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The Steinhaus property and Haar-null sets

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Abstract

It is shown that, if G is an uncountable Polish group and $A \subseteq G$ is a universally measurable set such that $A^{-1}A$ is meager, then the set $T_l(A) = \{\mu \in P(G) : \mu(gA) = 0 \text{ for all } g \in G\}$ is co-meager. In particular, if A is analytic and not left Haar-null, then $1 \in \mathrm{Int}(A^{-1}AA^{-1}A)$.

1. *Introduction*

The purpose of this paper is to show that there exists a satisfactory extension of the classical Steinhaus theorem for an arbitrary Polish group. In order to get the extension, one needs, first, to isolate the appropriate σ-ideal on which the result will be applied. For the class of abelian Polish groups this is the σ-ideal of Haar-null sets, defined by Christensen [2]. However, in non-abelian (and non-locally-compact) Polish groups this σ-ideal is no longer well behaved. Actually, by the results of Solecki in [11], the Steinhaus property of Haar-null sets fails in 'most' non-abelian Polish groups. Notice also that the conclusion of the Steinhaus theorem is rather strong. If $A \subseteq \mathbb{R}$ is of positive Lebesgue measure, then $A - A$ contains a neighborhood of 0. If we relax the conclusion to $A - A$ is not meager, then this is valid in every abelian Polish group.

Remark. We recall that a subset A of a topological space X is said to be *meager* (or *of first category*) if A is covered by a countable union of closed nowhere dense sets. The complement of a meager set is usually referred to as *co-meager*.

To state our result we need some definitions. Let G be a Polish group and let $A \subseteq G$ be a universally measurable set. The set A is said to be *Haar-null* if there exists $\mu \in P(G)$ (that is, μ is a Borel probability measure on G) such that $\mu(g_1Ag_2) = 0$ for all $g_1, g_2 \in G$. It is said to be *left Haar-null* if there exists $\mu \in P(G)$ such that $\mu(gA) = 0$ for all $g \in G$. By the results in [9, 11], the notions of a Haar-null and a left Haar-null set are distinct (however, they obviously agree on abelian groups). We let

$$T(A) = \{\mu \in P(G) : \mu(g_1Ag_2) = 0 \text{ for all } g_1, g_2 \in G\}$$

and

$$T_l(A) = \{\mu \in P(G) : \mu(gA) = 0 \text{ for all } g \in G\}.$$

It is easy to see that, if A is analytic, then both $T(A)$ and $T_l(A)$ are faces (that is, extreme convex subsets) of $P(G)$ with the Baire property. It follows, by [4, Theorem 4], that the sets $T(A)$ and $T_l(A)$ are either meager, or co-meager. A set A is said to be *generically Haar-null* if $T(A)$ is co-meager. The set A is said to be *generically left Haar-null* if $T_l(A)$ is co-meager.

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Remark. We recall that a subset A of a Polish space X is said to be *analytic* if there exists a continuous map $f : \mathbb{N}^{\mathbb{N}} \to X$ with $f(\mathbb{N}^{\mathbb{N}}) = A$. It is a classical result that every Borel subset of a Polish space is analytic. It is also well known that an analytic set that is not meager is actually co-meager in a non-empty open set.

For every Polish group G the class of generically left Haar-null subsets of G forms a σ-ideal. Notice that, if A is not generically left Haar-null, then A should not be considered as a small set (it is null only for a relatively small set of measures). This is indeed true, as the following theorem demonstrates.

Theorem A. *Let G be an uncountable Polish group and let A be a universally measurable subset of G. Assume that $A^{-1}A$ is meager. Then $T_l(A)$ is co-meager.*

Thus, if A is analytic and not generically left Haar-null (in particular, not left Haar-null), then $A^{-1}A$ is non-meager.

The locally compact abelian case of Theorem A can also be derived by the results of Laczkovich in [7], who proved that, if A is not covered by an F_σ Haar-measure zero set, then $A^{-1}A$ is co-meager in a neighborhood of the identity. To see that this implies Theorem A, one invokes [3, Proposition 5] that states that, if G is locally compact and $A \subseteq G$ is covered by an F_σ Haar-null set, then $T_l(A)$ is co-meager. Both Laczkovich's result and the result of Christensen [2] that Haar-null sets satisfy the Steinhaus property in abelian Polish groups are heavily dependent on the classical Steinhaus theorem. The proof of Theorem A follows quite different arguments. It is based on the fact that, if $\mathcal{H}$ is a dense G_δ and hereditary subset of $K(G)$, then this is witnessed in the probabilities of G.

