mathcal{G}$. This is needed for technical reasons in our proof, coming from the fact that subgroups of $\mathsf{PL}([0,1])$ always admit non-trivial homomorphisms to $(\mathbb{R}, +)$, which create difficulties in our attempts to identify the harmonic space with this choice. To get additional rigidity, the idea is to consider a larger pseudogroup $\mathfrak{D}$ of maps which are still piecewise linear but have a non-discrete set of breakpoints, with some control on their accumulation points. The precise definition is the following.

Definition 2.4.8. Let $\mathfrak{D}$ be the pseudogroup consisting of all maps $f\colon I \to J$ such that there exists a finite set $\Sigma \subset I$ such that the following are satisfied

1. $\Sigma \subset I \cap \mathbb{Z}[\frac{1}{2}]$ and $f(\Sigma) \subset J \cap \mathbb{Z}[\frac{1}{2}]$.
2. Every $x \in I \setminus \Sigma$ has a neighbourhood U such that $f|_U$ is PL-dyadic.
3. For every $x \in \Sigma$ we have $f(x + 2t) - f(x) = 2(f(x + t) - f(x))$ for every sufficiently small t (that is, f locally commutes with the doubling map).

The set of all self-homeomorphisms of $[0,1]$ in $\mathfrak{D}$ is a group, denoted $\mathscr{F}$. This group is a relative of Thompson's group F; it is still finitely generated, and the important difference is that it is perfect. With this choice we manage to prove the following.

Theorem 2.4.9 (Explicit version of Theorem 2.4.6)**.** *Let (X,φ,σ) be a freely reversible subshift, and assume that φ does not have isolated periodic points, and that σ does not preserve any φ-orbit. Let (Y,Φ) be the suspension flow of (X,φ).*

Then the group $\mathsf{T}_{\mathfrak{D}}(\varphi,\sigma)$ is finitely generated, and its harmonic space identifies with (Y,Φ) (up to equivariant flow equivalence).

This result implies Theorem 2.4.6, using a well-known characterisation of suspension flows of subshifts due to Bowen an Walters [BW72].

By choosing (X,φ,σ) appropriately, Theorem 2.4.9 produces examples of groups satisfying various new phenomena at the level of the space of representations $\mathrm{Rep}_{\mathrm{irr}}(G,\mathbb{R})$. In fact, for the groups $G=\mathsf{T}(\varphi,\sigma)$ we also prove that the retraction $r\colon \mathrm{Rep}_{\mathrm{irr}}(G,\mathbb{R})\to Y$ from Theorem 2.2.3 is "essentially" an open map (at least, one can pretend so in the discussion below [10, Proposition 6.1]). This has applications related to the study of rigidity and flexibility of representations, as it implies that the dynamics of the flow (Y,Φ) carries complete information on the accumulation points of semi-conjugacy classes in $\mathrm{Rep}_{\mathrm{irr}}(G,\mathbb{R})$. In particular, for the groups $\mathsf{T}(\varphi,\sigma)$ we have a complete control on which representations are (locally) C^0-rigid: both properties are equivalent in this case, and such representations correspond to the orbits of (X,φ) consisting of isolated points [10, Proposition 6.3]. In the opposite direction, choosing the susbshift (X,φ) to be minimal, we obtain following.

Corollary 2.4.10. *Let $G=\mathsf{T}(\varphi,\sigma)$, where (X,φ,σ) is as in Theorem 2.4.9. If (X,φ) is minimal, then any semi-conjugacy class is dense in $\mathrm{Rep}_{irr}(G;\mathbb{R})$.*

The density of all semi-conjugacy classes in $\mathrm{Rep}_{\mathrm{irr}}(G,\mathbb{R})$ can be seen as an extreme flexibility property, namely it means that every representation $\rho\in\mathrm{Rep}_{\mathrm{irr}}(G;\mathbb{R})$ admits arbitrarily small perturbations which are semi-conjugate to any other representation in $\mathrm{Rep}_{\mathrm{irr}}(G;\mathbb{R})$. There were no known examples of groups with this property. Their existence can be compared with a question of Navas, asking whether there exists a finitely generated left-orderable group acting minimally on its space of left orders, see [Nav18, Question 6]. The conclusion in Corollary 2.4.10 is an analogous property at the level of actions on the line; however there is no formal connection: the groups constructed here do not act minimally on their space of left-orders, and conversely this property does not seem to imply *a priori* the conclusion of Corollary 2.4.10 (because not all representations in $\mathrm{Rep}_{\mathrm{irr}}(G;\mathbb{R})$ on the line are dynamical realisations of orders).

As another example of application, we obtain examples of finitely generated groups that admit many non semi-conjugate actions on $\mathbb{R}$, yet any two *generic* actions in $\mathrm{Rep}_{\mathrm{irr}}(G,\mathbb{R})$ are semi-conjugate up to the orientation, in the following sense.

Corollary 2.4.11 (Groups with a generic representation)**.** *There exist finitely generated groups G such that $\mathrm{Rep}_{irr}(G;\mathbb{R})$ contains infinitely (countably or uncountably) many distinct semi-conjugacy classes, and there is $\rho\in\mathrm{Rep}_{irr}(G;\mathbb{R})$ such that the semi-conjugacy classes of ρ and of its conjugate by the reflection form a dense open set.*

Remark 2.4.12. Since we required semi-conjugacy to be orientation-preserving, it is not possible for a group G to have a representation $\rho\in\mathrm{Rep}_{\mathrm{irr}}(G;\mathbb{R})$ with an open dense (or even co-meager) semi-conjugacy class, as its conjugate by the reflection should have the same property.

Another application concerns the structure of the (path-)connected components of the space $\mathrm{Rep}_{\mathrm{irr}}(G,\mathbb{R})$. Recall from §2.2.2 that, following [MW17], we say that an action $\rho \in \mathrm{Rep}_{\mathrm{irr}}(G;\mathbb{R})$ is *path-rigid* if its path-component in $\mathrm{Rep}_{\mathrm{irr}}(G,\mathbb{R})$ coincides with its semi-conjugacy class. Theorem 2.4.9 is a source of groups with a rather extreme behaviour with regard to this notion.

Corollary 2.4.13 (Path-rigidity of all representations)**.** *Let $G = \mathsf{T}(\varphi,\sigma)$, where (X,φ,σ) is as in Theorem 2.4.9. Then the path-components of* $\mathrm{Rep}_{irr}(G;\mathbb{R})$ *coincide with the semi-conjugacy classes.*

Choosing (X,φ) to be minimal and combining with Corollary 2.4.10, there are finitely generated groups G such that all representations in $\mathrm{Rep}_{\mathrm{irr}}(G;\mathbb{R})$ are path-rigid, yet they all admit arbitrarily small perturbations semi-conjugate to any other representation; moreover the space $\mathrm{Rep}_{\mathrm{irr}}(G;\mathbb{R})$ is connected but not path-connected, and nowhere locally connected. In this example there is a dramatic difference between between path-continuous deformations of representations and more general small perturbations (and similarly between path-connected and connected components). There is no reason to expect otherwise a-priori, but when studying other spaces of actions on one manifolds, the difference between the two can be subtle and difficult to understand (see for instance Mann and Wolff [MW17], or, in higher regularity, Bonatti and Eynard [BEB16], Eynard and Navas [EBN21]).

It is an interesting question to find other constructions of groups acting on spaces with flows, and use them to prove a more general realisation result as Theorem 2.4.6 to flows on spaces of higher topological dimension.

Question 2.4.14. Which compact spaces endowed with a flow (Y,Φ) can be realised as the harmonic space of a finitely generated group (up to flow equivalence)?

Some cases of this question are part of the project of my current Ph.D. student M. Gilabert.

Chapter 3

Liouville property via conformal dimension

3.1 Introduction

This part presents the recent paper [11] in collaboration with V. Nekrashevych and T. Zheng. The general context of this work is the study of amenability of groups and the Liouville property for random walk (which was also the main topic of my Ph.D thesis).

Recall that a group G is amenable if it admits an invariant mean (that is a finitely additive probability measure, defined on all subsets of G). The choice of a probability measure μ on a group G determines a random walk. On says that the random walk has the *Liouville property* if its Poisson boundary is trivial, that is, every bounded μ-harmonic function on G is constant on the subgroup generated by the support of μ. When μ is generating, the Liouville property implies amenability of G in a very constructive way, namely the Cesaro averages of the convolutions μ^{*n} accumulate on an invariant mean. Conversely on every amenable group there exists *some* generating, symmetric measure generating a Liouville random walk, by a result of Kaimanovich–Vershik and Rosenblatt [KV83, Ros81]; however such a measure cannot in general be chosen to be finitely supported, or even to have finite entropy (Kaimanovich and Vershik [KV83], Erschler [Ers04]).

Many groups either belong to the class of *elementary amenable* groups (obtained from finite and abelian groups through group theoretic operations which preserve amenability) or are non-amenable for the much stronger reason that they admit non-abelian free subgroups. For a group which does not belong to any of these two classes, the amenability question might be challenging and remains an open problem in many cases. Many examples for which the question is non-trivial (and solved or unsolved) arise as groups of dynamical origin.

