Stabilizers of closed sets in the Urysohn space

Julien Melleray

Abstract
Building on earlier work of Katětov, Uspenskij proved in [9] that the group of isometries of Urysohn’s universal metric space \( U \), endowed with the pointwise convergence topology, is a universal Polish group (i.e it contains an isomorphic copy of any Polish group). Answering a question of Gao and Kechris, we prove here the following, more precise result: for any Polish group \( G \), there exists a closed subset \( F \) of \( U \) such that \( G \) is topologically isomorphic to the group of isometries of \( U \) which map \( F \) onto itself.

1 Introduction
In a posthumously published article ([7]), P.S Urysohn constructed a complete separable metric space \( U \) that is universal (meaning that it contains an isometric copy of every complete separable metric space), and \( \omega \)-homogeneous (i.e such that its isometry group acts transitively on isometric \( r \)-tuples contained in it).
In recent years, interest in the properties of \( U \) has greatly increased, especially since V.V Uspenskij, building on earlier work of Katětov, proved in [8] that the isometry group of \( U \) (endowed with the product topology) is a universal Polish group, that is to say any Polish group is isomorphic to a (necessarily closed) subgroup of it.
In [2], S. Gao and A.S Kechris used properties of \( U \) to study the complexity of the equivalence relation of isometry between certain classes of Polish metric spaces; as a side-product of their construction, they proved the beautiful fact that any Polish group is (topologically) isomorphic to the isometry

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group of some Polish space. A consequence of their construction is that, for any Polish group $G$, there exists a sequence $(X_n)$ of closed subsets of $U$ such that $G$ is isomorphic to $\text{Iso}(U, (X_n)) = \{ \varphi \in \text{Iso}(U) : \forall n (\varphi(X_n) = X_n) \}$. This led them to ask the following question (cf [2]): Can every Polish group be represented, up to isomorphism, by a group of the form $\text{Iso}(U, F)$ for a single subset $F \subseteq U$?

The purpose of this article is to provide a positive answer to this question by proving the following theorem:

**Theorem 1.1.** Let $G$ be a Polish group. There exists a closed set $F \subseteq U$ such that $G$ is (topologically) isomorphic to $\text{Iso}(F)$, and every isometry of $F$ is the restriction of a unique isometry of $U$; in particular, $G$ is isomorphic to $\text{Iso}(U, F)$.

This gives a somewhat concrete realization of any Polish group as a subgroup of $\text{Iso}(U)$.

The construction, which will be detailed in section 3, starts with a bounded Polish metric space $X$ such that $G$ is isomorphic to $\text{Iso}(X)$ (the isometry group of $X$, endowed with the product topology) (Gao and Kechris proved that such an $X$ always exist see [2]). Identifying $G$ with $\text{Iso}(X)$, we construct an embedding of $X$ in $U$ and a discrete, unbounded sequence $(x_n) \subseteq U$ such that $F = X \cup \{x_n\}$ has the desired properties (here we identify $X$ with its image via the embedding provided by our construction).

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## 2 Notations and definitions

If $(X, d)$ is a complete separable metric space, we say that it is a *Polish metric space*, and often write it simply $X$.

To avoid confusions, we say, if $(X, d)$ and $(X', d')$ are two metric spaces, that $f$ is an *isometric map* if $d(x, y) = d'(f(x), f(y))$ for all $x, y \in X$; if $f$ is onto, then we say that $f$ is an *isometry*.

A *Polish group* is a topological group whose topology is Polish. If $X$ is a separable metric space, then we denote its isometry group by $\text{Iso}(X)$, and
endow it with the product topology, which turns it into a second countable topological group, and into a Polish group if \( X \) is Polish (see [1] or [5] for a thorough introduction to the theory of Polish groups).

We say that a metric space \( X \) is \textit{finitely injective} iff for any finite subsets \( K \subseteq L \) and any isometric map \( \varphi: K \to X \) there exists an isometric map \( \tilde{\varphi}: L \to X \) such that \( \tilde{\varphi}|_K = \varphi \). Up to isometry, \( U \) is the only finitely injective Polish metric space (see [7]).