1.1. *Preliminaries*

Our general notation and terminology follows [5]. By $\mathbb{N} = \{0, 1, 2, \ldots\}$ we denote the natural numbers. For any Polish space X, we denote by $K(X)$ the hyperspace of all compact subsets of X with the Vietoris topology and by $P(X)$ the space of all Borel probability measures on X with the weak* topology. Both are Polish (see [5]). If d is a compatible complete metric of X, then by d_{H} we denote the Hausdorff metric on $K(X)$ associated to d, defined by

$$d_{\mathrm{H}}(K, C) = \inf\{\varepsilon > 0 : K \subseteq C_\varepsilon \text{ and } C \subseteq K_\varepsilon\},$$

where $A_\varepsilon = \{x \in X : d(x, A) \leqslant \varepsilon\}$ for every $A \subseteq X$. All balls in $K(X)$ are taken with respect to d_{H} and are denoted by B_{H}. In $P(X)$ we consider the so-called Lévy metric ρ, defined by

$$\begin{aligned}\rho(\mu, \nu) = \inf\{\varepsilon > 0 : \mu(A) \leqslant \nu(A_\varepsilon) + \varepsilon \text{ and } \nu(A) \leqslant \mu(A_\varepsilon) + \varepsilon \\ \text{for every compact (or Borel) subset } A \text{ of } X\}\end{aligned}$$

(see [1] for more details). All balls in $P(X)$ are taken with respect to ρ and are denoted by B_P. If G is a Polish group and $\mu, \nu \in P(G)$, then by $\mu * \nu$ we denote their convolution, defined by

$$\mu * \nu(A) = \int_G \mu(Ax^{-1}) d\nu(x).$$

A subset $\mathcal{H}$ of $K(X)$ is said to be *hereditary* if for every $K \in \mathcal{H}$ and every $C \in K(X)$ with $C \subseteq K$ then we have that $C \in \mathcal{H}$. All the other pieces of notation that we use are standard.

2. *Hereditary, dense G_δ sets and measures*

Throughout this section X will be a Polish space and $\mathcal{H}$ a hereditary, dense G_δ subset of $K(X)$. By d we denote a compatible complete metric of X.

LEMMA 1. *Let X and $\mathcal{H}$ be as above. Then there exists a sequence $(\mathcal{U}_n)$ of open, dense and hereditary subsets of $K(X)$ such that $\mathcal{H} = \bigcap_n \mathcal{U}_n$.*

Proof. Write $\mathcal{H} = \bigcap_n \mathcal{V}_n$, where each $\mathcal{V}_n$ is open and dense but not necessarily hereditary. Fix n and define

$$\mathcal{C}_n = \{K \in K(X) : \exists C \subseteq K \text{ compact with } C \notin \mathcal{V}_n\}.$$

It is easy to check that $\mathcal{C}_n$ is closed and $\mathcal{C}_n \cap \mathcal{H} = \varnothing$. Therefore, if we set $\mathcal{U}_n = K(X) \setminus \mathcal{C}_n$, then we see that the sequence $(\mathcal{U}_n)$ has all the desired properties. □

In what follows we will say that the sequence $(\mathcal{U}_n)$ obtained by Lemma 1 is the *normal form* of $\mathcal{H}$. We need the following lemmas.

LEMMA 2. *Let $\mathcal{U} \subseteq K(X)$ be open, dense and hereditary. Also let $x_0, \ldots, x_n$ be the distinct points in X and $r_1 > 0$. Then there exist $y_0, \ldots, y_n$ distinct points in X such that $d(x_i, y_i) < r_1$ for all $i \in \{0, \ldots, n\}$ and, moreover, $\{y_0, \ldots, y_n\} \in \mathcal{U}$.*

Proof. We may assume that $B(x_i, r_1) \cap B(x_j, r_1) = \varnothing$ for all $i, j \in \{0, \ldots, n\}$ with $i \neq j$. Let

$$\mathcal{V} = \left\{K : K \subseteq \bigcup_{i=0}^{n} B(x_i, r_1) \text{ and } K \cap B(x_i, r_1) \neq \varnothing\ \forall i = 0, \ldots, n\right\}.$$