A prototipical example of this situation is the class of *contracting self-similar groups*. These are groups G defined through an action on some rooted tree defined in by recursive rules. A famous example of such a group is the Grigorchuk group, which is the first example of non-elementary amenable group [Gri84] (for the much stronger reason that it has intermediate growth). From the early 2000s, the amenability of various classes of self-similar groups has been shown, in some cases by establishing the Liouville property for some suitably constructed, finitely supported random, starting with the work of Bartholdi and Virág [BV05] (an account of these results can be found in § 3.4). In fact, for a long time, this class (or close relatives) has been the only source of examples of non-elementary amenable groups.

The most studied class of self-similar groups are *contracting* self-similar groups. These arise in connection with dynamics, as *iterated monodromy groups* of spaces with an expanding (branched) self-covering $p\colon \mathcal{X} \to \mathcal{X}$. The iterated monodromy group is, roughly speaking, the faithful image of the fundamental group under its monodromy action associated to all iterates of p (see §3.2.2). An important class of examples of expanding self coverings are post-critically finite rational functions $f\colon \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$, over the Riemann sphere (which are expanding in restriction to their Julia set). Conversely to every contracting self-similar group G, one can construct a space with an expanding self-covering $p\colon \mathcal{J}_G \to \mathcal{J}_G$, called its *limit dynamical system*, and these constructions are dual to each other (in the case of the iterated monodromy group of a post-critically finite rational function f, the space $\mathcal{J}_G$ is homeomorphic to the Julia set of f). The study of iterated monodromy groups has been initiated in the early 2000s by V. Nekrashevych (motivated by a constructions of R. Pink), and has been developed since them by him and several other authors, see his books [Nek05, Nek22] for an account on the theory.

It remains a widely open question whether all contracting self-similar groups (equivalently, iterated monodromy groups of expanding self-coverings) are amenable. In [11] we provide new progress on this question. Our main result, Theorem 3.3.4 below, establishes the Liouville property (and hence amenability) for a class of contracting groups. That result applies to all classes of contracting groups whose amenability had previously been established, and to many new ones. An application is the following.

Theorem 3.1.1. *Let $f \in \mathbb{C}(z)$ be any post-critically finite rational function whose Julia set is not the whole sphere, and let G be its iterated monodromy group. Then for every probability measure μ on G with finite second moment, the μ-random walk has the Liouville property. In particular, G is amenable.*

The novelty in our approach is the use of *conformal dimension*, a fundamental invariant of metric spaces introduced by P. Pansu in the late 80s (see, for example [Pan89]), which is now widely studied in geometric group theory and dynamical systems. See [MT10] for a survey of its properties and applications. The conformal (and related Ahlfors-regular conformal) dimension are invariants of *quasi-symmetries* between metric spaces. Since quasi-isometries of hyperbolic spaces correspond to quasi-symmetries of their boundaries, the conformal dimension is a natural invariant of hyperbolic groups and their boundaries, and was originally exploited in this setting. More generally, considering metrics up to quasi-symmetry is specially useful and natural for self-similar metric spaces, and another natural class of examples are precisely metric spaces together with an expanding (branched) covering map $p : \mathcal{J} \to \mathcal{J}$. Their quasi-conformal geometry is the subject of the works [HP09, HP11, HP12] by P. Haïssinsky and K. Pilgrim, who also suggested the potential importance of the conformal dimension to the study of contracting self-similar groups. Theorem 3.3.4 roughly speaking says that if $p : \mathcal{J} \to \mathcal{J}$ is an expanding self-covering and if the Ahlfors-regular conformal dimension of $\mathcal{J}$ is less than 2, then the iterated monodromy group is Liouville. This is essentially the first result that connects the two aspects of the theory of self-similar groups alluded above, by relating directly the properties of an expanding self-covering (for instance, the Hausdorff dimension of the Julia set of a rational function) to the amenability of its iterated monodromy group.

3.2 Definitions

3.2.1 Contracting self-similar groups

For $d \geq 1$, consider the finite alphabet $\mathsf{X} = \{1, \ldots, d\}$. The set X^* of finite words in X^* is naturally a rooted tree, where each word w is connected by an edge to wx for every $x \in \mathsf{X}$. We denote by $\mathrm{Aut}(\mathsf{X}^*)$ the group of automorphisms of this rooted tree. Every such automorphism fixes the root (corresponding to the empty word $\varnothing$). The following definition provides a way to define group actions on X^* by recursive rules.

Definition 3.2.1. Given a group G, a *wreath recursion* is a homomorphism

$$\mathsf{r}\colon G \to \mathrm{Sym}(\mathsf{X}) \ltimes G^{\mathsf{X}}, \quad g \mapsto \sigma_g(g|_1, \cdots, g|_d).$$

Two wreath recursions $\mathsf{r}_1, \mathsf{r}_2\colon G \to \mathrm{Sym}(\mathsf{X}) \ltimes G^{\mathsf{X}}$ are *equivalent* if r_1 is equal to r_2 postcomposed by an inner automorphism of the wreath product $\mathrm{Sym}(\mathsf{X}) \ltimes G^{\mathsf{X}}$.

A wreath recursion gives rise to a natural action $G \to \mathrm{Aut}(\mathsf{X}^*)$, defined recursively by

$$g(xw) = \sigma_g(x) g|_x(w), \quad x \in \mathsf{X}, w \in \mathsf{X}^*.$$

Equivalent wreath recursions yield conjugate actions in $\mathrm{Aut}(\mathsf{X}^*)$. More explicitly the action of G on each level X^n can be determined by iterating the wreath recursion, by applying it again in all coordinates of G^{X}. This leads to a sequence of homomorphisms $G \to \mathrm{Sym}(\mathsf{X}^n) \ltimes G^{\mathsf{X}^n}, g \mapsto (\sigma_g^{(n)}, (g|_v)_{v \in \mathsf{X}^n})$, and the action of G satisfies

$$g(vw) = \sigma_g^{(n)}(v) g|_v(w), \quad \forall v \in \mathsf{X}^n, w \in \mathsf{X}^*.$$

The element $g|_v$ is called the *section* of g at v.

Definition 3.2.2. A (faithful) *self similar group* is a group G endowed with a wreath recursion $\mathsf{r}\colon G \to \mathrm{Sym}(\mathsf{X}) \ltimes G^{\mathsf{X}}$ such that the associated action of G on X^* is faithful.

Two self similar groups with wreath recursions $\mathsf{r}_i\colon G_i \to \mathrm{Sym}(\mathsf{X}_i) \ltimes G_i^{\mathsf{X}_i}$ are *equivalent* if there exists an isomorphism $\Phi\colon G_1 \to G_2$ and a bijection $\psi\colon \mathsf{X}_1 \to \mathsf{X}_2$ such that the wreath recursions become equivalent after identifying G_1 to G_2 via Φ and X_1 and X_2 via ψ.

In practice, to define a self-similar group, it is enough to define the wreath recursion at the level of a free group $\mathbb{F}_n$, by specifying the image of the generators, and then let G be the quotient of the induced action of $\mathbb{F}_n$ on X^*. For example, the Grigorchuk group G is generated by the elements a, b, c, d with wreath recursion over the alphabet $\mathsf{X} = \{0, 1\}$

$$a = (01)(e, e), \quad b = (01)(c, e), \quad d = (01)(d, e), \quad d = e(b, e).$$

Definition 3.2.3. A self-similar group (G, r) is *contracting* if there exists a finite set $\mathcal{N} \subset G$ such that for every $g \in G$, there exists $n \geq 1$ such that the section $g|_v \in \mathcal{N}$ for all words $v \in \mathsf{X}^n$. The smallest set $\mathcal{N}$ is called the *nucleus* of G.

The condition that G is contracting does not depend on the choice of the wreath recursion up to equivalence. (Its nucleus, however, might depend on it.)

To clarify the terminology, when $G \leq \mathrm{Aut}(\mathsf{X}^*)$ is finitely generated and self-similar, and $\ell(\cdot)$ is a word metric on G, the contracting condition is equivalent to the existence of constants $C > 0, \lambda \in (0, 1)$ and n_0 such that

$$\ell(g|_v) \leq \lambda \ell(g) + C \quad \forall g \in G, v \in X^{n_0}. \tag{3.1}$$

3.2.2 Iterated monodromy groups and limit spaces

The presentation here is close to [Nek11]. Let $\mathcal{M}$ be a path connected and locally path connected topological space. A *partial self covering* of $\mathcal{M}$ is a finite degree covering map

$$p\colon \mathcal{M}_1 \to \mathcal{M},$$

where $\mathcal{M}_1 \subset \mathcal{M}$ is a subset. Iterating a partial self-covering yields a sequence of partial self coverings $p^n\colon \mathcal{M}_n \to \mathcal{M}$, where $\mathcal{M}_n \subset \mathcal{M}_1$ is the set of points where p^n is defined.

