DYNAMICAL SIMPLICES AND FRAÏSSÉ THEORY

JULIEN MELLERAY

ABSTRACT. We simplify a criterion (due to Ibarlucía and the author) which characterizes dynamical simplices, that is, sets K of probability measures on a Cantor space X for which there exists a minimal homeomorphism of X whose set of invariant measures coincides with K. We then point out that this criterion is related to Fraïssé theory, and use that connection to provide a new proof of Downarowicz' theorem stating that any Choquet simplex is affinely homeomorphic to a dynamical simplex. The construction enables us to prove that there exist minimal homeomorphisms of a Cantor space which are speedup equivalent but not orbit equivalent, answering a question of D. Ash.

1. INTRODUCTION

In this paper, we continue investigations initiated in [IM] concerning *dynamical simplices*, that is, sets K of probability measures on a Cantor space X such that there exists a minimal (i.e. such that all orbits are dense) homeomorphism of X whose set of invariant measures coincides with K. Dynamical simplices are natural invariants of orbit equivalence and, in fact, a famous theorem of Giordano–Putnam–Skau [GPS] asserts that they are complete invariants of orbit equivalence for minimal homeomorphisms (see section 5 for details on orbit equivalence, speedups, and the Giordano–Putnam–Skau theorem). The main theorem of [IM] is the following.

Theorem (Ibarlucía–Melleray [IM])**.** *Let X be a Cantor space. A set $K \subset P(X)$ is a dynamical simplex if and only if:*

1. *K is compact and convex.*
2. *All elements of K are atomless and have full support.*
3. *K satisfies the* Glasner–Weiss condition*: whenever A, B are clopen subsets of X and $\mu(A) < \mu(B)$ for all $\mu \in K$, there exists a clopen $C \subsetneq B$ such that $\mu(A) = \mu(C)$ for all $\mu \in K$.*
4. *K is* approximately divisible*: for any clopen A, any integer n, and any $\varepsilon > 0$, there exists a clopen $B \subset A$ such that $\mu(A) - \varepsilon \leq n\mu(B) \leq \mu(A)$*

The first two conditions are obviously necessary for K to be the set of invariant measures for a minimal action of any group by homeomorphisms of X; the fact that the third one is necessary follows from a theorem of Glasner–Weiss [GW], hence the terminology we adopt here. When [IM] was completed, the status of approximate divisibility was more ambiguous: on the one hand, it played a key part in the arguments; on the other hand, this assumption seems rather technical, and it is not hard to see that when K has finitely many extreme points approximate divisibility is a consequence of the three other conditions in the above theorem. Thus it was asked in [IM] whether this assumption is really necessary; we prove here that it is in fact redundant, thus simplifying the characterization of a dynamical simplex and giving it its final form.

Theorem. *Let X be a Cantor space. Assume that $K \subset P(X)$ is compact and convex; all elements of K are atomless and have full support; and K satisfies the Glasner–Weiss condition. Then K is approximately divisible (hence K is a dynamical simplex).*

This theorem is obtained as a corollary of the following result, which is of independent interest.

Theorem. *Let X be a Cantor space, and G be a group of homeomorphisms of X such that each G-orbit is dense. Then the set of all G-invariant Borel probability measures on X is approximately divisible.*

The proof of this result is similar in spirit to some of the arguments used in [IM]. A more novel aspect of the work presented here is that we exploit a connection between dynamical simplices and Fraïssé theory, which enables us to build interesting examples. Along those lines, we obtain a new and rather elementary proof of a well-known theorem of Downarowicz.

Theorem (Downarowicz [D])**.** *For any metrizable Choquet simplex K, there exists a minimal homeomorphism φ of a Cantor space X such that K is affinely homeomorphic to the set of all φ-invariant Borel probability measures on X.*

An interesting aspect of the construction used to prove this result is its flexibility; we exploit this to prove the following theorem.