Then $\mathcal{V}$ is open. As $\mathcal{U}$ is open and dense, there exists $K \in \mathcal{V} \cap \mathcal{U}$. For every $i \in \{0, \ldots, n\}$ we select $y_i \in K \cap B(x_i, r_1)$. As $\mathcal{U}$ is hereditary, we see that $\{y_0, \ldots, y_n\} \in \mathcal{U}$. Clearly, $y_0, \ldots, y_n$ are as desired. □

LEMMA 3. *Let $\mathcal{U} \subseteq K(X)$ be open, dense and hereditary. Also let $\varepsilon > 0$. Then the set*

$$G_{\mathcal{U},\varepsilon} = \{\mu \in P(X) : \exists K \in \mathcal{U} \text{ with } \mu(K) \geqslant 1 - \varepsilon\}$$

is co-meager in $P(X)$.

Proof. Fix $\mathcal{U}$ and $\varepsilon > 0$ as above. We will show that for every $V \subseteq P(X)$ open there exists $W \subseteq V$ open such that $W \subseteq G_{\mathcal{U},\varepsilon}$. This completes the proof (actually, it implies that $G_{\mathcal{U},\varepsilon}$ contains a dense open set). Therefore, let $V \subseteq P(X)$ be open. As finitely supported measures are dense in $P(X)$, we may select $\nu = \sum_{i=0}^n a_i \delta_{x_i}$ and $r > 0$ such that the following hold:

(1) $a_i > 0$ for all $i \in \{0, \ldots, n\}$ and $\sum_{i=0}^n a_i = 1$;
(2) $B_P(\nu, r) \subseteq V$.

By Lemma 2, there exist $y_0, \ldots, y_n$ distinct points in X with $\{y_0, \ldots, y_n\} \in \mathcal{U}$ such that $d(x_i, y_i) < r/2$ for all $i \in \{0, \ldots, n\}$. We set $\mu = \sum_{i=0}^n a_i \delta_{y_i}$. Then it is easy to see that we have the following:

(3) $\rho(\mu, \nu) \leqslant r/2$.

Let $F = \{y_0, \dots, y_n\}$. As $\mathcal{U}$ is open and $F \in \mathcal{U}$ there exists $\theta > 0$ such that the following hold:

(4) $\theta < \min\{\varepsilon/3, r/3\}$;
(5) $B_{\mathrm{H}}(F, 2\theta) \subseteq \mathcal{U}$.

Then $W = B_P(\mu, \theta)$ is as desired. Indeed, by (2)–(4) it is clear that W is a subset of V. We only need to check that W is a subset of $G_{\mathcal{U},\varepsilon}$. Let $\lambda \in W$ be arbitrary. Then $\rho(\lambda, \mu) < \theta$ and so

$$1 = \mu(F) \leqslant \lambda(F_\theta) + \theta,$$

which gives that $\lambda(F_\theta) \geqslant 1 - \varepsilon/3$ by the choice of θ. By the inner regularity of λ, there exists $C \subseteq F_\theta$ compact such that $\lambda(C) \geqslant 1 - \varepsilon$. We set $K = C \cup F$. Then $d_H(K, F) \leqslant \theta$ and so, by (5), $K \in \mathcal{U}$. Moreover, $\lambda(K) \geqslant \lambda(C) \geqslant 1 - \varepsilon$. This implies that $\lambda \in G_{\mathcal{U},\varepsilon}$ and the proof is completed. □

Our aim in this section is to prove the following proposition.

PROPOSITION 4. *Let $\mathcal{H}$ be a hereditary, dense G_δ subset of $K(X)$. Then the set*

$$G_{\mathcal{H}} = \{\mu \in P(X) : \forall \varepsilon > 0\ \exists K \in \mathcal{H} \text{ with } \mu(K) \geqslant 1 - \varepsilon\}$$

is co-meager in $P(X)$.