Example 3.2.4. A complex rational function $f\colon \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is *post-critically finite* if the forward orbit of every critical point of f (namely a point z where f is not a local biholomorphism) is finite. The set $P_f = \{f^n(z)\colon n \geq 1, z \text{ critical point}\}$ is then called the post-critical set of f. Any post-critically finite rational function defines a partial self-covering $f\colon \mathcal{M}_1 \to \mathcal{M}$ is a partial self-covering, with $\mathcal{M} = \mathbb{P}^1(\mathbb{C}) \setminus P_f$ and $\mathcal{M}_\infty = \mathbb{P}^1(\mathbb{C}) \setminus f^{-1}(P_f)$.

Fix a base point $t \in \mathcal{M}$. The disjoint union $T = \sqcup_{n\geq 0} p^{-n}(t)$ is naturally a rooted tree with root t, where each point $x \in T$ is connected by an edge to $p(x)$. It is called the *tree of preimages* of p. The natural monodromy action of the fundamental group $\pi_1(\mathcal{M}, t)$ on each fiber $p^{-n}(t)$ defines an action by automorphisms on the tree T,

Definition 3.2.5. The quotient of $\pi_1(\mathcal{M}, t)$ by the kernel of its action on the tree of preimages is called the *iterated monodromy group* of the partial self-covering, and is denoted $\mathrm{IMG}(p)$.

Iterated monodromy groups are naturally self-similar groups. In fact it is possible to write en explicit wreath recursion which generates the action of $\pi_1(\mathcal{M}, t)$ on the tree of preimages. To this end, fix an alphabet $\mathsf{X} = \{1, \dots, d\}$ of cardinality equal to the degree d of the covering. Fix a bijection $\iota\colon \mathsf{X} \to p^{-1}(t)$, and for each $x \in \mathsf{X}$, choose a path ℓ_x in $\mathcal{M}$ connecting t to $\iota(x)$. Since $\pi_1(\mathcal{M}, t)$ acts on $p^{-1}(t)$, for every $\gamma \in \pi_1(\mathcal{M}, t)$ there exists a permutation $\sigma_\gamma \in \mathrm{Sym}(\mathsf{X})$ such that $\gamma \cdot \iota(x) = \iota(\sigma_\gamma(x))$. For a path $\gamma \in \pi_1(\mathcal{M}, t)$, let γ_x be a lift of γ starting at $\iota(x)$ (and ending at $\iota(\sigma_\gamma(x))$). Then the following formula defines a wreath recursion on $\pi_1(\mathcal{M}, t)$:

$$\pi_1(\mathcal{M}, t) \to \mathrm{Sym}(\mathsf{X}) \ltimes \pi_1(\mathcal{M}, t)^{\mathsf{X}}, \quad \gamma \mapsto \sigma_\gamma(\ell^{-1}_{\sigma_\gamma(1}\gamma_1\ell_1, \dots, \ell^{-1}_{\sigma_\gamma(d)}\gamma_d\ell_d).$$

It is possible to identify the tree $\mathsf{X}^*$with T equivariantly, so that the image of the associated action coincides with the iterated monodromy group. A change in any of the choices made (the basepoint t, the identification of $p^{-1}(t)$ with a finite alphabet X, and the paths ℓ_x) yields an equivalent self-similar group.

Nekrashevych gave in [Nek05] a definition of an expanding partial self-covering $p\colon \mathcal{M}_1 \to \mathcal{M}$, and showed that the iterated monodromy group p of an expanding self-covering is contracting. Roughly speaking p is expanding if it locally expands a *length structure* on $\mathcal{M}_1$ (namely a distance with respect to which it is possible to measure the length of curves; for instance a Riemannian metric in the case of a manifold). The precise definition is a bit technical and I shall not repeat it here, but I shall just mention that as a relevant special case, the iterated monodromy group of every post-critically finite rational function $f \in \mathbb{C}(z)$ is contracting (the fact that f is expanding follows from Pick's theorem).

A converse construction allows to associate to every contracting group a compact space together with an expanding map.

Definition 3.2.6. The *limit space* $\mathcal{J}_G$ of a contracting self-similar group (G, r) is the quotient of the set of left-infinite sequences $\mathsf{X}^{-\omega} := \{\cdots x_2 x_1, x_i \in \mathsf{X}\}$ by the equivalence relation that identifies $\cdots x_2 x_1$ and $\cdots y_2 y_1$ if there exists a sequence (g_n) taking values in a finite subset of G such that $g_n(x_n \cdots x_1) = y_n \cdots y_1$ for every n in $\mathbb{N}$.

The shift $\cdots x_2 x_1 \mapsto \cdots x_3 x_2$ passes to the quotient to a continuous map $\mathsf{s}\colon \mathcal{J}_G \to \mathcal{J}_G$, called the *limit dynamical system* of G.

Changing the wreath recursion to an equivalent one yields an homeomorphic space $\mathcal{J}_G$ and a topologically conjugate limit dynamical system.

One way to think about the space $\mathcal{J}_G$ is to interpret it in terms of Schreier graphs. Namely, if we fix a finite generating set S of G, we may consider the sequence of $\Gamma(G, \mathsf{X}^n)$ of Schreier graphs of the action of G on the levels of the tree. Then two sequences $\cdots x_2 x_1$ and $\cdots y_2 y_1$, define the same point in $\mathcal{J}_G$ if and only if the sequences $x_n \cdots x_1$ and $y_n \cdots y_1$ stay at uniformly bounded distance in $\Gamma(G, \mathsf{X}^n)$. We warn however that the graphs $\Gamma(G, \mathsf{X}^n)$ do not converge to $\mathcal{J}_G$ in a metric sense in general.

Another result of Nekrashevych [Nek05] asserts that when $G = \mathrm{IMG}(f)$ is the iterated monodromy group of a postcritically finite rational map $f \in \mathbb{C}(z)$, then $\mathcal{J}_G$ is homeomorphic to the Julia set of f, and the homeomorphism conjugates the limit dynamical system $(\mathcal{J}_G, \mathsf{s})$ to the restriction of f to it. It follows that $\mathrm{IMG}(f)$ is a complete invariant of the topological dynamics of f on its Julia set.

In general, a contracting group G can essentially be recovered as the iterated monodromy group of its limit dynamical system $\mathsf{s}\colon \mathcal{J}_G \to \mathcal{J}_G$. However this is not quite litteraly true; to make this statement true one needs to endow $\mathcal{J}_G$ with an additional structure of *orbispace* and develop the whole theory in this setting, see [Nek22]. Modulo this point, the class of contracting self-similar groups and iterated monodromy groups of expanding partial self-coverings coincide.

3.3 Conformal dimension and the main result

Let $G \to \mathrm{Sym}(\mathsf{X}) \ltimes G^{\mathsf{X}}$ be a contracting self-similar group. For $v \in \mathsf{X}^n$, we denote by $\mathcal{T}_v \subset \mathcal{J}_G$ the image of $\mathsf{X}^{-\omega} v$ (namely the set of sequences of the form $\cdots x_{n+2} x_{n+1} v$). Sets of the form $\mathcal{T}_v$ are called the (nth-levels) *tiles* of the limit space $\mathcal{J}_G$. For $\xi \in \mathcal{J}_G$ and $n \geq 1$, let us denote $\mathcal{T}^+(\xi, n)$ the union of all level n tiles $\mathcal{T}_v$ such that $\mathcal{T}_v \cap \mathcal{T}_w \neq \varnothing$ for some nth level tile $\mathcal{T}_w$ containing ξ.

Although we did not specify any metric on $\mathcal{J}_G$ yet, the set $\mathcal{T}^+(\xi, n)$ should be thought as analogues of balls. It is natural to consider the class of metrics for which they are "approximately round", in the following sense.

Definition 3.3.1. A metric d on $\mathcal{J}_G$ is quasiconformal if it induces the topology of $\mathcal{J}_G$ and there exists $C > 0$ such that for every $\xi \in \mathcal{J}_G$ and every n there exists $r > 0$ such that

$$B(\xi, r) \subset \mathcal{T}^+(\xi, n) \subset B(\xi, Cr),$$

where $B(\xi, r)$ denotes the ball with respect to d.

A distance d on a space Y is Ahlfors-regular if there exists a Borel measure μ on Y and constants $\alpha > 0$ and $C > 0$ such that $C^{-1} R^\alpha \leq \mu(B(\xi, R)) \leq C R^\alpha$ for every $\xi \in Y$ and $R > 0$. The exponent α is equal to the Hausdorff dimension of d.

Definition 3.3.2. For a contracting self-similar group G, the Ahlfors-regular conformal dimension of the limit space $\mathcal{J}_G$ is the infimum of the Hausdorff dimension of the Ahlfors-regular quasi-conformal metrics on $\mathcal{J}_G$, and is denoted $\mathrm{ARdim}(\mathcal{J}_G)$.

