Risk indicators with several lines of business: comparison, asymptotic behavior and applications to optimal reserve allocation

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Plan

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Context I.

European rules ⇒ companies have to better understand their risks and take into account dependencies between risks / branches, ERM setting.

- Main risk drivers for the overall company have been identified,
- Capital to be allocated (free or investment capital, or global solvency capital requirement) between lines of business or activity branches.

Avoid as far as possible that some lines of business become insolvent too often.

⇒ Minimize a risk indicator.

Context II.

We study the expected total *“orange area”* and *"violet area"* in the discrete time framework inspired by Solvency II and ORSA (Own Risk and Solvency Assessment) related issues $\Rightarrow$ optimality of the global reserve allocation may be obtained by minimizing the expected sum of the penalties that each line of business would have to pay due to its temporary potential insolvency.

Multivariate risk indicator

- d lines of business,
- n periods,
- u is the capital to be allocated, u^k will be allocated to the kth line of business,
- R_j^k is the reserve of line k, at time j:

$$R_j^k = u^k + Y_j^k,$$

with Y_j^k the aggregate premium minus the aggregate claim amount for the kth branch, during the jst period:

$$Y_j^k = \sum_{i=1}^{j} (c^k - X_i^k).$$

Multivariate risk indicator

We look for the optimal allocation $(u^1, \ldots, u^d)$ that minimizes I_1 the *orange area*, or I the *stopped orange area* or J the *violet area*, under the constraint that $u^1 + \cdots + u^d = u$:

$$I_1 = \sum_{k=1}^{d} \sum_{j=1}^{n} \mathbb{E}\left(g_k(R_j^k) \mathbb{1}_{\{R_j^k < 0\}} \mathbb{1}_{\{R_j^1 + \cdots + R_j^d \geq 0\}} \right)$$

$$I = \sum_{k=1}^{d} \sum_{j=1}^{n \wedge \tau} \mathbb{E}\left(g_k(R_j^k) \mathbb{1}_{\{R_j^k < 0\}} \mathbb{1}_{\{R_j^1 + \cdots + R_j^d \geq 0\}} \right)$$

$$J = \sum_{k=1}^{d} \sum_{j=1}^{n} \mathbb{E}\left(g_k(R_j^k) \mathbb{1}_{\{R_j^k < 0\}} \mathbb{1}_{\{R_j^1 + \cdots + R_j^d \leq 0\}} \right)$$

Multivariate risk indicator

$$I_1 = \sum_{k=1}^{d} \sum_{j=1}^{n} \mathbb{E}\left(g_k(R_j^k) \mathbb{1}_{\{R_j^k < 0\}} \mathbb{1}_{\{R_j^1 + \cdots + R_j^d \geq 0\}} \right)$$

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$$J = \sum_{k=1}^{d} \sum_{j=1}^{n} \mathbb{E}\left(g_k(R_j^k) \mathbb{1}_{\{R_j^k < 0\}} \mathbb{1}_{\{R_j^1 + \cdots + R_j^d \leq 0\}} \right)$$

Where τ is the *ruin time*: $\tau = \inf\left\{ j \in \mathbb{N}^*, \ R_j^1 + \cdots + R_j^d < 0 \right\}$ and $g_k : \mathbb{R} \to \mathbb{R}$ a C^1 convex function, with $g_k(0) = 0$, $g_k(x) \geq 0$. This is a penalty function to the kth branch when it becomes insolvable.

Multivariate risk indicator

$$I_1 = \sum_{k=1}^{d}\sum_{j=1}^{n} \mathbb{E}\left(g_k(R_j^k) \mathbb{1}_{\{R_j^k<0\}} \mathbb{1}_{\{R_j^1+\cdots+R_j^d\geq 0\}} \right)$$

$$I = \sum_{k=1}^{d}\sum_{j=1}^{n\wedge\tau} \mathbb{E}\left(g_k(R_j^k) \mathbb{1}_{\{R_j^k<0\}} \mathbb{1}_{\{R_j^1+\cdots+R_j^d\geq 0\}} \right)$$

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Remark: $g_k(x) = |x|$ is a possible choice (then we consider the ruin amount). Then, the indicator is the expected sum of penalties that each line of business would have to pay due to its temporary potential insolvency.