Theorem. *There exist two dynamical simplices K, L of a Cantor space X, and homeomorphisms $g, h \in \mathrm{Homeo}(X)$ such that $g_*K \subset L$ and $h_*L \subset K$, yet there is no $f \in \mathrm{Homeo}(X)$ such that $f_*K = L$.*

This answers a question recently raised by Ash [A] and shows that the relation of speedup equivalence is strictly coarser than the relation of orbit equivalence.

The paper is organized as follows: we first prove that approximate divisibility is redundant in the characterization of dynamical simplices. Then we outline some basics of Fraïssé theory and explain why this theory is relevant to the study of dynamical simplices. We exploit that connection to prove Downarowicz's theorem on realizability of Choquet simplices as dynamical simplices, and adapt this construction to show that there exist minimal homeomorphisms which are speedup equivalent but not orbit equivalent.

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2. Approximate divisibility of simplices of invariant measures

Throughout this section X stands for a Cantor space, and $P(X)$ denotes the space of Borel probability measures on X, with its usual topology (induced by the maps $\mu \mapsto \mu(A)$ as A runs over all clopen subsets of X). Our objective in this section is to prove the following theorem, and then deduce from it that approximate divisibility is redundant in the characterization of a dynamical simplex.

Theorem 2.1. *Let G be a group of homeomorphisms of a Cantor space X such that each G-orbit is dense. Then the set K_G of all G-invariant Borel probability measures on X is approximately divisible.*

For the remainder of this section, we fix G as above.

Definition 2.2. Fix an integer N, and a clopen set A. A *N-dividing partition* of A is a finite clopen partition $(U_{i,j})_{i\in I, j\in\{0,\dots,n_i\}}$ of A such that $n_i \geq N$ for all i and there exists $g^i_1,\dots,g^i_{n_i}$ in G such that $g^i_j(U_{i,0}) = U_{i,j}$ for all i and all $j \in \{1,\dots,n_i\}$.

By analogy with Kakutani–Rokhlin partitions we say that $\{U_{i,j} \colon 0 \leq j \leq n_i\}$ is a column of the partition, with base $U_{i,0}$ (though the actual ordering of the partition does not matter here).

There are two operations on N-dividing partitions which will be useful to us. Given an N-dividing partition $\mathcal{U}$, a column $\mathcal{C}$ of $\mathcal{U}$ with base $U_{i,0}$, and a clopen partition $V_1,\dots,V_k$ of $U_{i,0}$, we can form a new partition by replacing $\mathcal{C}$ with columns $(V_1, g^i_1(V_1),\dots,g^i_{n_i}(V_1)),\dots,(V_k, g^i_1(V_k),\dots,g^i_{n_i}(V_k))$. The other operation is that, when $\mathcal{U}$ covers a clopen set A, V is a clopen subset disjoint from A, $U_{i,j}$ is an atom of $\mathcal{U}$ and there exists $g \in G$ such that $g(U_{i,j}) = V$, then we may extend $\mathcal{U}$ to a N-dividing partition that covers $A \cup V$ by setting $U_{i,n_i+1} = V$ and $g^i_{n_i+1} = g g^i_j$; we say that we have added V on top of the i-th column of $\mathcal{U}$.

If $\mathcal{U}'$ is obtained from $\mathcal{U}$ by applying these two operations finitely many times, we say that $\mathcal{U}'$ *refines* $\mathcal{U}$.

Lemma 2.3. *Let $\mathcal{U} = (U_{i,j})_{i\in I, j\in\{0,\dots,n_i\}}$ be a N-dividing partition of some clopen A, and $(V_0, h_1(V_0),\dots,h_m(V_0))$ be a N-dividing partition of some clopen set B. Then there exists a N-dividing partition $\mathcal{U}'$ which refines $\mathcal{U}$, $W_0 \subseteq V_0$ which is disjoint from all the elements of $\mathcal{U}'$, and such that the union of $\mathcal{U}'$ and $(W_0, h_1(W_0),\dots,h_m(W_0))$ still covers $A \cup B$.*

Note that the only difficulty is that A and B may not be disjoint, and the two N-dividing partitions may overlap, requiring a bit of care.