Remark 3.3.3. Usually the (Ahlfors-regular) conformal dimension of a metric space (Y, d) is defined as the infimum of the Hausdorff dimensions of the (Ahlfors-regular) metrics which are quasi-symmetric to d, see e.g. [MT10]. A typical situation is when Y is the boundary of a Gromov hyperbolic space, and d is a visual metric. In fact, the limit space $\mathcal{J}_G$ of a contracting group can be identified with the Gromov boundary of a Gromov-hyperbolic graph, called the *self-similarity graph* whose vertex set is X^* and where two vertices v, w are connected by an edge in one of the following situations: $w = xv$ for some $x \in \mathsf{X}$ (vertical edges) and $w = h(v)$ for some h in the nucleus (horizontal edges). One can show that any metric quasi-symmetric to a visual metric on $\mathcal{J}_G$ is quasi-conformal in the sense of Definition 3.3.1. Conversely under a mild assumption on G (that it is *self-replicating*), it turns out that the two notions coincide (in particular this is true for the iterated monodromy group of a post-critically finite rational function). In that case Definition 3.3.2 coincides with the more standard definition. We do not know whether this is true for all contracting groups, for which our definition might give a smaller number, but it is the more natural definition for our main result.

We are finally ready to state the main result from [11].

Theorem 3.3.4 ([11])**.** *Let G be a finitely generated contracting self-similar group such that* $\mathrm{ARdim}(\mathcal{J}_G) < 2$. *Then for every symmetric probability measure μ on G with a finite $\alpha-$moment for some $\alpha >$* $\mathrm{ARdim}(\mathcal{J}_G)$*, the μ-random walk is Liouville. In particular every such group is amenable.*

If G is the iterated monodromy group of a post-critically finite rational function $f \in \mathbb{C}(z)$, results of P. Haïssinsky and K. Pilgrim [HP09] imply that the natural homeomorphisms between the limit space $\mathcal{J}_G$ and the Julia set $\mathcal{J}_f$ is a quasi-symmetry, with respect to the visual metric on $\mathcal{J}_G$ and the spherical metric on $\mathcal{J}_f$. Thus $\mathrm{ARdim}(\mathcal{J}_G)$ is bounded above by the Hausdorff dimension of $\mathcal{J}_f$. If $\mathcal{J}_f \subsetneq \mathbb{P}^1(\mathbb{C})$, its Hausdorff dimension is always strictly less than 2 by a result of McMullen [McM00]. Thus Theorem 3.3.4 implies Theorem 3.1.1.

Let me give a rough idea of how the proof of Theorem 3.3.4 proceeds (at least for finitely supported measures). As often in the realm of groups of dynamical origin, to understand the random walk on G, one proceeds in two steps by first obtaining a very good control for the random walk induced on the Schreier graphs of some natural non-free action, and then trying to find some argument to lift the conclusion to G. Here the Schreier graphs are the finite Schreier graphs $\Gamma(G, \mathsf{X}^n)$ on the finite levels of the tree. These can be thought as the adjacency graphs of the sequence of partitions of the limit space $\mathcal{J}_G$ into tiles. The assumption that $\mathrm{ARdim}(\mathcal{J}_G) < 2$ delivers us a metric d on $\mathcal{J}_G$ with Hausdorff dimension $\alpha < 2$. The fact that Lipschitz functions on $\mathcal{J}_G$ separate points can be used to produce functions on these graphs. Using this way to produce functions, one can estimate the ℓ^2-capacity (or effective conductance) between large subsets of $\Gamma(G, \mathsf{X}^n)$ that approximate disjoint regions on the limit space $\mathcal{J}_G$, and show that it must tend to 0 as $n \to \infty$. This is reminiscent of connections between conformal dimension and potential theory due to Carrasco [CP13] and Kigami [Kig20]. This implies that the random walk on $\Gamma(G, \mathsf{X}^n)$ must take a long time to travel back and forth between regions that approximate disjoint parts of $\mathcal{J}_G$. This is the first step. The second step consists in showing that if the random walk in G goes far, then the induced random walk on $\Gamma(G, \mathsf{X}^n)$ must constantly go back and forth between disjoint regions of $\mathcal{J}_G$. This uses crucially the theory of *contracting models* developed by Nekrashevych [Nek22], which allows to see $\mathcal{J}_G$ as an inverse limit of finite graphs in a metric sense (the graphs $\Gamma(G, \mathsf{X}^n)$ are not suited to this purpose). The concluding argument uses the entropy criterion from [KV83]

3.3.1 ℓ^p-contraction and moment conditions

The possibility of proving Theorem 3.3.4 for measures with a moment condition is based on the study of another invariant of contracting groups, namely *critical exponent*. Assume that G is a finitely generated contracting group, with a fixed word-metric $\ell(\cdot)$. Recall that the condition that G is contracting can be characterised by the contraction of the length of sections of elements as in (3.1). As a natural generalisation of this condition, for $p \in (0, \infty]$, we say that a contracting group satisfies ℓ^p-*contraction* if there exists some $\eta \in (0,1)$ and $n > 0$ and $C > 0$ such that every $g \in G$ satisfies

$$\ell(g) \le \eta(\sum_{v \in \mathsf{X}^n} \ell(g|_v)^p)^{1/p} + C.$$

Then ℓ^∞-contraction is equivalent to the standard contraction (3.1). Another relevant special case is ℓ^1-contraction (usually called *sum contraction*), which was observed by Grigorchuk to imply subexponential growth of G (he used this to show that his groups have intermediate growth [Gri84])

One can show that for every contracting group, the set of p such that G satisfies ℓ^p contraction is an interval $(p_c(G), \infty]$, for some $p_c(G) < \infty$.

Definition 3.3.5 ([Nek22])**.** For a contracting self-similar group G, the infimum of the values of p such that G satisfies ℓ^p-contraction is called the *critical exponent* of G, and is denoted $p_c(G)$.

Finally for a finitely generated group G, let us denote by $\mathrm{Cr}_{\mathrm{Liouv}}(G)$ the infimum of α such that the μ-random walk is Liouville for all symmetric measure μ on G with a finite α-moment. Then the main results in [11] can be summarized as the following chain of inequalities (which implies Theorem 3.3.4).

Theorem 3.3.6. *Let G be a finitely generated contracting group such that* $\mathrm{ARdim}(\mathcal{J}_G) < 2$*. Then*

$$\mathrm{Cr}_{\mathrm{Liouv}}(G) \le p_c(G) \le \mathrm{ARdim}(\mathcal{J}_G).$$

The inequality $\mathrm{Cr}_{\mathrm{Liouv}}(G) \le p_c(G)$ is sharp in all the (rather limited) cases where both quantities are known. In particular it is sharp for the Grigorchuk group, for which $\mathrm{Cr}_{\mathrm{Liouv}}(G)$ has been computed by Erschler and Zheng [EZ20] (a matching upper bound for $p_c(G)$ follows from older work of Bartholdi [Bar98]).

The inequality $p_c(G) \le \mathrm{ARdim}(\mathcal{J}_G)$ does not require the assumption that $\mathrm{ARdim}(\mathcal{J}_G) < 2$, and was known to V. Nekrashevych before our collaboration. This inequality lead him to conjecture that $\mathrm{ARdim}(\mathcal{J}_G) < 2$ should imply the Liouville property for G (at least for finitely supported symmetric measures), a conjecture that he stated during a conference at IHP in spring 2022, where the collaboration with T. Zheng and myself started. His conjecture was partly inspired by a result from my Ph.D thesis [16], that the topological full group of a subshift with strictly subquadratic complexity function is Liouville for symmetric finitely supported measures. The complexity of a subshift $X \subset A^{\mathbb{Z}}$ is the function $p_X(n)$ that measures the number of finite words of length n that appear as subwords of words in X. The number $p_c(G)$ can in fact be interpreted as the exponent of an analogue of the complexity function for the G-action on the tree. Eventually our proof of Theorem 3.3.4 turned out to be quite different. The role of $p_c(G)$ turned out to be not so central in the case of finitely supported measures, but crucial to extend the result to measures with a moment condition to control the long jumps of the random walk.

3.4 Examples and comparison with previous results

An historically important example is the iterated monodromy group of the polynomial z^2-1, also known as the *basilica group*. The group $\mathrm{IMG}(z^2-1)$ is generated by two elements a, b given by the following explicit wreath recursion on the binary alphabet $\mathsf{X} = \{0, 1\}$.

$$a = ((01), (b, e)), \quad b = (e, (a, e)). \tag{3.2}$$

In fact Grigorchuk and Żuk defined this group (in terms of its wreath recursion) even before the definition of iterated monodromy group was formulated, and asked if it is amenable [GŻ02]. Bartholdi and Virág answered this in the affirmative in [BV05]; their argument (as streamlined by Kaimanovich [Kai05]) consists in showing the Liouville property for the simple random walk associated to the uniform measure μ on the standard generating set above. Their proof relies on this specific choice of a random walk via the fact that it is *self-similar* in the sense of [Kai05], namely after iterating the wreath recursion its sections follow the same law along some sequence of stopping times. This is a rather restrictive condition which depends on the form of the wreath recursion, and most self-similar groups do not admit a self-similar random walk.