Multivariate risk indicator

$$I_1 = \sum_{k=1}^{d}\sum_{j=1}^{n}\mathbb{E}\left(g_k(R_j^k)\mathbb{1}_{\{R_j^k<0\}}\mathbb{1}_{\{R_j^1+\cdots+R_j^d\geq 0\}}\right)$$

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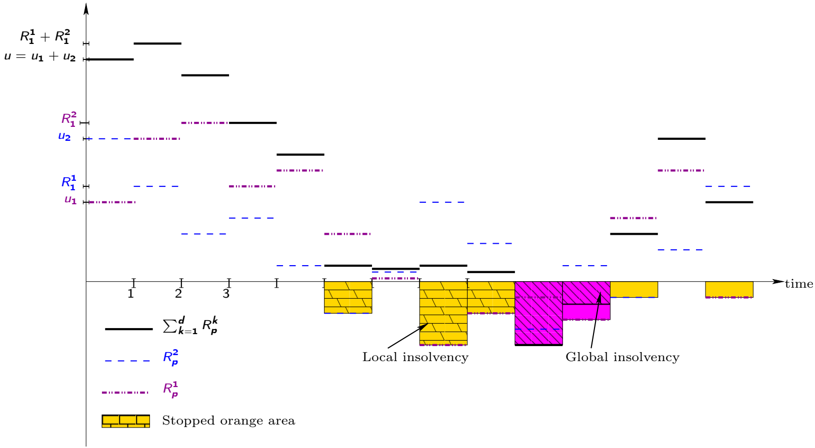
$$J = \sum_{k=1}^{d}\sum_{j=1}^{n}\mathbb{E}\left(g_k(R_j^k)\mathbb{1}_{\{R_j^k<0\}}\mathbb{1}_{\{R_j^1+\cdots+R_j^d\leq 0\}}\right)$$

Time horizon n could correspond to 1 year in the Solvency II SCR computation problem, or to 3, 5, 10 or 15 years in the ORSA framework.

There may be dependencies: *vectorial* (with respect to $k = 1, \ldots, d$) and/or *temporal* (with respect to $j = 1, \ldots, n$).

The orange and violet areas



Other attempt to allocate the u^i's

- *Classical* allocations: Variance based allocation, proportional allocation,
- Euler's allocation (VaR based),
- Decomposition of the aggregated TVaR (Cai and Li, Chiragiev and Landsman, Bargès Cossette and Marceau),
- General framework by J. Dhaene, A. Tsanakas, E.A. Valdez, S. Vanduffel: *capital allocation by using minimization principles.*
- Using asymptotic behavior of the ruin probabilities (Biard),

General framework

General framework considered by J. Dhaene, A. Tsanakas, E.A. Valdez, S. Vanduffel

Different capital allocations must in some sense correspond to different questions that can be asked within the context of risk management.

The indicator should be designed for a specific purpose.

General framework

General framework considered by J. Dhaene, A. Tsanakas, E.A. Valdez, S. Vanduffel : minimize

$$\sum_{j=1}^{d} \mathbb{E}\left[\xi_j D\left(\frac{X_j - u^j}{v_j}\right)\right]$$

with X_j the loss of the jth branch. D is a deviation function, *Dhaene et al.* proposed to take $D(x) = x^2$ or $D(x) = |x|$.
If we take $\xi_j = \mathbf{1}_{\{S \leq u\}}$ (resp. $\xi_j = \mathbf{1}_{\{S \geq u\}}$) and $D(x) = x^+$, then we recover our indicator orange area (resp. violet area) in the case $p = 1$, with $S = \sum_{j=1}^{d} X_j$.

Specific models (independent exponential).

Two lines of business ($d = 2$) and one-period problem ($n = 1$).
No impact of the premiums c^k, $\Rightarrow$ $u^k \leftrightarrow u^k + c^k$.

X_j^k independent exponential laws

- $X_1^1 \rightsquigarrow \mathcal{E}(\mu^1)$,
- $X_1^2 \rightsquigarrow \mathcal{E}(\mu^2)$.

No explicit expression for u^1 and u^2 but an implicit forms.

- $\mu^2 = \alpha\mu^1$ with $\alpha \geq 1$,
- $u^1 = \beta u$ with $0 \leq \beta \leq 1$.