Proof. As above we denote by (g^i_j) some elements of G witnessing that $\mathcal{U}$ is a N-dividing partition. Let $V = V_0 \cap A$; if $V = \emptyset$ we have nothing to do as $W_0 = V_0$ works. Otherwise we set $W_0 = V_0 \setminus V$. We then let $\mathcal{P}$ denote the clopen partition of V generated by $V \cap h_1^{-1}(A),\dots,V \cap h_m^{-1}(A)$. Note that for any atom C of $\mathcal{P}$ and all j, $h_j(C)$ is either contained in A or disjoint from it. Then, using the maps g^i_j, we may refine $\mathcal{U}$ (by partitioning the base of each column and replicating the corresponding partition along the column via g^i_j) to form a new N-dividing partition $\mathcal{V}$ which refines $\mathcal{U}$ and is finer than $\mathcal{P} \cup \{A \setminus V\}$. This implies that, for any atom C of $\mathcal{V}$, either C is contained in V or is disjoint from V; and if C is contained in V then each of $C, h_1(C),\dots,h_m(C)$ is either contained in A or disjoint from it.

Given one of the columns of $\mathcal{V}$ which meets V, with base $V_{i,0}$, list the atoms $C_1,\dots,C_k$ in this column that are contained in V; for each $l \in \{1,\dots,k\}$ let $J_l \subseteq \{1,\dots,m\}$ denote the set of all indices j such that $h_j(C_l) \cap A = \emptyset$. We add each $h_j(C_l)$ for $j \in J_l$ on top of our column (if J_l is empty the colum is not modified), which is fine since these sets are pairwise disjoint, are also disjoint from all elements of $\mathcal{V}$, and there is an element of G mapping $V_{i,0}$ onto $h_j(C_l)$.

Once we have done this for all the columns of $\mathcal{V}$ which intersect V, we have produced an N-dividing partition $\mathcal{U}'$ which refines $\mathcal{U}$ and covers V as well as each $h_i(V)$. Hence the union of $\mathcal{U}'$ and $(W_0, h_1(W_0), \dots, h_m(W_0))$ still covers $A \cup B$, as required. □

Lemma 2.4. *Fix an integer N, and assume that A, B are clopen subsets such that there exists a N-dividing partition of A and a N-dividing partition of B. Then there is a N-dividing partition of $A \cup B$.*

Proof. By induction on the number of columns of the N-dividing partition of B, it is enough to consider the case where it is of the form $(V_0, h_1(V_0), \dots, h_m(V_0))$ for some clopen V_0 and some $m \geq N$. Using the previous lemma m times, we produce N-dividing partitions $\mathcal{U}_i$ which refine U, and clopen sets W_i contained in V_0 such that

- For all $i < m$, $\mathcal{U}_{i+1}$ refines $\mathcal{U}_i$ and $W_{i+1} \subseteq W_i$;
- For all $i \leq m$ the elements of $\mathcal{U}_i$ do not intersect $W_i, h_1(W_i), \dots, h_i(W_i)$;
- For all $i \leq m$ the union of $\mathcal{U}_i$ and $(W_i, h_1(W_i), \dots, h_m(W_i))$ covers $A \cup B$.