The method of Bartholdi and Virág was generalised to show amenability of more classes of self-similar groups, defined in terms of a slow growth condition of the wreath recursion $G \to \mathrm{Sym}(\mathsf{X}) \ltimes G^{\mathsf{X}}$. Consider a self-similar group G which is *finite state*, namely for every $g \in G$, the set of sections $\{g|_v, v \in \mathsf{X}^*\}$ is a finite subset of G (this is clearly true if G is contracting). For $g \in G$, the *activity function* of g is the function $\alpha_g(n)$ that counts the number of vertices $v \in \mathsf{X}^n$ such that $g|_v \neq 1$. Then $\alpha_g(n)$ can be computed by some linear recursion, so its value is either bounded by a polynomial of some integer degree $d \geq 0$, or it grows exponentially. The elementary relation $\alpha_{gh}(n) \leq \alpha_g(n) + \alpha_h(n)$ shows that the set of G of elements of polynomial activity of degree at most d is a subgroup. If all elements of G (equivalently some generating set) have this property, we say that G has *polynomial activity* of degree d, else we say that G has *exponential activity*.

The simplest class from this perspective are groups of *bounded* activity (that is, polynomial of degree $d = 0$). The basilica group belongs to this class, as well as the iterated monodromy group of every post-critically finite polynomial $f \in \mathbb{C}[z]$, by [BN08, Nek09]. Bartholdi, Kaimanovich and Nekrashevych showed that every self-similar group of bounded activity is amenable [BKN10]. Their proof proceeds by embedding all such groups in a family of "mother groups" designed to admit a self-similar random walk, and apply the method of [BV05] (we later showed with Amir, Angel and Viráf [14] that all symmetric finitely supported random walk on a group of bounded activity are Liouville). Amir, Angel and Virág applied the same strategy (overcoming considerably more technical difficulties) to show amenability of automata groups of linear activity [AAV13]. Note that groups of bounded activity are always contracting, but this is not true for polynomial activity of degree $d \geq 1$.

A different method for proving amenability is based on the notion of extensively amenable actions, introduced by Juschenko and Monod [JM13]. Using this method, a combination of results of Juschenko, Nekrashevych and de la Salle [JNS16] and Nekrashevych, Pilgrim and Thurston [NPT20] implies that all contracting self-similar groups having a wreath recursion of polynomial activity are amenable. The method of extensive amenability does not go through establishing the Liouville property for random walk, although random walk is still present in it, as one of the main requirement to run this method is recurrence of the random walk on induced on an orbit under some action of the group.

The results of [NPT20] also imply the following.

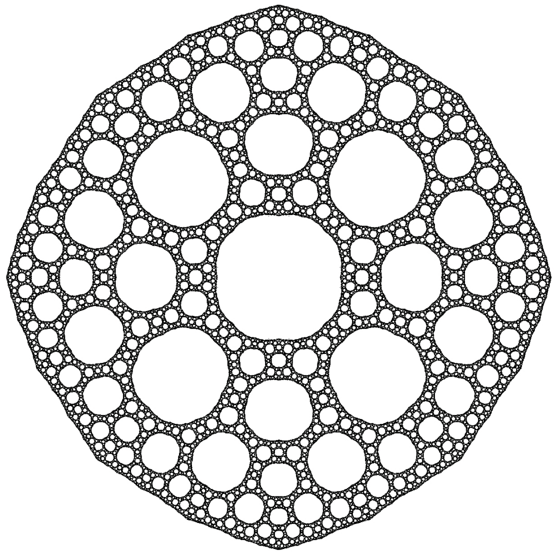

Figure 3.1: The Julia set of $z^2 - \frac{1}{16z^2}$

Theorem 3.4.1 ([NPT20])**.** *If G is a contracting self-similar group of polynomial activity, then* $\mathrm{ARdim}(\mathcal{J}_G) = 1$*.*

Thus Theorem 3.3.6 gives the following.

Corollary 3.4.2. *Let G be a finitely generated contracting group with a wreath recursion of polynomial activity. Then for every symmetric measure μ on G having a finite $(1+\varepsilon)$-moment for some $\varepsilon > 0$, the μ-random walk is Liouville.*

Thus in this case amenability was known using the method of extensive amenability; in contrast the Liouville property was known only in some more restricted cases and for finitely supported symmetric measures.

Beyond this case, the main novelty of Theorem 3.3.4 is that it can be applied to contracting groups G such that $\mathrm{ARdim}(\mathcal{J}_G) > 1$, in which case amenability could not be established before. A special case of this situation are iterated monodomy groups post-critically finite rational function $f \in \mathbb{C}(z)$ whose Julia set is homeomorphic to the Sierpiński carpet. The conformal dimension of such a Julia set can be arbitrarily close to 2, see [HP12]. In this case the wreath recursion of the group necessarily has exponential activity, and might be too complicated to work with directly. A new aspect of Theorem 3.3.4 is that, while all previous results were based on the analysis of some explicit wreath recursion, the proof of Theorem 3.3.4 avoids completely such computations and is based instead entirely on the theory of iterated monodromy groups.

I conclude with a concrete example of wreath recursion for the iterated monodromy group of a rational function with Sierpiński carpet Julia set. Consider the rational function $f(z) = z^2 - \frac{1}{16z^2}$. It's Julia set is shown in Figure 3.1 and is homeomorphic to the Sierspiński carpet.

An explicit wreath recursion for $G = \mathrm{IMG}(f)$ is computed in [11, §6.2]. Taking as alphabet $\mathsf{X} = \{1, 2, 3, 4\}$, a generating set of the group G is given by

$$\begin{aligned} a &= e(b, 1, c, 1), \\ b &= (23)(14)(e, e, e, e) \\ c &= (12)(34)(a, a^{-1}, 1, 1). \end{aligned}$$

Chapter 4

List of publications

Below is a list of my publications and prepublications organised by area.

Work presented in the memoir

Chapter 1

[1] A. Le Boudec and N. Matte Bon. Subgroup dynamics and C^*-simplicity of groups of homeomorphisms. *Ann. Sci. Éc. Norm. Supér. (4)*, 51(3):557–602, 2018.

[2] N. Matte Bon. Rigidity properties of topological full groups of pseudogroups over the Cantor set. *arXiv preprint*, 2019.

[3] A. Le Boudec and N. Matte Bon. A commutator lemma for confined subgroups and applications to groups acting on rooted trees. *Trans. Amer. Math. Soc.*, 376(10):7187–7233, 2023.

[4] (P.-E. Caprace, A. Le Boudec, and N. Matte Bon. Piecewise strongly proximal actions, free boundaries and the Neretin groups. *Bull. Soc. Math. France*, 150(4):773–795, 2022.

[5] A. Le Boudec and N. Matte Bon. Growth of actions of solvable groups, 2022. arXiv preprint, submitted.

[6] A. Le Boudec and N. Matte Bon. Confined subgroups and high transitivity. *Ann. H. Lebesgue*, 5:491–522, 2022.

Chapter 2

[7] N. Matte Bon and M. Triestino. Groups of piecewise linear homeomorphisms of flows. *Compos. Math.*, 156(8):1595–1622, 2020.

[8] J. Brum, N. Matte Bon, C. Rivas, and M. Triestino. Locally moving groups acting on the line and $\mathbb{R}$-focal actions, 2021. monograph (on arXiv), submitted (161pp).

[9] J. Brum, N. Matte Bon, C. Rivas, and M. Triestino. Solvable groups and affine actions on the line, 2022. arXiv preprint, submitted.

[10] J. Brum, N. Matte Bon, C. Rivas, and M. Triestino. A realisation result for moduli spaces of group actions on the line, 2023. arXiv preprint, submitted.

Chapter 3

[11] N. Matte Bon, N. Nekrashevych, and T. Zheng. Liouville property for groups and conformal dimension. arXiv preprint, submitted, 2023.

Other work (not discussed or only briefly mentioned)

(T) indicates a paper which was written during my Ph.D. thesis.

On Chabauty dynamics (topic of Chapter 1)

[12] N. Matte Bon and T. Tsankov. Realizing uniformly recurrent subgroups. *Ergodic Theory Dynam. Systems*, 40(2):478–489, 2020.

[13] A. Le Boudec and N. Matte Bon. Locally compact groups whose ergodic or minimal actions are all free. *Int. Math. Res. Not. IMRN*, (11):3318–3340, 2020.

The paper [18] below is not directly concerned with this topic but is somewhat related.

Related to groups actions on one-manifolds (topic of Chapter 2)

The papers [17,18] listed below have a component in this topic, although I wouldn't describe it as their primary category.