Let \((X, d)\) be a metric space; we say that \( f: X \to \mathbb{R} \) is a \textit{Katětov map} iff
\[
\forall x, y \in X \ |f(x) - f(y)| \leq d(x, y) \leq f(x) + f(y).
\]
These maps correspond to one-point metric extensions of \( X \). We denote by \( E(X) \) the set of all Katětov maps on \( X \) and endow it with the sup-metric, which turns it into a complete metric space.

That definition was introduced by Katětov in [4], and it turns out to be pertinent to the study of finitely injective spaces, since one can easily see by induction that a non-empty metric space \( X \) is finitely injective if, and only if,
\[
\forall x_1, \ldots, x_n \in X \ \forall f \in \mathcal{E}(\{x_1, \ldots, x_n\}) \ \exists z \in X \ \forall i = 1 \ldots n \ d(z, x_i) = f(x_i).
\]
If \( Y \subseteq X \) and \( f \in \mathcal{E}(Y) \), define \( \hat{f}: X \to \mathbb{R} \) (the Katětov extension of \( f \)) by
\[
\hat{f}(x) = \inf\{f(y) + d(x, y): y \in Y\}.
\]
Then \( \hat{f} \) is the greatest 1-Lipschitz map on \( X \) which is equal to \( f \) on \( Y \); one checks easily (see for instance [4]) that \( \hat{f} \in E(X) \) and \( f \mapsto \hat{f} \) is an isometric embedding of \( E(Y) \) into \( E(X) \).

To simplify future definitions, we will say that if \( f: X \to \mathbb{R} \) and \( S \subseteq X \) are such that
\[
\forall x \in X \ f(x) = \inf\{f(s) + d(x, s): s \in S\}, \text{ then } S \text{ is a support of } f, \text{ or that } S \text{ controls } f.
\]
Notice that if \( S \) controls \( f \in E(X) \) and \( S \subseteq T \), then \( T \) controls \( f \).

Also, \( X \) isometrically embeds in \( E(X) \) via the Kuratowski map \( x \mapsto f_x \), where \( f_x(y) = d(x, y) \). A crucial fact for our purposes is that
\[
\forall f \in E(X) \ \forall x \in X \ d(f, f_x) = f(x).
\]

Thus, if one identifies \( X \) with its image in \( E(X) \) via the Kuratowski map, \( E(X) \) is a metric space containing \( X \) and such that all one-point metric
extensions of $X$ embed isometrically in $E(X)$. 

We now go on to sketching Katětov’s construction of $U$; we refer the reader to [2], [3], [7] and [9] for a more detailed presentation and proofs of the results we will use below. 

Most important for the construction is the following 

**Theorem 2.1.** (Urysohn) If $X$ is a finitely injective metric space, then the completion of $X$ is also finitely injective. 

Since $U$ is, up to isometry, the unique finitely injective Polish metric space, this proves that the completion of any separable finitely injective metric space is isometric to $U$. 

The basic idea of Katětov’s construction works like this: if one lets $X_0 = X$, $X_{i+1} = E(X_i)$ then, identifying each $X_i$ to a subset of $X_{i+1}$ via the Kuratowski map, let $Y$ be the inductive limit of the sequence $X_i$. 

The definition of $Y$ makes it clear that $Y$ is finitely injective, since any $\{x_1, \ldots, x_n\} \subseteq Y$ must be contained in some $X_m$, so that for any $f \in E(\{x_1, \ldots, x_n\})$ there exists $z \in X_m$ such that $d(z, x_i) = f(x_i)$ for all $i$. 

Thus, if $Y$ were separable, its completion would be isometric to $U$, and one would have obtained an isometric embedding of $X$ into $U$. 

The problem is that $E(X)$ is in general not separable: at each step, we have added too many functions. 

Define then $E(X, \omega) = \{f \in E(X) : f \text{ is controlled by some finite } S \subseteq X\}$. 