In the end, $\mathcal{U}_m$ is disjoint from $(W_m, h_1(W_m), \dots, h_m(W_m))$, so we may just add $(W_m, h_1(W_m), \dots, h_m(W_m))$ as a new column to $\mathcal{U}_m$ to obtain the desired N-dividing partition of $A \cup B$. □

Proposition 2.5. *For any integer N and any clopen A there exists a N-dividing partition of A.*

Proof. Since by assumption the G-orbit of any $x \in A$ is dense, its intersection with A is infinite and we may find some clopen subset $U_0 \ni x$ and elements $g_1, \dots, g_N$ of G such that $U_0, g_1(U_0), \dots, g_N(U_0)$ are disjoint and contained in A. Thus there exists a covering of A by clopen sets which can each be covered by a N-dividing partition, and by compactness there exists such a covering which is finite. Then we obtain the desired N-dividing partition of A from Lemma 2.4. □

Proof of Theorem 2.1. Fix a nonempty clopen subset A of X, an integer n and $\varepsilon > 0$. Then pick a N-dividing partition of A for some $N > \frac{n}{\varepsilon}$; let us denote it as before by $(U_{i,j})_{i \in I, j \in \{0,\dots,n_i\}}$ .For each i, we write $n_i + 1 = np_i + q_i$ for some $q_i \in \{0, \dots, n-1\}$ and then define, for all $k \in \{0, \dots, n-1\}$,

$$B_k = \bigsqcup_{i \in I} \bigsqcup_{l=0}^{p_i - 1} U_{i,k+nl} \ .$$

By construction, for all $\mu \in K_G$ we have $\mu(B_0) = \mu(B_1) = \ldots = \mu(B_{n-1})$ (because there exists an element of G mapping B_0 to B_i for all $i \in \{0, \dots, n-1\}$); hence $n\mu(B_0) \leq \mu(A)$ since $B_0, \dots, B_{n-1}$ are disjoint and contained in A. Also by construction, for any $\mu \in K_G$ we have

$$\mu(A \setminus \bigsqcup_{k=0}^{n-1} B_k) \leq \sum_{i \in I} q_i \mu(U_{i,0}) \leq (n-1)\mu(\bigcup_{i \in I} U_{0,i})$$

In any N-dividing partition of A, the union of the bases of the columns must have μ-measure less that $\frac{\mu(A)}{N}$ for all $\mu \in K_G$, so what we just obtained implies that

$$\mu(A) - n\mu(B_0) \leq \frac{n-1}{N} \mu(A) \leq \varepsilon \ .$$

Thus, $B = B_0$ is such that $\mu(A) - \varepsilon \leq n\mu(B) \leq \mu(A)$, proving that K_G is approximately divisible. $\square$

We are now ready to prove that the assumption of approximate divisibility in the definition of a dynamical simplex is redundant.

Corollary 2.6. *Assume that $K \subset P(X)$ is compact and convex; all elements of K are atomless and have full support; and K satisfies the Glasner–Weiss condition. Then K is approximately divisible.*

Proof. We pick K as above, and let $H = \{h \in \mathrm{Homeo}(X) \colon \forall \mu \in K\ h_*\mu = \mu\}$. We first note that the argument used in the proof Proposition 2.6 in [GW] shows that the action of H on X is transitive. Hence it follows from Theorem 2.1 that the simplex K_H of all G-invariant Borel probability measures on X is approximately divisible. Since K is contained in K_H, K is also approximately divisible (we note that it then follows from the arguments of [IM] that $K = K_H$ but this is not needed here). $\square$

3. Dynamical simplices as Fraïssé limits

In this section, we develop the connection between dynamical simplices and Fraïssé theory. We introduce the basic concepts of Fraïssé theory in our specific context. Since we do not assume familarity with this theory, we give some background details; we refer the reader to [H] for a more thorough discussion.

For the moment, we fix a set E (later, E will be a Choquet simplex), and the objects we consider are Boolean algebras endowed with a family of finitely additive probability measures $(\mu_e)_{e\in E}$. We let $\mathcal{B}_E$ denote the class of all these objects: an element of $\mathcal{B}_E$ is of the form $(A, (\mu_e)_{e\in E})$ where A is a Boolean algebra and each μ_e is a finitely additive probability measure on A. Since there should be no risk of confusion, we simply write $A \in \mathcal{B}_E$ and use the same letter A also to denote the underlying Boolean algebra of the structure we are considering.