Specific models (independent exponential).

β with $u^1 = \beta u$,

Orange area case:

$$\begin{aligned} f(\alpha, \beta u, \mu^1) &= -(\alpha - 1)e^{-\alpha\mu^1 u(1-\beta)} + (1+\alpha)e^{-\mu^1 u(\alpha-\alpha\beta+\beta)} \\ &\quad +(\alpha-1)e^{-\mu^1 u\beta} - \alpha e^{-\mu^1 u} - e^{-\alpha u\mu^1} = 0. \end{aligned}$$

Violet area case:

$$(\alpha+1)e^{-u\mu^1(\alpha-\alpha\beta+\beta)} - \alpha e^{-u\mu^1} - e^{-\alpha u\mu^1} = 0.$$

Specific models (conditionally independent exponential).

Conditionally independent exponential laws: X_j^k are independent exponential laws conditionally to $\Theta \rightsquigarrow \Gamma(a, b)$. More precisely,

- $X_1^1 | \Theta \rightsquigarrow \mathcal{E}(\Theta)$,
- $X_1^2 | \Theta \rightsquigarrow \mathcal{E}(\alpha\Theta)$, $\alpha \geq 1$.

The X_j^k are correlated gpd.

NB: model previously studied by H. Albrecher, C. Constatinescu and S. Loisel (some explicit formulas for the ruin probability).

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The X_j^k are correlated gpd.

Integrating equations above $\Longrightarrow$ implicit expression for β: Orange area case:

$$(\alpha - 1)s(\beta) - \alpha s(1) + (\alpha + 1)s(\alpha - \alpha\beta + \beta)$$
$$-(\alpha - 1)s(\alpha(1 - \beta)) - s(\alpha) = 0$$

Violet area case:

$$(\alpha + 1)s(\alpha - \alpha\beta + \beta) - \alpha s(1) - s(\alpha) = 0$$

where $s(x) = (1 + x\frac{u}{b})^{-a}$.

Quantitative behavior

Case independent exponential

Orange area

- $\beta \leq \frac{\alpha}{\alpha+1}$
- β is increasing with respect to α
- $\beta \longrightarrow \frac{\alpha}{\alpha+1}$ as u goes to infinity.

Violet area

- $\beta \leq 1$
- β is increasing with respect to α
- $\beta \longrightarrow 1$ as u goes to infinity.

Quantitative behavior

Case conditionally independent exponential

Orange area

- $\beta \leq \frac{\alpha}{\alpha+1}$
- β is increasing with respect to α
- $\beta \longrightarrow \beta_0$ as u goes to infinity, with $\beta_0 < \frac{\alpha}{\alpha+1}$ solution to:

$$(\alpha-1)\beta^{-a} - \alpha + (\alpha+1)(\alpha - \alpha\beta + \beta)^{-a} - (\alpha-1)(\alpha(1-\beta))^{-a} - \alpha^{-a} = 0.$$

Violet area

- $\beta \leq 1$
- β is increasing with respect to α
- $\beta \longrightarrow \beta_1$ as u goes to infinity, with $\beta_1 < 1$ solution to:

$$\alpha + \alpha^{-a} - (\alpha+1)(\alpha - \alpha\beta + \beta)^{-a} = 0.$$

Quantitative behavior

Case independent GPD

Orange area

- No exact result
- $\beta \longrightarrow \frac{\alpha}{\alpha+1}$ as u goes to infinity.

Violet area

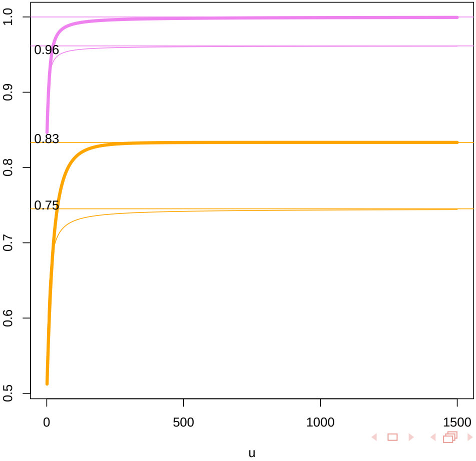
- No exact result
- $\beta \longrightarrow 1$ as u goes to infinity.