On amenability and the Liouville property (topic of Chapter 3)

[14] (T) G. Amir, O. Angel, N. Matte Bon, and B. Virág. The Liouville property for groups acting on rooted trees. *Ann. Inst. Henri Poincaré Probab. Stat.*, 52(4):1763–1783, 2016.

[15] (T) K. Juschenko, N. Matte Bon, N. Monod, and M. d. l. Salle. Extensive amenability and an application to interval exchanges. *Ergodic Theory Dynam. Systems*, 38(1):195–219, 2018.

[16] (T) N. Matte Bon. Subshifts with slow complexity and simple groups with the Liouville property. *Geom. Funct. Anal.*, 24(5):1637–1659, 2014.

On commensurated actions

[17] Y. Lodha, N. Matte Bon, and M. Triestino. Property FW, differentiable structures and smoothability of singular actions. *J. Topol.*, 13(3):1119–1138, 2020.

On transitivity degrees for group actions

[18] A. Le Boudec and N. Matte Bon. Triple transitivity and non-free actions in dimension one. *J. Lond. Math. Soc. (2)*, 105(2):884–908, 2022.

On abstract group theory, miscellaneous

[19] E. Gorokhovsky, N. Matte Bon, and O. Tamuz. A quantitative Neumann lemma for finitely generated groups, 2022. arXiv preprint, to appear in Israel Journal of Math.

[20] A. Le Boudec and N. Matte Bon. Some torsion-free metabelian groups with few subquotients, 2022. arXiv preprint, to appear in Math. Proc. Cambridge Philos. Soc.

[21] N. Matte Bon. A remark on groups defined by slowly growing trees., 2017. Appendix to: "Amir, G. and Kozma, G., Groups with minimal harmonic functions as small as you like", arXiv preprint, to appear in Groups, Geometry and Dynamics.

On aspects of groups of dynamical origin not in the previous categories

[22] (T) N. Matte Bon. Topological full groups of minimal subshifts with subgroups of intermediate growth. *J. Mod. Dyn.*, 9:67–80, 2015.

[23] (T) N. Matte Bon. Full groups of bounded automaton groups. *J. Fractal Geom.*, 4(4):425–458, 2017.

Bibliography

[AAV13] Gideon Amir, Omer Angel, and Bálint Virág, *Amenability of linear-activity automaton groups*, Journal of the European Mathematical Society **15** (2013), no. 3, 705–730.

[AGV14] Miklós Abért, Yair Glasner, and Bálint Virág, *Kesten's theorem for invariant random subgroups*, Duke Math. J. **163** (2014), no. 3, 465–488.

[Bar98] Laurent Bartholdi, *The growth of Grigorchuk's torsion group*, Internat. Math. Res. Notices **20** (1998), 1049–1054.

[Bas72] H. Bass, *The degree of polynomial growth of finitely generated nilpotent groups*, Proc. London Math. Soc. (3) **25** (1972), 603–614.

[BEB16] Christian Bonatti and Hélène Eynard-Bontemps, *Connectedness of the space of smooth actions of $\mathbb{Z}^n$ on the interval*, Ergodic Theory Dynam. Systems **36** (2016), no. 7, 2076–2106.

[BG02] Laurent Bartholdi and Rostislav I. Grigorchuk, *On parabolic subgroups and Hecke algebras of some fractal groups*, Serdica Math. J. **28** (2002), no. 1, 47–90.

[BKKO17] Emmanuel Breuillard, Mehrdad Kalantar, Matthew Kennedy, and Narutaka Ozawa, *C^*-simplicity and the unique trace property for discrete groups*, Publ. Math. Inst. Hautes Études Sci. **126** (2017), 35–71.

[BKN10] Laurent Bartholdi, Vadim A. Kaimanovich, and Volodymyr V. Nekrashevych, *On amenability of automata groups*, Duke Math. J. **154** (2010), no. 3, 575–598.

[BM02] M. Burger and N. Monod, *Continuous bounded cohomology and applications to rigidity theory*, Geom. Funct. Anal. **12** (2002), no. 2, 219–280.

[BMNR17] Christian Bonatti, Ignacio Monteverde, Andrés Navas, and Cristóbal Rivas, *Rigidity for C^1 actions on the interval arising from hyperbolicity I: solvable groups*, Math. Z. **286** (2017), no. 3-4, 919–949.

[BN08] Laurent Bartholdi and Volodymyr V. Nekrashevych, *Iterated monodromy groups of quadratic polynomials I*, Groups, Geometry, and Dynamics **2** (2008), no. 3, 309–336.

[Bri99] Matthew G. Brin, *The ubiquity of Thompson's group F in groups of piecewise linear homeomorphisms of the unit interval*, J. London Math. Soc. (2) **60** (1999), no. 2, 449–460.

[BS16a] Robert Bieri and Ralph Strebel, *On groups of PL-homeomorphisms of the real line*, Mathematical Surveys and Monographs, vol. 215, American Mathematical Society, Providence, RI, 2016.

[BS16b] ———, *On groups of PL-homeomorphisms of the real line*, Mathematical Surveys and Monographs, vol. 215, American Mathematical Society, Providence, RI, 2016.

[BV05] Laurent Bartholdi and Bálint Virág, *Amenability via random walks*, Duke Math. J. **130** (2005), no. 1, 39–56.

[BW72] Rufus Bowen and Peter Walters, *Expansive one-parameter flows*, J. Differential Equations **12** (1972), 180–193.

[Cha12] V. Chaynikov, *Properties of hyperbolic groups: Free normal subgroups, quasiconvex subgroups and actions of maximal growth*, ProQuest LLC, Ann Arbor, MI, 2012. Thesis (Ph.D.)–Vanderbilt University.

[CK93] Ian M. Chiswell and Peter H. Kropholler, *Soluble right orderable groups are locally indicable*, Canad. Math. Bull. **36** (1993), no. 1, 22–29.

[CM14] Pierre-Emmanuel Caprace and Nicolas Monod, *Relative amenability*, Groups Geom. Dyn. **8** (2014), no. 3, 747–774.

[CM23] Lei Chen and Kathryn Mann, *Structure theorems for actions of homeomorphism groups*, Duke Math. J. **172** (2023), no. 5, 915–962.

[Cor11] Y. Cornulier, *On the Cantor-Bendixson rank of metabelian groups*, Annales de l'Institut Fourier **61** (2011), no. 2, 593–618.

[Cor14] Yves Cornulier, *Groupes pleins-topologiques [d'après Matui, Juschenko, Monod,...]*, Astérisque **361** (2014), Exp. No. 1064. Séminaire Bourbaki. Vol. 2012/2013.

[CP13] Matias Carrasco Piaggio, *On the conformal gauge of a compact metric space*, Ann. Sci. Éc. Norm. Supér. (4) **46** (2013), no. 3, 495–548 (2013).

[CW22] P.-E. Caprace and G. Willis, *A totally disconnected invitation to locally compact groups*, 2022. preprint, to appear in the 2022 ICM proceedings.

[Den32] Arnaud Denjoy, *Sur les courbes définies par les équations différentielles à la surface du tore.*, J. Math. Pures Appl. (9) **11** (1932), 333–375 (French).

[Der13] Bertrand Deroin, *Almost-periodic actions on the real line*, Enseign. Math. (2) **59** (2013), no. 1-2, 183–194.

[DH20] B. Deroin and S. Hurtado, *Non left-orderability of lattices in higher rank semi-simple lie groups*, arXiv preprint (2020).

[Dix90] J. Dixon, *Most finitely generated permutation groups are free*, Bull. London Math. Soc. **22** (1990), no. 3, 222–226.

[DKN07] Bertrand Deroin, Victor Kleptsyn, and Andrés Navas, *Sur la dynamique unidimensionnelle en régularité intermédiaire*, Acta Math. **199** (2007), no. 2, 199–262.

[DKNP13] Bertrand Deroin, Victor Kleptsyn, Andrés Navas, and Kamlesh Parwani, *Symmetric random walks on* $\mathrm{Homeo}^+(\mathbf{R})$, Ann. Probab. **41** (2013), no. 3B, 2066–2089.

[DNR16] Bertrand Deroin, Andrés Navas, and Cristóbal Rivas, *Groups, orders, and dynamics*, 2016.

[EBN21] H. Eynard-Bontemps and A. Navas, *(Arc-)connectedness for the space of $\mathbb{Z}^d$-actions by c^2 diffeomorphisms on 1-dimensional manifolds*, 2021. arXiv preprint.

[EG16] A. Eisenmann and Y. Glasner, *Generic IRS in free groups, after Bowen*, Proc. Amer. Math. Soc. **144** (2016), no. 10, 4231–4246.

[Ele18] Gábor Elek, *Uniformly recurrent subgroups and simple C^*-algebras*, J. Funct. Anal. **274** (2018), no. 6, 1657–1689.

[Eps70] D. B. A. Epstein, *The simplicity of certain groups of homeomorphisms*, Compositio Math. **22** (1970), 165–173.

[Ers04] Anna Erschler, *Liouville property for groups and manifolds*, Invent. Math. **155** (2004), no. 1, 55–80.