Allow us to point out, for those who are used to Fraïssé theory, that elements of $\mathcal{B}_E$ really are first order structures in disguise, in a language where each μ_e is coded by unary predicates μ_e^r which are interpreted via $\mu_e(a) = r \leftrightarrow a \in \mu_e^r$; this is for instance done in [KR] (with just one measure in the language but this makes no essential difference). We adopt somewhat unusual conventions in the hope of improving readability in our particular case.

Note that inside $\mathcal{B}_E$ we have natural notions of isomorphism, embedding, and substructure. Below, we call elements of $\mathcal{B}_E$ *E-structures*.

Definition 3.1. Let $\mathcal{K}$ be a subclass of $\mathcal{B}_E$. One says that $\mathcal{K}$ satisfies :

(1) the *hereditary property* if whenever $A \in \mathcal{K}$ and B embeds in A then also $B \in \mathcal{K}$.
(2) the *joint embedding property* if for any $A, B \in \mathcal{K}$ there exists $C \in \mathcal{K}$ such that both A and B embed in C.
(3) the *amalgamation property* if, for any $A, B, C \in \mathcal{K}$ and embeddings $\alpha \colon A \to B$, $\beta \colon A \to C$ there exists $D \in \mathcal{K}$ and embeddings $\alpha' \colon B \to D$, $\beta' \colon C \to D$ such that $\alpha' \circ \alpha = \beta' \circ \beta$.

All the classes we will consider contain the trivial boolean algebra $\{0, 1\}$, with $\mu_e(0) = 0$ and $\mu_e(1) = 1$ for all e, which is a substructure of all elements of $\mathcal{B}_E$. Thus the joint embedding property will be implied by the amalgamation property.

The hereditary property is usually easy to check; the amalgamation property is typically much trickier, and is intimately related to *homogeneity*.

Definition 3.2. Given $A \in \mathcal{B}_E$, we say that A is *homogeneous* if any isomorphism between finite substructures of A extends to an isomorphism of A.

Here we recall that an isomorphism between E-structures is an isomorphism of the underlying Boolean algebras which preserves the values of each μ_e.

Definition 3.3. Let $A \in \mathcal{B}_E$. The *age* of A is the class of all elements of $\mathcal{B}_E$ which are isomorphic to a finite substructure of A.

Note that the age of any E-structure satisfies the hereditary property and the joint embedding property.

Proposition 3.4. *Assume that $M \in \mathcal{B}_E$ is homogeneous. Then its age satisfies the amalgamation property.*

Proof. Let A, B, C belong to the age of M and $\alpha\colon A \to B$ and $\beta\colon A \to C$ be embeddings. Given the definition of an age, we may assume that $B, C \subset M$. The isomorphism $\alpha\beta^{-1}\colon \beta(A) \to \alpha(A)$ extends to an automorphism f of M by homogeneity. Let D be the substructure of M generated by B and $f(C)$ (which is finite since a finitely generated Boolean algebra is finite), and α' the inclusion map from B to D, β' the restriction of f to C. Then for all $a \in A$ we have $\beta'\beta(a) = \alpha\beta^{-1}\beta(a) = \alpha(a) = \alpha'\alpha(a)$. □

There exists a form of converse to the previous result, which applies to countable E-structures (that is, the underlying Boolean algebra is countable).

Definition 3.5. A subclass $\mathcal{K}$ of $\mathcal{B}_E$ is a *Fraïssé class* if :

- All elements of $\mathcal{K}$ are finite
- $\mathcal{K}$ satisfies the hereditary property, the joint embedding property and the amalgamation property.
- $\mathcal{K}$ contains countably many structures up to isomorphism.

We already know that the age of a countable, homogeneous structure is a Fraïssé class.