Comparisons

Chose parameters a and b of the CIE model in such a way that $\mathbb{E}(\mu^1) = \frac{1}{20}$, specifically $a = 1$ and $b = 20$.
Chose the μ^1 parameter of the IE model equal to $\frac{1}{20}$.
$\alpha = 5$ for the two models.

Comparisons

The bold lines are for the independent exponential model, the simple lines are for the conditionally independent.



Simulations

We consider

- independent exponential laws, conditionally independent exponential laws, independent gpd distributions.
- $n = 1$ observation of several periods of length 1,
- for $d = 2$, we have compared the results given by the stochastic algorithm with the theoretical values,
- in higher dimension ($d = 10$), simulation study
- multi-periodic simulation ($n > 1$).

Simulations

Tool for the simulation study: a Kiefer-Wolfowitz version of the stochastic mirror algorithm $\Rightarrow$ efficient algorithm to find the optimal solution.

Advantages of the stochastic algorithms approach:

- no parametric hypothesis on the law of the X_i's,
- dependence allowed over one period,
- high dimension (d) allowed.

Simulation in dimension 2

We have chosen $\mu^1 = \frac{1}{20}$. In order to get the estimation of the minimum, we have performed 10 times the stochastic algorithm on data of length 15 000. For $\alpha = 5$, we have taken $u = 50$, we compare with the theoretical value using the mean squared error (mse).

Orange area case

	$\alpha = 5$					
	IE model		CIE model		GPDI model	
	$\hat{u}^1$	$\hat{u}^2$	$\hat{u}^1$	$\hat{u}^2$	$\hat{u}^1$	$\hat{u}^2$
mean	38.37	11.63	36.8	13.2	35.83	14.17
sd dev	0.085	0.085	0.115	0.115	0.133	0.133
th.	38.46	11.54	36.84	13.16	non available	
$\sqrt{\text{mse}}$	0.121	0.121	0.119	0.119	non available	

Simulation in dimension 2

We have chosen $\mu^1 = \frac{1}{20}$. In order to get the estimation of the minimum, we have performed 10 times the stochastic algorithm on data of length 15 000. For $\alpha = 5$, we have taken $u = 50$, we compare with the theoretical value using the mean squared error (mse).

Violet area case

	$\alpha = 5$					
	IE model		CIE model		GPDI model	
	$\hat{u}^1$	$\hat{u}^2$	$\hat{u}^1$	$\hat{u}^2$	$\hat{u}^1$	$\hat{u}^2$
mean	46.83	3.17	47.18	2.81	48.13	1.87
sd dev	0.35	0.35	0.136	0.136	0.114	0.114
th.	49.08	0.92	48.36	1.64	non available	
$\sqrt{\text{mse}}$	2.28	2.28	0.92	0.92	non available	

Models in dimension 10.

A block of correlated GPD and a block of independent GDP, in dimension 10 (Mixed Model).

- $X_1^1 \rightsquigarrow \mathcal{E}(\Theta)$, $X_1^i \rightsquigarrow \mathcal{E}(\alpha\Theta)$, $i = 2, \ldots,\ 5$, where $\Theta \rightsquigarrow \Gamma(a, b)$.
- $X_1^1 \rightsquigarrow GPD(\frac{1}{a}, \frac{b}{a})$; $X_1^i \rightsquigarrow GPD(\frac{1}{a}, \frac{b}{\alpha a})$, $i = 6, \ldots,\ 10$.

- Conditional exponential: $X_1^1 \rightsquigarrow \mathcal{E}(\Theta)$, $X_1^i \rightsquigarrow \mathcal{E}(\alpha\Theta)$, $i = 2, \ldots, 10$.
- Independent GPD: $X_1^1 \rightsquigarrow GPD(\frac{1}{a}, \frac{b}{a})$; $X_1^i \rightsquigarrow GPD(\frac{1}{a}, \frac{b}{\alpha a})$, $i = 2, \ldots, 10$.

These three models have the same marginal laws. We have chosen $\alpha = 5$ and $u = 80$. We have performed our stochastic algorithm 10 times on data sets of length 20 000 for the orange area and 23 000 for the violet area.