$E(X, \omega)$ is easily seen to be separable if $X$ is, and the Kuratowski map actually maps $X$ into $E(X, \omega)$, since each $f_x$ is controlled by $\{x\}$. Notice also that, if $\{x_1, \ldots, x_n\} \subseteq X$ and $f \in E(\{x_1, \ldots, x_n\})$, then its Katětov extension $\hat{f}$ is in $E(X, \omega)$, and $d(\hat{f}, f_x) = f(x_i)$ for all $i$. 

Thus, if one defines this time $X_0 = X$, $X_{i+1} = E(X_i, \omega)$, then the inductive limit $Y$ of $\cup X_i$ is separable and finitely injective, hence its completion $Z$ is isometric to $U$, and $X \subseteq Z$. 

The most interesting property of this construction is that it enables one to keep track of the isometries of $X$: indeed, any $\varphi \in Iso(X)$ is the restriction of a unique isometry $\hat{\varphi}$ of $E(X, \omega)$, and the mapping $\varphi \mapsto \hat{\varphi}$ from $Iso(X)$ into $Iso(E(X, \omega))$ is an isomorphic embedding of topological groups. 

That way, we obtain for all $i$ continuous embeddings $\Psi^i : Iso(X) \to Iso(X_i)$, such that $\Psi^{i+1}(\varphi)|_{X_i} = \Psi^i(\varphi)$ for all $i$ and all $\varphi \in Iso(X)$. 

This in turns defines a continuous embedding from $Iso(X)$ into $Iso(Y)$, and since extension of isometries defines a continuous embedding from the
group of isometry of any metric space into that of its completion (see [8]),
we actually have a continous embedding of $Iso(X)$ into the isometry group
of $Z$, that is to say $Iso(\mathbb{U})$ (and the image of any $\varphi \in Iso(X)$ is actually an
extension of $\varphi$ to $Z$).

3 Proof of the main theorem

To prove theorem 1.1, we will use ideas very similar to those used in [2];
all the notations are the same as in section 2.
We will need an additional definition, which was introduced in [2]:
If $X$ is a metric space and $i \geq 1$, let
$$E(X,i) = \{ f \in E(X) : f \text{ has a support of cardinality } \leq i \}$$

We endow $E(X,i)$ with the sup-metric.
Gao and Kechris proved the following result, of which we will give a new,
slightly simpler proof:

**Theorem 3.1. (Gao-Kechris)**
If $X$ is a Polish metric space and $i \geq 1$ then $E(X,i)$ is a Polish metric
space.

**Proof:**
Notice first that the separability of $E(X,i)$ is easy to prove; we will prove
its completeness by induction on $i$.
The proof for $i = 1$ is the same as in [2]; we include it for completeness.
First, let $(f_n)$ be a Cauchy sequence in $E(X,1)$.
It has to converge uniformly to some Katětov map $f$, and it is enough to
prove that $f \in E(X,1)$.
By definition of $E(X,1)$, there exists a sequence $(y_n)$ such that
$$\forall x \in X \ f_n(x) = f_n(y_n) + d(y_n,x) \quad (*)$$
But then let $\varepsilon > 0$, and let $M$ be big enough that $m,n \geq M \Rightarrow d(f_n,f_m) \leq \varepsilon$.
Then, for $m,n \geq M$, one has
$$2d(y_n,y_m) = (f_n(y_m) - f_m(y_m)) + (f_m(y_n) - f_n(y_n)) \leq 2\varepsilon.$$
This proves that \((y_n)\) is Cauchy, hence has a limit \(y\).

One easily checks that \(f(y) = \lim f_n(y_n)\), so that (*) gives us, letting \(n \to \infty\),

\[\forall x \in X \quad f(x) = f(y) + d(y, x)\]

That does the trick for \(i = 1\); suppose now we have proved the result for 
\(1 \ldots i - 1\), and let \((f_n)\) be a Cauchy sequence in \(E(X, i)\).

By definition, there are \(y^n_1, \ldots, y^n_i\) such that:

\[\forall x \in X \quad f_n(x) = \min_{1 \leq j \leq i} \{f_n(y^n_j) + d(y^n_j, x)\} \quad (**).

Once again, \((f_n)\) converges uniformly to some Katětov map \(f\), and we want
to prove that \(f \in E(X, i)\).