Theorem 3.6 (Fraïssé). *Given a Fraïssé subclass $\mathcal{K}$ of $\mathcal{B}_E$, there exists a unique (up to isomorphism) countable homogeneous E-structure whose age is equal to $\mathcal{K}$. This structure is called the* Fraïssé limit *of $\mathcal{K}$.*

This E-structure can be built by repeated amalgamation of elements of $\mathcal{K}$; to prove both the existence and uniqueness, one can use the following characterization, which follows from a back-and-forth argument.

Proposition 3.7. *Given a Fraïssé subclass $\mathcal{K}$ of $\mathcal{B}_E$, a E-structure M is isomorphic to the Fraïssé limit of $\mathcal{K}$ iff the age of M coincides with $\mathcal{K}$ and M satisfies the* Fraïssé property, *namely: whenever $A \subseteq M$, $B \in \mathcal{K}$ and $\alpha\colon A \to B$ is an embedding, there exists an embedding $\beta\colon B \to M$ such that $\beta\alpha(a) = a$ for all $a \in A$.*

The following lemma is an easy consequence of [IM, Proposition 2.7] and is crucial for our purposes.

Lemma 3.8. *For any two clopen partitions $(A_i)_{i=1,\dots,n}$, $(B_i)_{i=1,\dots,n}$ of X such that $\mu(A_i) = \mu(B_i)$ for all $i \in \{1,\dots,n\}$ and all $\mu \in K$, there exists $g \in G$ such that $g(A_i) = B_i$ for all $i \in \{1,\dots,n\}$.*

Given a dynamical simplex K on a Cantor space X, we can see the clopen algebra of X, endowed with all the measures in K, as a K-structure, and Lemma 3.8 states that any isomorphism between finite Boolean subalgebras extends to an automorphism of the whole structure, that is, this structure is homogeneous. Thus, its age is a Fraïssé class.

We are now concerned with the other direction: building dynamical simplices from Fraïssé classes of E-structures. We will use the fact that the clopen algebra $\mathrm{Clopen}(X)$ of a Cantor space X is the unique (up to isomorphism) atomless countable Boolean algebra; and that the set of finitely additive probability measures on $\mathrm{Clopen}(X)$ is naturally identified with the compact space $P(X)$ of Borel probability measures on X.

From now on, we only consider Fraïssé subclasses $\mathcal{K}$ of $\mathcal{B}_E$ which satisfy the nontriviality condition that, given any $A \in \mathcal{K}$, there is an embedding $\alpha \colon A \to B \in \mathcal{K}$ such that for all atoms $a \in A$ $\alpha(a)$ is not an atom of B. We call such classes *suitable*.

Given any suitable Fraïssé subclass $\mathcal{K}$ of $\mathcal{B}_E$, the underlying Boolean algebra of the limit of $\mathcal{K}$ is a countable atomless Boolean algebra, which we see as the algebra of clopen sets of a Cantor space $X_{\mathcal{K}}$ by Stone duality; then for any $e \in E$ we see the measure μ_e as a probability measure on $X_{\mathcal{K}}$. We denote by $S(\mathcal{K})$ the closed convex hull of $\{\mu_e\}_{e \in E}$ inside $P(X_{\mathcal{K}})$.

Proposition 3.9. *Let $\mathcal{K}$ be a Fraïssé class of E-structures. Then $S(\mathcal{K})$ is a dynamical simplex if and only if the following conditions are satisfied:*

1. *For any $A \in \mathcal{K}$, and any nonzero $a \in A$, $\inf_e \mu_e(a) > 0$.*
2. *For any $A \in \mathcal{K}$, any $a \in A$ and any $\varepsilon > 0$, there exists $B \in \mathcal{K}$ and $a_1, \dots, a_n, a' \in B$ such that $\bigvee a_i = a'$, $\mu_e(a_i) \le \varepsilon$ for all i and all $e \in E$, and $\mu_e(a') = \mu_e(a)$ for all $e \in E$.*
3. *For any $A, B \in \mathcal{K}$ and any $a \in A$, $b \in B$, if $\mu_e(a) \le \mu_e(b) - \varepsilon$ for all $e \in E$ and some $\varepsilon > 0$ then there exists $C \in \mathcal{K}$ and $a', b' \in C$ such that a' is contained in b' and $\mu_e(a) = \mu_e(a')$, $\mu_e(b) = \mu_e(b')$ for all $e \in E$.*