Models in dimension 10.

	mixed model		cond. expo.		indep. GPD	
	mean	sd dev.	mean	sd dev.	mean	sd dev.
u^1	20.56	0.15	21.48	0.23	18.42	0.219
u^2	5.78	0.043	6.49	0.123	6.88	0.138
u^3	5.77	0.05	6.52	0.133	6.88	0.151
u^4	5.8	0.059	6.49	0.116	6.82	0.16
u^5	5.79	0.052	6.52	0.116	6.82	0.132
u^6	7.25	0.059	6.5	0.11	6.81	0.113
u^7	7.25	0.071	6.5	0.085	6.89	0.154
u^8	7.31	0.071	6.5	0.13	6.82	0.14
u^9	7.25	0.066	6.49	0.121	6.82	0.143
u^{10}	7.26	0.078	6.49	0.121	6.83	0.153

Models in dimension 10.

	mixed model		cond. expo.		indep. GPD	
	mean	sd dev.	mean	sd dev.	mean	sd dev.
u^1	41.4	0.23	37.68	0.31	41.99	0.254
u^2	5.2	0.07	4.68	0.068	4.19	0.044
u^3	5.19	0.08	4.72	0.076	4.23	0.046
u^4	5.18	0.07	4.67	0.069	4.2	0.073
u^5	5.16	0.07	4.73	0.0.63	4.2	0.094
u^6	3.57	0.08	4.68	0.053	4.22	0.048
u^7	3.56	0.06	4.72	0.094	4.23	0.069
u^8	3.58	0.05	4.69	0.046	4.24	0.07
u^9	3.56	0.05	4.69	0.04	4.25	0.064
u^{10}	3.59	0.06	4.72	0.069	4.24	0.051

$n > 1$ and 5 lines of business

Simulation only for models with independence in time.
The premiums c^k are taken into account (5% of the expectation of the branch).
We have performed 10 simulations of length 15 000 for the orange area case and 16 000 for the violet area case, for our three models (independent GPD, conditionally exponential, mixed model), with $\alpha = 5$, $u = 80$, $a = 3$, $b = 60$, $n = 4$.

$n > 1$ and 5 lines of business

- Mixed Model (MM): the first 3 lines are conditionally exponential and the last two lines are independent GPD (parameters are the same as above).
- Comparison with the correlated Pareto and independent GPD models with the same margin.

Orange area

	MM		CIE		IGPD	
	mean	sd dev.	mean	sd dev.	mean	sd dev.
u^1	31.33	0.36	32.68	0.28	30.1	0.31
u^2	10.63	0.16	11.86	0.14	12.49	0.17
u^3	10.6	0.24	11.78	0.24	12.51	0.23
u^4	13.74	0.27	11.83	0.15	12.4	0.14
u^5	13.7	0.27	11.85	0.12	12.5	0.2

$n > 1$ and 5 lines of business

- Mixed Model (MM): the first 3 lines are conditionally exponential and the last two lines are independent GPD (parameters are the same as above).
- Comparison with the correlated Pareto and independent GPD models with the same margin.

Violet area

	MM		CIE		IGPD	
	mean	sd dev.	mean	sd dev.	mean	sd dev.
u^1	77.64	0.24	73.63	0.326	79.6	0.157
u^2	1.11	0.11	1.62	0.119	0.1	0.041
u^3	1.1	0.23	1.52	0.194	0.11	0.042
u^4	0.066	0.02	1.58	0.327	0.09	0.033
u^5	0.072	0.02	1.65	0.203	0.11	0.045

Conclusion

- An attempt to provide an allocation based on the minimization of a risk indicator. The indicator should be specifically designed for its use.
- The orange area is NOT suited for allocation of SCR, which should be done according to the risk contribution of each branch, while the violet area is more suited for allocation of SCR / economic capital.
- Efficient algorithm to compute the allocation, even in high dimension.
- Trackable on several periods.

Further practical work

- Analysis on several periods to be done.
- Rule of the penalty functions g_k.
- Rule of the various parameters $(u, \alpha, \ldots)$, how to construct a control card?
- Impact of the dependency on time.
- Preliminary simulations done for the *stopped* orange area seem to indicate that there is no significant differences with the orange area, unless in the mixed case. This kind of behavior should be analyzed.
- ...