Thanks to the induction hypothesis, we can assume that there is \(\delta > 0\) such
that for all \(n\) and all \(k \neq j \leq i\) one has \(d(y^n_j, y^n_k) \geq 2\delta\) (if not, a subsequence
of \((f_n)\) can be approximated by a Cauchy sequence in \(E(X, i - 1)\), and the
induction hypothesis applies).

Let \(d_n = \min \{f_n(x) : x \in X\}\).

Then \((d_n)\) is Cauchy, so it has a limit \(d \geq 0\); up to some extraction, and
if necessary changing the enumeration of the sequence, we can assume that
there is \(p \geq 1\) and \(\delta' > 0\) such that:

- \(\forall j \leq p \quad f_n(y^n_j) \to d\)
- \(\forall j > p \quad \forall n \quad f_n(y^n_j) > d + \delta'\).

Let \(\varepsilon > 0\), \(\alpha = \min(\delta, \delta', \varepsilon)\) and choose \(M\) big enough that \(n, m \geq M \Rightarrow d(f_n, f_m) < \frac{\alpha}{4}\) and \(|f_n(y^n_j) - d| < \frac{\alpha}{4}\) for all \(j \leq p\).

Then, for \(n, m \geq M\) and \(j \leq p\) one has:

\(f_n(y^n_j) < d + \frac{\alpha}{2}\), so there exists \(k \leq p\) such that \(f_n(y^n_j) = f_n(y^n_k) + d(y^n_k, y^n_j)\).

Such a \(y^n_k\) has to be at a distance strictly smaller than \(\delta\) from \(y^n_j\): there is at
most one \(y^n_k\) that can work, and there is necessarily one. Thus, one obtains,
as in the case \(i = 1\), that \(d(y^n_k, y^n_j) \leq \varepsilon\).

This means that one can assume, choosing an appropriate enumeration, that
for \(k \leq p\) each sequence \((y^n_k)_n\) is Cauchy, hence has a limit \(y_k\).

Define then \(\tilde{f}_n: x \mapsto \min_{1 \leq k \leq p} \{f_n(y^n_k) + d(x, y^n_k)\}\).

\(\tilde{f}_n \in E(X, p)\), and one checks easily, since \(y^n_k \to y_k\) for all \(k \leq p\), that \((\tilde{f}_n)\)
converges uniformly to \(\tilde{f}\), where \(\tilde{f}(x) = \min_{1 \leq k \leq p} \{f_k + d(x, y_k)\}\).

If \(p = i\) then we are finished; otherwise, notice that, using again the induc-
tion hypothesis, we may assume that there is $\eta > 0$ such that
\[ \forall n \forall j > p \quad f_n(y^n_j) < \tilde{f}_n(y^n_j) - \eta \quad (***) . \]
Now define $\tilde{g}_n$ by $\tilde{g}_n(x) = \min\{f_n(y^n_j) + d(x, y^n_j)\}$.

Choose $M$ such that $n, m \geq M \Rightarrow d(f_n, f_m) < \frac{\eta}{4}$ and $d(\tilde{f}_n, \tilde{f}_m) < \frac{\eta}{4}$.

Then (*** ) shows that for all $n, m \geq M$ and all $j > p$,
\[ f_m(y^n_j) \leq f_n(y^n_j) + \frac{\eta}{4} \leq \tilde{f}_n(y^n_j) - \frac{3\eta}{4} \leq \tilde{f}_m(y^n_j) - \frac{\eta}{2} , \]
so that $f_m(y^n_j) = f_m(y^n_j) + d(y^n_j, y^n_k)$ for some $k > p$.

Consequently, for $m, n \geq M$ and $j > p$, $f_m(y^n_j) = \tilde{g}_m(y^n_j); \text{ by definition, } f_m(y^n_j) = \tilde{g}_m(y^n_j)$.

This proves that for all $n, m \geq M$ one has $d(\tilde{g}_n, \tilde{g}_m) \leq d(f_n, f_m)$, so that $(\tilde{g}_n)$ is Cauchy in $E(X, i - p)$, hence has a limit $\tilde{g} \in E(X, i - p)$ by the induction hypothesis.