Proof. Let $(\mathcal{U}_n)$ be the normal form of $\mathcal{H}$. For every $n, m \in \mathbb{N}$ let

$$G_{n,m} = \left\{\mu \in P(X) : \exists K \in \mathcal{U}_n \text{ with } \mu(K) \geqslant 1 - \frac{1}{m+1}\right\}.$$

By Lemma 3, we have that $G_{n,m}$ is co-meager. Hence, so is $\bigcap_{n,m} G_{n,m}$. We claim that $G_{\mathcal{H}} = \bigcap_{n,m} G_{n,m}$. This completes the proof. It is clear that $G_{\mathcal{H}} \subseteq \bigcap_{n,m} G_{n,m}$. Conversely, fix $\mu \in \bigcap_{n,m} G_{n,m}$ and let $\varepsilon > 0$ be arbitrary. Choose a sequence (ε_n) of positive reals such that

$$\sum_{n \in \mathbb{N}} \varepsilon_n < \frac{\varepsilon}{2}.$$

Choose also a sequence (m_n) of natural numbers with $1/(m_n + 1) \leqslant \varepsilon_n$ for every $n \in \mathbb{N}$. As

$$\mu \in \bigcap_{n,m} G_{n,m} \subseteq \bigcap_n G_{n,m_n},$$

we may select a sequence (K_n) in $K(X)$ such that the following hold:

(1) $K_n \in \mathcal{U}_n$;
(2) $\mu(K_n) \geqslant 1 - 1/(m_n + 1) \geqslant 1 - \varepsilon_n$.

For every $n \in \mathbb{N}$ we let $F_n = \bigcap_{i=0}^n K_i$ and we set $F = \bigcap_n K_n$. Then $F_n \downarrow F$. Notice that $F \in \mathcal{U}_n$ as $F \subseteq F_n \subseteq K_n \in \mathcal{U}_n$ and $\mathcal{U}_n$ is hereditary. Hence, $F \in \bigcap_n \mathcal{U}_n = \mathcal{H}$. Moreover, by (2) above, we have

$$\mu(F_n) = \mu(K_0 \cap \ldots \cap K_n) \geqslant 1 - \sum_{k=0}^n \varepsilon_k.$$

As $F_n \downarrow F$, we obtain that

$$\mu(F) = \lim_{n \in \mathbb{N}} \mu(F_n) \geqslant 1 - \sum_{n \in \mathbb{N}} \varepsilon_n \geqslant 1 - \varepsilon.$$

This shows that $\mu \in G_{\mathcal{H}}$, as desired. □

3. *Left Haar-null sets in Polish groups*

Our aim is to give the proof of Theorem A stated in the introduction.

Proof of Theorem A. Let G be an uncountable Polish group and let A be a universally measurable subset of G such that $A^{-1}A$ is meager. We select a sequence (C_n) of closed, nowhere dense subsets of G with the following properties:

(i) $1 \notin C_n$ for all $n \in \mathbb{N}$;
(ii) $A^{-1}A \setminus \{1\} \subseteq \bigcup_n C_n$.

For every $n \in \mathbb{N}$ let

$$\mathcal{U}_n = \{K \in K(G) : K^{-1}K \cap C_n = \varnothing\}.$$

Clearly, every $\mathcal{U}_n$ is hereditary. Moreover, as the function $f : K(G) \to K(G)$ defined by $f(K) = K^{-1}K$ is continuous, we see that every $\mathcal{U}_n$ is open.

Claim 5. For every $n \in \mathbb{N}$ the set $\mathcal{U}_n$ is dense in $K(G)$.

Proof. As finite sets are dense in $K(G)$, it is enough to show that for every finite subset $\{x_0, \dots, x_l\}$ of G and every $r > 0$ there exist $y_0, \dots, y_l$ distinct points in G with

$$\{y_i^{-1}y_j : i, j \in \{0, \dots, l\} \text{ with } i \neq j\} \cap C_n = \varnothing$$

such that $d(x_i, y_i) \leqslant r$ for all $i \in \{0, \dots, l\}$ (here d is simply a compatible complete metric of G). The points $y_0, \dots, y_l$ will be chosen by recursion. We set $y_0 = x_0$. Assume that $y_0, \dots, y_k$ have been chosen for some $k < l$ so that $\{y_i^{-1}y_j : i, j \in \{0, \dots, k\} \text{ with } i \neq j\} \cap C_n = \varnothing$. For every $g \in G$ the functions $x \mapsto gx^{-1}$ and $x \mapsto gx$ are homeomorphisms. It follows that the set $F_k = \bigcup_{i=0}^{k}(y_i C_n^{-1} \cup y_i C_n)$ is a closed set with empty interior. Hence, there exists $y_{k+1} \in B(x_{k+1}, r)$ such that $y_{k+1} \notin F_k \cup \{y_0, \dots, y_k\}$. This implies that for every $i \in \{0, \dots, k\}$ we have $y_{k+1}^{-1}y_i \notin C_n$ and $y_i^{-1}y_{k+1} \notin C_n$. This completes the recursive selection and the proof of the claim is completed. □

