Computing the complexity of the relation of isometry between separable Banach spaces

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Abstract

We compute here the Borel complexity of the relation of isometry between separable Banach spaces, using results of Gao, Kechris [1] and Mayer-Wolf [4]. We show that this relation is Borel bireducible to the universal relation for Borel actions of Polish groups.

1 Introduction

Over the past fifteen years or so, the theory of complexity of Borel equivalence relations has been a very active field of research; in this paper, we compute the complexity of a relation of geometric nature, the relation of (linear) isometry between separable Banach spaces. Before stating precisely our result, we begin by quickly recalling the basic facts and definitions that we need in the following of the article; we refer the reader to [3] for a thorough introduction to the concepts and methods of descriptive set theory. Before going on with the proof and definitions, it is worth pointing out that this article mostly consists in putting together various results which were already known, and deduce from them after an easy manipulation the complexity of the relation of isometry between separable Banach spaces. Since this problem has been open for a rather long time, it still seems worth it to explain how this can be done, and give pointers to the literature. Still, it seems to the author that this will be mostly of interest to people with a knowledge of descriptive set theory, so below we don't recall descriptive set-theoretic notions but explain quickly what Lipschitz-free Banach spaces are and how that theory works.

Notations definitions.

We use the standard notations and terminology from descriptive set theory (see [3]). The only slight variation from usual terminology is the notion of

Polish metric space, i.e a metric separable complete metric space (X,d).
If X is a metric space, we denote by $Iso(X)$ its isometry group, endowed with the pointwise convergence topology.
Up to isometry, Urysohn's universal metric space $\mathbb{U}$, first constructed by Urysohn in [6], is the only Polish metric space with is both *universal* (any separable metric space embeds in it) and *ultrahomogeneous* (any isometry between two finite subsets F_1, F_2 of $\mathbb{U}$ extends to an isometry of $\mathbb{U}$). This space has generated an increasing interest over the past few years; for more information and bibliographical references, we refer the reader to [1] or the more recent [5]. Here, we use the Urysohn space because of results by Gao and Kechris [1]. To state these, we first need to point out that, since any Polish metric space is isometric to some closed set $F \in \mathcal{F}(\mathbb{U})$, one may consider $\mathcal{F}(\mathbb{U})$ (with the Effros Borel structure), as being the (Borel) space of Polish spaces. One easily checks that the relation of isometry $\simeq_i$ (defined on $\mathcal{F}(\mathbb{U})$, endowed with the Effros Borel structure) is analytic, where

$$(P \simeq_i P') \Leftrightarrow (P \text{ and } P' \text{ are isometric}) \ .$$

To compute the exact complexity of this relation, Gao and Kechris considered the relation $\simeq_i^{\mathbb{U}}$ defined, for $P, P' \in \mathcal{F}(\mathbb{U})$, by

$$(P \simeq_i^{\mathbb{U}} P') \Leftrightarrow (\exists \varphi \in Iso(\mathbb{U})\ \varphi(P) = P') \ .$$

Using a variation of Katětov's construction of $\mathbb{U}$ (cf. [2]), they proved that $(\simeq_i) \sim_B (\simeq_i^{\mathbb{U}})$, and that $\simeq_i^{\mathbb{U}}$ is Borel bireducible to the universal relation for relations induced by a Borel action of a Polish group.

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2 Lipschitz-free Banach spaces

We begin by briefly presenting the basic facts of the theory of Lipschitz-free Banach spaces; we follow the second chapter of [7]. The interested reader may find more informations about these spaces and bibliographical references in [7].

If (X,d,e) is a pointed metric space, one lets $Lip_0(X,d,e)$ denote the space of Lipschitz functions on X that map e to 0.
One defines a norm on $Lip_0(X,d,e)$ by setting

$$\|f\| = \inf\{k \in \mathbb{R} \colon f \text{ is } k-\text{Lipschitz}\} \ .$$

It is worth noting that, if one chooses another basepoint e', then $Lip_0(X,d,e)$ and $Lip_0(X,d,e')$ are isometric, one possible isometry being given by the mapping $f \mapsto f - f(e')$.
In the following, when no confusion is possible, we forget about d and e and simply write $Lip_0(X)$; we write $[Lip_0(X)]_1$ to denote the closed unit ball of $Lip_0(X)$.
If (X,d) is a metric space, we say that $m\colon X \to \mathbb{R}$ is a *molecule* if m has a finite support, and $\sum_{x\in X} m(x) = 0$. For $p,q \in X$, one may define a molecule m_{pq} by setting $m_{pq} = \chi_{\{p\}} - \chi_{\{q\}}$, where χ_X stands for the characteristic function of X.
For any molecule m, one may find points $p_i, q_i \in X$ and reals a_i such that $m = \sum_{i=1}^{n} a_i m_{p_i q_i}$; we let $\|m\| = \inf\{\sum_{i=1}^{n} |a_i| d(p_i,q_i)\colon m = \sum_{i=1}^{n} a_i m_{p_i q_i}\}$.
Then $\|\cdot\|$ is a seminorm on the space of molecules; the *Lipschitz-free space over* X, denoted by $F(X)$, is the completion (relative to this seminorm) of the space of molecules modulo null vectors (there are actually no null vectors, as we will see shortly). This space is also known in the litterature as the *Arens-Eells space* of X, or *Banach-Kantorovitch* space associated to X.
The following fact is the basis of the theory of Lipschitz-free Banach spaces:

Fact.(see e.g [7]) The spaces $F(X)^*$ and $Lip_0(X)$ are isometric.
The natural isometry $T\colon F(X)^* \to Lip_0(X)$ is defined by

$$\forall x \in X \ (T\phi)(x) = \phi(m_{xe}) \ .$$

(here e is any point in X; recall that, if $e \neq e' \in X$, then $Lip_0(X,d,e)$ and $Lip_0(X,d,e')$ are isometric).
The inverse S of T is defined by the formula

$$(Sf)(m) = \sum_{x\in X} f(x)m(x) \ .$$

Therefore, the Hahn-Banach theorem implies that, for any molecule m,

$$\|m\| = \sup\Big\{ \sum_{x\in X} f(x)m(x)\colon f \in [Lip_0(X)]_1 \Big\} \ .$$

It is interesting to notice that this means that $\|m\|$ is determined by the distances between points in the support of m and the values of m on its support. Indeed, if $Y \subset X$ is a subspace of the metric space X, it is easy to see that any 1-Lipschitz real-valued map on Y extends to a 1-Lipschitz real-valued map on X.

This may be used to check that $||.||$ is in fact a norm on the space of molecules, and that the family $\{m_{xe}\}_{x\in X}$ is linearly independent. Furthermore, one checks easily that $||m_{xy}|| = d(x,y)$ for all $x, y \in X$. Hence, if e is any point in X, then the mapping $x \mapsto m_{xe}$ is an isometric embedding of X in $F(X)$, such that the closed linear span of the image of X is equal to the whole space $F(X)$. In the following, we will be mainly interested in the Lipschitz-free space over the Urysohn space $\mathbb{U}$. Using the results and remarks above, one sees that, if $P \subset \mathbb{U}$ is a Polish metric space containing 0, then the closed linear span of P (in $F(\mathbb{U})$) is (isometric to) the Lipschitz-free space over P. Notice that if X is a closed subset of the separable Banach space B, then the mapping

$$F \in \mathcal{F}(X) \mapsto \overline{span(F)} \in \mathcal{F}(B)$$

is Borel (where both $\mathcal{F}(X)$ and $\mathcal{F}(B)$ are endowed with the Effros Borel structure). So, if one identifies the class of Polish spaces to the set of closed subsets of $\mathbb{U}$ containing 0, and the class of separable Banach spaces to the set of closed subspaces of $F(\mathbb{U})$, then we may see the mapping $X \mapsto F(X)$ as a Borel mapping between two standard Borel spaces.

In our context, one question about Lipschitz-free Banach spaces is of special interest: if X is a Polish metric space, how much of its metric structure is "encoded" in $F(X)$? In other words, if one knows that X, Y are Polish metric spaces such that $F(X), F(Y)$ are isometric, can we find a relation between the metric structures of X, Y ?
Before saying more about this, it is convenient to introduce a new definition: We say that $f \colon X \to Y$ is a *dilatation* if there exists $\lambda > 0$ such that

$$d(f(x), f(x')) = \lambda d(x, x') \ .$$

It is clear that if there is a dilatation from X onto Y then $F(X)$ and $F(Y)$ are isometric. The converse is false in general, but a beautiful result of Mayer-Wolf imply that it holds for a rather large class of spaces; what is interesting for us is that the relation of isometry between all Polish spaces reduces to that of isometry between spaces in the aforementioned subclass.
Weaver [7] says that a Polish space P is *concave* if, for all $p \neq q \in P$, the molecule $\frac{m_{pq}}{d(p,q)}$ is an extreme point in the unit ball of $F(X)^{**}$ (here we use the canonical embedding of $F(X)$ into its bidual).