But then, (** ) shows that, for all $x \in X$, $f(x) = \min(\tilde{f}(x), \tilde{g}(x))$, and this concludes the proof. 

If $Y$ is a nonempty, closed and bounded subset of a metric space $X$, define
\[ E(X, Y) = \{ f \in E(X) \colon \exists d \in \mathbb{R}^+ \forall x \in X \ f(x) = d + d(x, Y) \} \]
$E(X, Y)$ is closed in $E(X)$, and is isometric to $\mathbb{R}^+$.

Now we can go on to the

**Proof of theorem 1.1.**

Essential to our proof is the fact that for every Polish group $G$ there exists a Polish space $(X, d)$ such that $G$ is isomorphic to the group of isometries of $X$ (This result was proved by Gao and Kechris, see [2]).

So, let $G$ be a Polish group, and $X$ be a metric space such that $G$ is isomorphic to $Iso(X)$.

One can assume that $X$ contains more than two points, and $(X, d)$ is bounded, of diameter $d_0 \leq 1. (\text{If not, define } d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}). \text{ Then } (X, d') \text{ is now a bounded Polish metric space with the same topology as } X, \text{ and the isometries of } (X, d') \text{ are exactly the isometries of } (X, d) \text{.} \)
Let $X_0 = X$, and define inductively bounded Polish metric spaces $X_i$, of diameter $d_i$, by:

$$X_{i+1} = \left\{ f \in E(X_i, i) \cup \bigcup_{j < i} E(X_i, X_j) : \forall x \in X_i \ f(x) \leq 2d_i \right\}$$

(We endow $X_{i+1}$ with the sup-metric; since $X_i$ canonically embeds isometrically in $X_{i+1}$ via the Kuratowski map, we assume that $X_i \subseteq X_{i+1}$).

Remark that $d_i \to +\infty$ with $i$, and that each $X_i$ is a Polish metric space.

Let then $Y$ be the completion of $\bigcup_{i \geq 0} X_i$.

The definition of $\bigcup X_i$ makes it easy to see that it is finitely injective, so that $Y$ is isometric to $U$.

Also, any isometry $g \in G$ extends to an isometry of $X_i$, and for any $i$ and $g \in G$ there is a unique isometry $g^i$ of $X_i$ such that $g^i(X_j) = X_j$ for all $j \leq i$ and $g^i|_{X_0} = g$ (same proof as in [4]).

Remark also that the mappings $g \mapsto g^i$, from $G$ to $Iso(X_i)$, are continuous (see [8]).

All this enables us to assign to each $g$ an isometry $g^*$ of $Y$, given by $g^*|_{X_i} = g^i$, and this defines a continuous embedding of $G$ into $Iso(Y)$ (see again [8] for details).

It is important to remark here that, if $f \in X_{i+1}$ is defined by $f(x) = d + d(x, X_j)$ for some $d \geq 0$ and some $j < i$, then $g^*(f) = f$ for all $g \in G$ (This was the aim of the definition of $X_i$; adding ”many” points that are fixed by the action of $G$).

Notice that an isometry $\varphi$ of $Y$ is equal to $g^*$ for some $g \in G$ if, and only if, $\varphi(X_n) = X_n$ for all $n$.

The idea of the construction is then simply to construct a closed set $F$ such that $\varphi(F) = F$ if, and only if, $\varphi(X_n) = X_n$ for all $n$. To achieve this, we will build $F$ as a set of carefully chosen ”witnesses”.

The construction proceeds as follows:

First, let $(k_i)_{i \geq 1}$ be an enumeration of the non-negative integers where every number appears infinitely many times.

Using the definition of the sets $X_i$, we choose recursively for all $i \geq 1$ points $a_i \in \bigcup_{n \geq 1} X_n$ (the witnesses), non-negative reals $e_i$, and a nondecreasing sequence of integers $(j_i)$ such that:
- $e_i \geq 4, \forall i \geq 1 \ e_{i+1} > 4e_i$.
- $\forall i \geq 1 \ j_i \geq k_i, \ a_i \in X_{j_i+1}$ and $\forall x \in X_{j_i}, d(a_i, x) = e_i + d(x, X_{k_i-1})$
- $\forall i \geq 1 \ \forall g \in G \ g^*(a_i) = a_i$.