It follows by the above claim that the set $\mathcal{H} = \bigcap_n \mathcal{U}_n$ is a hereditary, dense G_δ subset of $K(G)$ and that $(\mathcal{U}_n)$ is a normal form of $\mathcal{H}$. Notice that, if $K \in \mathcal{H}$, then $K^{-1}K \cap A^{-1}A = \{1\}$. By Proposition 4, we have that the set

$$B_1 = \{\mu \in P(G) : \forall \varepsilon > 0 \ \exists K \in \mathcal{H} \text{ with } \mu(K) \geqslant 1 - \varepsilon\}$$

is co-meager. Our assumption that G is uncountable implies that the Polish group G viewed as a topological space is perfect. Hence, the set of all non-atomic Borel probability measures on G is co-meager in $P(G)$ (see [**6**, **8**]). It follows that the set

$$B_2 = \{\mu \in P(G) : \mu \text{ is non-atomic and } \mu \in B_1\}$$

is co-meager in $P(G)$. We will show that $B_2 \subseteq T_l(A)$. This completes the proof. We need the following fact (its easy proof is left to the reader).

Fact 6. *Let $\mu \in P(G)$. Then $\mu \in T_l(A)$ if and only if for every $\nu \in P(G)$ we have $\nu * \mu(A) = 0$.*

Fix $\mu \in B_2$. By the above fact, in order to verify that $\mu \in T_l(A)$ we have to show that $\nu * \mu(A) = 0$ for every $\nu \in P(G)$. Therefore, let $\nu \in P(G)$ be arbitrary. Also, let $\varepsilon > 0$ be arbitrary.

As $\mu \in B_2 \subseteq B_1$, there exists $K \in \mathcal{H}$ with $\mu(K) \geqslant 1 - \varepsilon$. Then

$$\nu * \mu(A) = \int_G \nu(Ay^{-1})d\mu(y) \leqslant \int_K \nu(Ay^{-1})d\mu(y) + \mu(G \setminus K)$$
$$\leqslant \int_K \nu(Ay^{-1})d\mu(y) + \varepsilon.$$

We set $I = \{y \in K : \nu(Ay^{-1}) > 0\}$.

Claim 7. The set I is countable.

Proof. Notice that, if $y, z \in I$ with $y \neq z$, then $Ay^{-1} \cap Az^{-1} = \varnothing$. For, if not, then we would have that $1 \neq y^{-1}z \in K^{-1}K \cap A^{-1}A$, which contradicts the fact that $K \in \mathcal{H}$. It follows that the family $\{Ay^{-1} : y \in I\}$ is a family of pairwise disjoint sets of positive ν-measure. Hence, I is countable, as claimed. □

The measure μ is non-atomic as $\mu \in B_2$. Hence, by Claim 7, we see that $\mu(I) = 0$. It follows that

$$\int_K \nu(Ay^{-1})d\mu(y) = \int_I \nu(Ay^{-1})d\mu(y) \leqslant \mu(I) = 0,$$

and so $\nu * \mu(A) \leqslant \varepsilon$. Since ε is arbitrary, this implies that $\nu * \mu(A) = 0$. The proof of Theorem A is completed. □

Combining Theorem A with Pettis' theorem (see [**5**, Theorem 9.9]) we obtain the following corollary.

Corollary 8. *Let G be an uncountable Polish group and let A be an analytic subset of G. If A is not generically left Haar-null (in particular, if A is not left Haar-null), then $1 \in \mathrm{Int}(A^{-1}AA^{-1}A)$.*

Clearly, Theorem A implies that, in non-locally-compact groups, compact sets are generically left Haar-null. Another application of this form concerns the size of analytic subgroups of Polish groups. Specifically, we have the following corollary that may be considered as the non-locally-compact analog of Laczkovich's theorem [**7**].

Corollary 9. *Let G be an uncountable Polish group and let H be an analytic subgroup of G with empty interior. Then H is generically left Haar-null.*

What about Haar-null sets? We would like to remark on the possibility of extending Theorem A to Haar-null sets, instead of merely left Haar-null. As it has been shown by Solecki in [**11**], that the Steinhaus property of the σ-ideal of Haar-null sets fails in a large number of Polish groups (in a sense, it fails for most non-abelian Polish groups). Precisely, by [**11**, Theorem 6.1], if (H_n) is a sequence of countable groups such that infinitely many of them are not FC (see [**11**] for the definition of FC groups), then one can find a closed set $A \subseteq \prod_n H_n$ that is not Haar-null and $A^{-1}A$ is meager. Therefore, there is no analog of Theorem A for Haar-null sets in arbitrary Polish groups. Yet there is one if we further assume that the group G satisfies the following non-singularity condition.