[EZ20] Anna Erschler and Tianyi Zheng, *Growth of periodic Grigorchuk groups*, Invent. Math. **219** (2020), no. 3, 1069–1155.

[FF03] Benson Farb and John Franks, *Groups of homeomorphisms of one-manifolds. III. Nilpotent subgroups*, Ergodic Theory Dynam. Systems **23** (2003), no. 5, 1467–1484.

[Fil82] R. P. Filipkiewicz, *Isomorphisms between diffeomorphism groups*, Ergodic Theory Dynam. Systems **2** (1982), no. 2, 159–171.

[FJ04] Charles Favre and Mattias Jonsson, *The valuative tree*, Vol. 1853, Berlin: Springer, 2004 (English).

[FLMMS22] Pierre Fima, François Le Maître, Soyoung Moon, and Yves Stalder, *A characterization of high transitivity for groups acting on trees*, Discrete Anal. (2022), Paper No. 8, 63.

[Gel18] Tsachik Gelander, *A view on invariant random subgroups and lattices*, Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. II. Invited lectures, 2018, pp. 1321–1344.

[GG08] T. Gelander and Y. Glasner, *Countable primitive groups*, Geom. Funct. Anal. **17** (2008), no. 5, 1479–1523.

[GG13a] S. Garion and Y. Glasner, *Highly transitive actions of* $\mathrm{Out}(F_n)$, Groups Geom. Dyn. **7** (2013), no. 2, 357–376.

[GG13b] T. Gelander and Y. Glasner, *An Aschbacher-O'Nan-Scott theorem for countable linear groups*, J. Algebra **378** (2013), 58–63.

[GGS22] Tsachik Gelander, Yair Glasner, and Gregory Soifer, *Maximal subgroups of countable groups: a survey*, Dynamics, geometry, number theory—the impact of Margulis on modern mathematics, 2022, pp. 169–210.

[Ghy87a] Étienne Ghys, *Groupes d'homéomorphismes du cercle et cohomologie bornée*, The Lefschetz centennial conference, Part III (Mexico City, 1984), 1987, pp. 81–106.

[Ghy87b] ———, *Groupes d'homéomorphismes du cercle et cohomologie bornée*, The Lefschetz centennial conference, Part III (Mexico City, 1984), 1987, pp. 81–106.

[GM98] Robert Guralnick and Kay Magaard, *On the minimal degree of a primitive permutation group*, J. Algebra **207** (1998), no. 1, 127–145.

[GNS00] R. I. Grigorchuk, V. V. Nekrashevich, and V. I. Sushchanskiĭ, *Automata, dynamical systems, and groups*, Tr. Mat. Inst. Steklova **231** (2000), no. Din. Sist., Avtom. i Beskon. Gruppy, 134–214.

[GPS99] Thierry Giordano, Ian F. Putnam, and Christian F. Skau, *Full groups of Cantor minimal systems*, Israel J. Math. **111** (1999), 285–320.

[GR18] Nancy Guelman and Cristóbal Rivas, *Quasi-invariant measures for some amenable groups acting on the line*, Algebr. Geom. Topol. **18** (2018), no. 2, 1067–1076.

[Gri00] R. I. Grigorchuk, *Just infinite branch groups*, New horizons in pro-p groups, 2000, pp. 121–179.

[Gri11] ———, *Some problems of the dynamics of group actions on rooted trees*, Tr. Mat. Inst. Steklova **273** (2011), 72–191.

[Gri84] ———, *Degrees of growth of finitely generated groups and the theory of invariant means*, Izv. Akad. Nauk SSSR Ser. Mat. **48** (1984), no. 5, 939–985.

[Gro93a] M. Gromov, *Asymptotic invariants of infinite groups*, Geometric group theory, Vol. 2 (Sussex, 1991), 1993, pp. 1–295.

[Gro93b] ———, *Asymptotic invariants of infinite groups*, Geometric group theory, Vol. 2 (Sussex, 1991), 1993, pp. 1–295.

[GS87] Étienne Ghys and Vlad Sergiescu, *Sur un groupe remarquable de difféomorphismes du cercle*, Comment. Math. Helv. **62** (1987), no. 2, 185–239.

[Gui73] Y. Guivarc'h, *Croissance polynomiale et périodes des fonctions harmoniques*, Bull. Soc. Math. France **101** (1973), 333–379.

[GW15] Eli Glasner and Benjamin Weiss, *Uniformly recurrent subgroups*, Recent trends in ergodic theory and dynamical systems, 2015, pp. 63–75.

[GŻ02] Rostislav I. Grigorchuk and Andrzej Żuk, *On a torsion-free weakly branch group defined by a three state automaton*, 2002, pp. 223–246. International Conference on Geometric and Combinatorial Methods in Group Theory and Semigroup Theory (Lincoln, NE, 2000).

[Hig54] Graham Higman, *On infinite simple permutation groups*, Publ. Math. Debrecen **3** (1954), 221–226.

[HL19] James Hyde and Yash Lodha, *Finitely generated infinite simple groups of homeomorphisms of the real line*, Invent. Math. **218** (2019), no. 1, 83–112.

[HLNR21] James Hyde, Yash Lodha, Andrés Navas, and Cristóbal Rivas, *Uniformly perfect finitely generated simple left orderable groups*, Ergodic Theory Dynam. Systems **41** (2021), no. 2, 534–552.

[HO16] M. Hull and D. Osin, *Transitivity degrees of countable groups and acylindrical hyperbolicity*, Israel J. Math. **216** (2016), no. 1, 307–353.

[HO17] Uffe Haagerup and Kristian Knudsen Olesen, *Non-inner amenability of the Thompson groups T and V*, J. Funct. Anal. **272** (2017), no. 11, 4838–4852.

[Höl01] Otto Hölder, *Die Axiome der Quantität und die Lehre vom Maß*, Leipz. Ber. **53** (1901), 1–64 (German).

[HP09] Peter Haïssinsky and Kevin M. Pilgrim, *Coarse expanding conformal dynamics*, Astérisque **325** (2009), viii+139 pp.

[HP11] ———, *Finite type coarse expanding conformal dynamics*, Groups Geom. Dyn. **5** (2011), no. 3, 603–661.

[HP12] ———, *Quasisymmetrically inequivalent hyperbolic Julia sets*, Rev. Mat. Iberoam. **28** (2012), no. 4, 1025–1034.

[Hur15] Sebastian Hurtado, *Continuity of discrete homomorphisms of diffeomorphism groups*, Geom. Topol. **19** (2015), no. 4, 2117–2154.

[HZ97] B. Hartley and A. E. Zalesskiĭ, *Confined subgroups of simple locally finite groups and ideals of their group rings*, J. London Math. Soc. (2) **55** (1997), no. 2, 210–230.

[Jac19] L. Jacoboni, *Metabelian groups with large return probability*, Ann. Inst. Fourier (Grenoble) **69** (2019), no. 5, 2121–2167.

[JdlS15] Kate Juschenko and Mikael de la Salle, *Invariant means for the wobbling group*, Bull. Belg. Math. Soc. Simon Stevin **22** (2015), no. 2, 281–290.

[JM13] Kate Juschenko and Nicolas Monod, *Cantor systems, piecewise translations and simple amenable groups*, Ann. of Math. (2) **178** (2013), no. 2, 775–787.

[JNS16] Kate Juschenko, Volodymyr Nekrashevych, and Mikael de la Salle, *Extensions of amenable groups by recurrent groupoids*, Invent. Math. **206** (2016), no. 3, 837–867.

[Kai05] Vadim A. Kaimanovich, *"Münchhausen trick" and amenability of self-similar groups*, Internat. J. Algebra Comput. **15** (2005), no. 5-6, 907–937.

[Kec21] Alexander S. Kechris, *The theory of countable Borel equivalence relations*, 2021. Preliminary version available at `http://www.math.caltech.edu/~kechris/`.

[Ken15] M. Kennedy, *An intrinsic characterization of C^*-simplicity*, Ann. Sci. Éc. Norm. Supér, to appear (2015).

[Kig20] Jun Kigami, *Geometry and analysis of metric spaces via weighted partitions*, Lecture Notes in Mathematics, vol. 2265, Springer, Cham, 2020.

[Kit12] D. Kitroser, *Highly-transitive actions of surface groups*, Proc. Amer. Math. Soc. **140** (2012), no. 10, 3365–3375.

[KK17] M. Kalantar and M. Kennedy, *Boundaries of reduced C^*-algebras of discrete groups*, J. Reine Angew. Math. **727** (2017), 247–267.

[KKL19] Sang-hyun Kim, Thomas Koberda, and Yash Lodha, *Chain groups of homeomorphisms of the interval*, Ann. Sci. Éc. Norm. Supér. (4) **52** (2019), no. 4, 797–820.

[KKM19] Sang-hyun Kim, Thomas Koberda, and Mahan Mj, *Flexibility of group actions on the circle*, Lecture Notes in Mathematics, vol. 2231, Springer, Cham, 2019.