(This is possible, since at step $i$ it is enough to fix $e_i > \max(4e_{i-1}, \text{diam}(X_{k_i}))$, then find $j_i \geq \max(1 + j_{i-1}, k_i)$ such that $\text{diam}(X_{j_i}) \geq e_i$, and define $a_i \in X_{j_i+1}$ by the equation above; then, by definition of $g^*$ and of $a_i$, one has $g^*(a_i) = a_i$ for all $g \in G$)

Let now $F = X_0 \cup \{a_i\}_{i \geq 1}$; since $X_0$ is complete, and $d(a_i, X_0) = e_i \to +\infty$, $F$ is closed.
We claim that for all $\varphi \in \text{Iso}(Y)$, one has

$$(\varphi(F) = F) \iff (\varphi \in G^*).$$

The definition of $F$ makes one implication obvious.
To prove the converse, we need a lemma:

**Lemma 3.2.** If $\varphi \in \text{Iso}(F)$, then $\varphi(X_0) = X_0$, so that $\varphi(a_i) = a_i$ for all $i$. Moreover, there exists $g \in G$ such that $\varphi = g^*|_F$.

Admitting this lemma for a moment, it is now easy to conclude:
Notice that lemma 3.2 implies that $G$ is isomorphic to the isometry group of $F$, and that any isometry of $F$ extends to $Y$.
Thus, to conclude the proof of theorem 1.1, we only need to show that the extension of a given isometry of $F$ to $Y$ is unique. As explained before, it is enough to show that, if $\varphi \in \text{Iso}(Y)$ is such that $\varphi(F) = F$, then $\varphi(X_n) = X_n$ for all $n \geq 0$.
So, let $\varphi \in \text{Iso}(Y)$ be such that $\varphi(F) = F$.
It is enough to prove that $\varphi(X_n) \supseteq X_n$ for all $n \in \mathbb{N}$ (since this will also be true for $\varphi^{-1}$), so assume that this is not true, i.e. there is some $n \in \mathbb{N}$ and $x \notin X_n$ such that $\varphi(x) \in X_n$.
Let $\delta = d(x, X_n) > 0$ (since $X_n$ is complete), and pick $y \in \bigcup X_m$ such that $d(x, y) \leq \frac{\delta}{4}$.
Then $y \in X_m \setminus X_n$ for some $m > n$; now choose $i$ such that $k_i = n + 1$ and $j_i \geq m$.
Then we know that

$$d(\varphi(y), \varphi(a_i)) = d(y, a_i) = e_i + d(y, X_n) \geq e_i + \frac{3\delta}{4},$$

and
\[ d(a_i, \varphi(y)) \leq d(a_i, \varphi(x)) + d(x, y) \leq e_i + \frac{\delta}{4}, \text{ so that } d(\varphi(a_i), a_i) \geq \frac{\delta}{2}, \] and this contradicts lemma 3.2.

It only remains to give the

**Proof of lemma 3.2:**

Since we assumed that \( X_0 \) has more than two points and \( \text{diam}(X_0) \leq 1 \), the definition of \( F \) makes it clear that

\[ \forall x \in F \ (x \in X_0) \Leftrightarrow (\exists y \in F: 0 < d(x, y) \leq 1) \]

The right part of the equivalence is invariant by isometries of \( F \), so this proves that \( \varphi(X_0) = X_0 \) for any \( \varphi \in \text{Iso}(F) \). In turn, this easily implies that \( \varphi(a_i) = a_i \) for all \( i \geq 1 \).

Thus, if one lets \( g \in G \) be such that \( g|_{X_0} = \varphi|_{X_0} \), we have shown that \( \varphi = g^*|_F \).  

\[ \diamond \]

**References**


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Equipe d’Analyse Fonctionnelle, Université Paris 6
Boîte 186, 4 Place Jussieu, Paris Cedex 05, France.
e-mail: melleray@math.jussieu.fr