(C) For every analytic and meager subset A of G, the conjugate saturation $[A] = \{x : \exists g \in G\ \exists a \in A \text{ with } x = gag^{-1}\}$ of A is meager.

Clearly, every abelian Polish group satisfies (C). Moreover, we have the following proposition.

PROPOSITION 10. *Let G_1 and G_2 be Polish groups. If both G_1 and G_2 satisfy* (C), *then so does $G_1 \times G_2$.*

Proof. Let $A \subseteq G_1 \times G_2$ be analytic and meager. By the Kuratowski–Ulam theorem (see [5, Theorem 8.41]), we have that

$$\forall^* x \in G_1 \text{ the section } A_x = \{y \in G_2 : (x, y) \in A\} \text{ of } A \text{ is meager.}$$

As G_2 satisfies (C), by another application of the Kuratowski–Ulam theorem we obtain that

$$A_1 = \{(x, z) : \exists g_2, y \in G_2 \text{ with } (x, y) \in A \text{ and } y = g_2 z g_2^{-1}\}$$

is analytic and meager. With the same reasoning, we see that the set

$$A_2 = \{(w, z) : \exists g_1, x \in G_1 \text{ with } (x, z) \in A_1 \text{ and } x = g_1 w g_1^{-1}\}$$

is analytic and meager too. Noticing that $A_2 = [A]$, the result follows. □

For groups that satisfy (C) we have the following strengthening of Theorem A.

PROPOSITION 11. *Let G be an uncountable Polish group that satisfies (C). If A is an analytic subset of G such that $A^{-1}A$ is meager, then $T(A)$ is co-meager.*

Proof. The proof is similar to the proof of Theorem A, and so we shall only indicate the necessary changes. Let $A \subseteq G$ be analytic such that $A^{-1}A$ is meager. Notice that $A^{-1}A$ is analytic. The group G satisfies (C). It follows that the set $[A^{-1}A]$ is meager too. Arguing as in the proof of Theorem A, this implies that there exists a co-meager set B_2 of non-atomic Borel probability measures on G such that, for every $\mu \in B_2$ and every $\varepsilon > 0$, there exists $K \subseteq G$ compact with $\mu(K) \geqslant 1 - \varepsilon$ and $K^{-1}K \cap [A^{-1}A] = \{1\}$. We claim that $B_2 \subseteq T(A)$. To this end, it is enough to show that for every $\mu \in B_2$, every $\nu \in P(G)$ and every $x \in G$ we have $\nu * \mu(Ax) = 0$. Let $\varepsilon > 0$ be arbitrary and choose $K \subseteq G$ compact as described above. Then

$$\nu * \mu(Ax) \leqslant \int_K \nu(Axy^{-1}) d\mu(y) + \varepsilon.$$

We set $I = \{y \in K : \nu(Axy^{-1}) > 0\}$. Observe that, if $y, z \in I$ with $y \neq z$, then $(Axy^{-1}) \cap (Axz^{-1}) = \varnothing$ (for, if not, then we would have that $1 \neq y^{-1}z \in K^{-1}K \cap [A^{-1}A]$). By the countable chain condition of ν, we obtain that I is countable and the result follows. □

REMARK 1. The σ-ideal of generically left Haar-null sets is a quite satisfactory σ-ideal of measure-theoretic small sets in arbitrary Polish groups. Besides Theorem A, this is also supported by the results in [3] asserting that every analytic and generically left Haar-null subset A of G can be covered by a *Borel* set B with the same property. The fact that this ideal is well behaved is also reflected in the complexity of the collection of all closed generically left Haar-null sets (in the Effros–Borel structure). It is much better than the one of closed Haar-null sets, at least in abelian Polish groups. Specifically, it follows by the results of Solecki in [10] that, in non-locally-compact abelian Polish groups, the σ-ideal of closed generically Haar-null sets is $\mathbf{\Pi}^1_1$-complete. The corresponding collection of closed Haar-null sets is much more complicated (it is both $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$-hard).

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