[Kop70] Nancy Kopell, *Commuting diffeomorphisms*, Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968), 1970, pp. 165–184.

[KV83] V. A. Kaĭmanovich and A. M. Vershik, *Random walks on discrete groups: boundary and entropy*, Ann. Probab. **11** (1983), no. 3, 457–490.

[LB16] Adrien Le Boudec, *Groups acting on trees with almost prescribed local action*, Comment. Math. Helv. **91** (2016), no. 2, 253–293.

[LBT23] A.. Le Boudec and T. Tsankov, *Continuity of the stabilizer map on maximally highly proximal flows*, arXiv preprint (2023).

[LM18] F. Le Maître, *Highly faithful actions and dense free subgroups in full groups*, Groups Geom. Dyn. **12** (2018), no. 1, 207–230.

[Man15] Kathryn Mann, *Spaces of surface group representations*, Invent. Math. **201** (2015), no. 2, 669–710.

[Man18] ———, *Rigidity and flexibility of group actions on the circle*, Handbook of group actions. Vol. IV, 2018, pp. 705–752.

[Mat06] Hiroki Matui, *Some remarks on topological full groups of Cantor minimal systems*, Internat. J. Math. **17** (2006), no. 2, 231–251.

[Mat15] ———, *Topological full groups of one-sided shifts of finite type*, J. Reine Angew. Math. **705** (2015), 35–84.

[Mat86] Shigenori Matsumoto, *Numerical invariants for semiconjugacy of homeomorphisms of the circle*, Proc. Amer. Math. Soc. **98** (1986), no. 1, 163–168.

[Mat87] ———, *Some remarks on foliated S^1 bundles*, Invent. Math. **90** (1987), no. 2, 343–358.

[McD77] T. P. McDonough, *A permutation representation of a free group*, Quart. J. Math. Oxford Ser. (2) **28** (1977), no. 111, 353–356.

[McM00] Curtis T. McMullen, *Hausdorff dimension and conformal dynamics. II. Geometrically finite rational maps*, Comment. Math. Helv. **75** (2000), no. 4, 535–593.

[Mil68] J. Milnor, *Growth of finitely generated solvable groups*, J. Differential Geometry **2** (1968), 447–449.

[Mon13] Nicolas Monod, *Groups of piecewise projective homeomorphisms*, Proc. Natl. Acad. Sci. USA **110** (2013), no. 12, 4524–4527.

[Mor06] Dave Witte Morris, *Amenable groups that act on the line*, Algebr. Geom. Topol. **6** (2006), 2509–2518.

[MR18] Kathryn Mann and Cristóbal Rivas, *Group orderings, dynamics, and rigidity*, Ann. Inst. Fourier (Grenoble) **68** (2018), no. 4, 1399–1445.

[MS81] G. A. Margulis and G. A. Soĭfer, *Maximal subgroups of infinite index in finitely generated linear groups*, J. Algebra **69** (1981), no. 1, 1–23.

[MT10] John M. Mackay and Jeremy T. Tyson, *Conformal dimension*, University Lecture Series, vol. 54, American Mathematical Society, Providence, RI, 2010. Theory and application.

[MT23] Kathryn Mann and Michele Triestino, *On the action of the $\Sigma(2,3,7)$ homology sphere group on its space of left-orders*, Fund. Math. **261** (2023), no. 3, 297–302.

[MW17] K. Mann and M.. Wolff, *Rigidity and geometricity for surface group actions on the circle*, arXiv preprint, to appear in Geometry and Topology (2017).

[Nav04a] Andrés Navas, *Groupes résolubles de difféomorphismes de l'intervalle, du cercle et de la droite*, Bull. Braz. Math. Soc. (N.S.) **35** (2004), no. 1, 13–50.

[Nav04b] ———, *Quelques groupes moyennables de difféomorphismes de l'intervalle*, Bol. Soc. Mat. Mexicana (3) **10** (2004), no. 2, 219–244 (2005).

[Nav10] ———, *On the dynamics of (left) orderable groups*, Ann. Inst. Fourier (Grenoble) **60** (2010), no. 5, 1685–1740.

[Nav11] ———, *Groups of circle diffeomorphisms*, Spanish, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 2011.

[Nav18] ———, *Group actions on 1-manifolds: a list of very concrete open questions*, Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. III. Invited lectures, 2018, pp. 2035–2062.

[Nek05] Volodymyr Nekrashevych, *Self-similar groups*, Mathematical Surveys and Monographs, vol. 117, American Mathematical Society, Providence, RI, 2005.

[Nek09] ———, *Combinatorics of polynomial iterations*, Complex dynamics – families and friends, 2009, pp. 169–214.

[Nek11] ———, *Iterated monodromy groups*, Groups St Andrews 2009 in Bath. Volume 1, 2011, pp. 41–93.

[Nek18] ———, *Palindromic subshifts and simple periodic groups of intermediate growth*, Ann. of Math. (2) **187** (2018), no. 3, 667–719.

[Nek19] V. Nekrashevych, *Simple groups of dynamical origin*, Ergodic Theory Dynam. Systems **39** (2019), no. 3, 707–732.

[Nek22] Volodymyr Nekrashevych, *Groups and topological dynamics*, Graduate Studies in Mathematics, vol. 223, American Mathematical Society, Providence, RI, 2022.

[NPT20] Volodymyr Nekrashevych, Kevin Pilgrim, and Dylan Thurston, *On amenability of iterated monodromy groups*, 2020. (preprint).

[NR10] Andrés Navas and Cristóbal Rivas, *Describing all bi-orderings on Thompson's group F*, Groups Geom. Dyn. **4** (2010), no. 1, 163–177.

[Ols15] A. Yu. Olshanskii, *On pairs of finitely generated subgroups in free groups*, Proc. Amer. Math. Soc. **143** (2015), no. 10, 4177–4188.

[Ono29] L. Onofri, *Teoria delle sostituzioni che operano su una infinità numerabile di elementi*, Ann. Mat. Pura Appl. **7** (1929), no. 1, 103–130.

[Pan89] Pierre Pansu, *Dimension conforme et sphère à l'infini des variétés à courbure négative*, Ann. Acad. Sci. Fenn. Ser. A I Math. **14** (1989), no. 2, 177–212.

[Pix77] Dennis Pixton, *Nonsmoothable, unstable group actions*, Trans. Amer. Math. Soc. **229** (1977), 259–268.

[Pla75] Joseph F. Plante, *Foliations with measure preserving holonomy*, Ann. of Math. (2) **102** (1975), no. 2, 327–361.

[Pla83] ———, *Solvable groups acting on the line*, Trans. Amer. Math. Soc. **278** (1983), no. 1, 401–414.

[Pla84] ———, *Subgroups of continuous groups acting differentiably on the half-line*, Ann. Inst. Fourier (Grenoble) **34** (1984), no. 1, 47–56.

[PT76] Joseph F. Plante and William P. Thurston, *Polynomial growth in holonomy groups of foliations*, Comment. Math. Helv. **51** (1976), no. 4, 567–584.

[Riv10] Cristóbal Rivas, *On spaces of Conradian group orderings*, J. Group Theory **13** (2010), no. 3, 337–353.

[Ros81] Joseph Rosenblatt, *Ergodic and mixing random walks on locally compact groups*, Math. Ann. **257** (1981), no. 1, 31–42.

[Rub89] Matatyahu Rubin, *On the reconstruction of topological spaces from their groups of homeomorphisms*, Trans. Amer. Math. Soc. **312** (1989), no. 2, 487–538.

[Rub96] ———, *Locally moving groups and reconstruction problems*, Ordered groups and infinite permutation groups, 1996, pp. 121–157.

[Sal21] V. Salo, *Graph and wreath products in topological full groups of full shifts*, 2021. preprint.

[SU33] O. Schreier and S. Ulam, *Über die permutationsgruppe der natürlichen zahlenfolge*, Studia Math. **4** (1933), 134–141.

[SZ93] S. K. Sehgal and A. E. Zalesskiĭ, *Induced modules and some arithmetic invariants of the finitary symmetric groups*, Nova J. Algebra Geom. **2** (1993), no. 1, 89–105.

[Tsu95] Takashi Tsuboi, *Homological and dynamical study on certain groups of Lipschitz homeomorphisms of the circle*, J. Math. Soc. Japan **47** (1995), no. 1, 1–30.

[Vee77] William A. Veech, *Topological dynamics*, Bull. Amer. Math. Soc. **83** (1977), no. 5, 775–830.

[Whi63] James V. Whittaker, *On isomorphic groups and homeomorphic spaces*, Ann. of Math. (2) **78** (1963), 74–91.

[Wol68] J. A. Wolf, *Growth of finitely generated solvable groups and curvature of Riemannian manifolds*, J. Differential Geometry **2** (1968), 421–446.

[Zal95] A. E. Zalesskiĭ, *Group rings of simple locally finite groups*, Finite and locally finite groups (Istanbul, 1994), 1995, pp. 219–246.