Theorem. (*Mayer-Wolf [4]*) *Let P and P' be two concave Polish metric spaces, and assume that $F(P)$ and $F(P')$ are isometric. Then there exists a dilatation from P onto P'.*

Weaver also shows that there are many concave Polish metric spaces:

Theorem. (*Weaver [7]*) *Let (P,d) be a Polish metric space. Then $(P,\sqrt{d})$ is concave.*

(actually (P,d^{α}) is concave for any $\alpha \in]0,1[$; we only need this fact for $\alpha=\frac{1}{2}$)
Intuitively, by replacing d by $\sqrt{d}$ (which is easily checked to be a complete distance, compatible with the topology of P), one has "uniformly eliminated" the equality case in the triangle inequality, and this fact is enough to study precisely the structure of the isometries of $F(P)$.
A very simple, yet very important for our construction, fact is that, if (P,d) and (P',d') are two metric spaces, then (P,d) and (P',d') are isometric if, and only if, $(P,\sqrt{d})$ and $(P',\sqrt{d'})$ are isometric.
This shows that one may reduce the relation of isometry between Polish metric spaces to that of isometry between *concave* Polish metric spaces.

3 The construction

As we saw above, the results of Mayer-Wolf and Weaver seem to imply that one may reduce the relation of dilatation between concave Polish metric spaces to the relation of isometry between separable Banach spaces. Furthermore, the relation of isometry between Polish spaces reduces to that of isometry between concave metric spaces.
Now, let us explain in detail how to do this in a Borel way. First, fix the diameter; for this, notice that the results of Gao and Kechris easily imply that the relation induced by the left-translation action of $Iso(\mathbb{U})$ on the subset of $\mathcal{F}(\mathbb{U})$ made up of all the unbounded closed subsets of $\mathbb{U}$ is Borel bireducible to the universal relation for relations induced by a Borel action of a Polish group. Denote $\mathcal{A}$ the set of such subspaces, and $\backsimeq_i$ the corresponding relation; then replace the distance d on $\mathbb{U}$ by $d_1=\dfrac{d}{1+d}$. The sets of $\mathcal{A}$ correspond to the set of closed subsets of $(\mathbb{U},d_1)$ of diameter (exactly) 1, and the isometry relation is unchanged. Then change again the distance, setting this time $d_2=\sqrt{d_1}$: once again the isometry relation is unchanged, and the sets in $\mathcal{A}$ are naturally identified with the subsets of $(\mathbb{U},d_2)$ of diameter exactly 1.
Embed now $Y=(\mathbb{U},d_2)$ in $\mathbb{U}$ (with its usual distance) in such a way that all isometries of Y extend to isometries of $\mathbb{U}$ (this may be done using Katětov's construction, see [2]). Identifying now $\mathcal{A}$ with the corresponding subset of $\mathcal{F}(\mathbb{U})$, we see that $\mathcal{A}$ is made up of concave metric spaces of diameter 1, that $\mathcal{A}$ is standard Borel, and that the relation induced by the left translation action of $\{\varphi \in Iso(\mathbb{U})\colon \varphi(Y)=Y\}$ on $\mathcal{A}$ (which we again denote $\backsimeq_i$, since this is really the same relation) is Borel bireducible to the universal relation for relations induced by an action of a Polish group.

For all $P \in \mathcal{A}$, denote now $\Phi(P)$ the closed linear span of P in $F(\mathbb{U})$. Recall that Φ is Borel, and $\Phi(P)$ is linearly isometric to the Lipschitz-free Banach space over P.

Theorem 3.1. *Let $P, P' \in \mathcal{A}$. Then the following assertions are equivalent:*
(1) $P \backsimeq_i P'$.
(2) $\Phi(P)$ and $\Phi(P')$ are isometric .
(3) There is a linear isometry of $F(\mathbb{U})$ which maps $\Phi(\mathbf{d})$ onto $\Phi(\mathbf{d}')$.

Proof:
(2) $\Rightarrow$ (1) is a direct consequence of the results of Mayer-Wolf, since all the sets in $\mathcal{A}$ are concave metric spaces and have the same diameter.
(3) $\Rightarrow$ (2) is a triviality, and (1) $\Rightarrow$ (3) is a consequence of the fact that any isometry of $\mathbb{U}$ mapping P onto P' extends to a linear isometry of $F(\mathbb{U})$ mapping the closed linear span of P onto that of P'. $\Diamond$

This shows that Φ is a Borel reduction of $\backsimeq_i$ to the relation of linear isometry between Banach spaces. Since two Banach spaces are linearly isometric if and only if they are isometric (as Polish metric spaces), this shows that there is also a reduction in the other direction. Given the results of Gao and Kechris [1], this is enough to compute the exact complexity of the relation of isometry between separable Banach spaces.

Theorem 3.2. *The relation of isometry between separable Banach spaces is Borel bi-reducible to the universal relation for relations induced by a Borel action of a Polish group.*

To conclude this article, it might be worth pointing out that the complexity of the relation of linear *isomorphism* between separable Banach spaces has recently been determined by Ferenczi, Louveau and Rosendal. They proved that this relation is Borel bireducible to the universal relation for *analytic* equivalence relations; this means that it is vastly more complicated that the relation of isometry between separable Banach spaces.

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