Implicit in the above Proposition is that, under its assumptions, $\mathcal{K}$ is automatically suitable: suitability directly follows from the combination of conditions (1) and (2), and the amalgamation property of a Fraïssé class.

Proof. The fact that (1) and (2) are necessary is well-known and easily checked (see for instance Proposition 2.5 of [IM]); (3) directly follows from the Glasner–Weiss property.

Conversely, assume that all three conditions above are satisfied. It is clear that (3) (and homogeneity of a Fraïssé limit) implies that $S(\mathcal{K})$ satisfies the Glasner–Weiss property. Now fix $\nu \in S(\mathcal{K})$. To see that ν has full support, pick some nonempty clopen U in $X_{\mathcal{K}}$; then by (1) there exists $\varepsilon > 0$ such that $\mu_e(U) \ge \varepsilon$ for all $e \in E$, hence also $\nu(U) \ge \varepsilon$. Finally, the combination of (2) and (3) (and homogeneity) implies that, given any clopen $U \subset X_{\mathcal{K}}$ and any $\varepsilon > 0$ there exists a covering of U by clopens $U_1, \dots, U_n$ such that $\mu_e(U_i) \le \varepsilon$ for all i and all $e \in E$, hence also $\nu(U_i) \le \varepsilon$ for all i, and this in turn implies that ν is atomless. □

4. A FRAÏSSÉ THEORETIC PROOF OF DOWNAROWICZ'S THEOREM

Given a Choquet simplex Q, we denote by $A(Q)$ the space of all real-valued continuous, convex, affine maps on Q. For $F \subset A(Q)$ we denote by F^+ the elements of F taking nonnegative values, and by F_1^+ the elements of $A(Q)$ with values in the half-open interval $]0,1]$.

Definition 4.1. We say that a subset F of $A(Q)$ satisfies the *finite sum property* if, whenever $f, f_1, \dots, f_n, g_1, \dots, g_m$ are elements of F^+ such that $f = \sum_{i=1}^n f_i = \sum_{j=1}^m g_j$ there exist $h_{i,j} \in F^+$ satisfying

$$\forall j \in \{1, \dots, n\} \ \sum_{k=1}^m h_{j,k} = f_j \ \text{ and } \forall k \in \{1, \dots, m\} \ \sum_{j=1}^n h_{j,k} = g_j \ .$$

It is an important fact that, when Q is a Choquet simplex, $A(Q)$ satisfies the finite sum property, which is an equivalent formulation of the Riesz decomposition property of $A(Q)$ (see [L] or [E] and Theorem 2.5.4, Chapter I.2 of [FL]; [P] and [FL] are good references for the facts we use here about Choquet simplices).

We need a version of the finite sum property for maps with values in $]0,1]$, which is easy to establish.

Lemma 4.2. *Let Q be a Choquet simplex. Assume that $F \subseteq A(Q)$ is a $\mathbf{Q}$-vector subspace which contains the constant function 1 and has the finite sum property. Assume also that f, $(f_j)_{i=1,\dots,n}$, $(g_k)_{k=1,\dots,m}$ are elements of F_1^+ such that*

$$\sum_{j=1}^n f_j = f = \sum_{k=1}^m g_k \ .$$

Then there exist elements $h_{j,k}$ of F_1^+ such that

$$\forall j \in \{1, \dots, n\} \ \sum_{k=1}^m h_{j,k} = f_j \ \text{ and } \forall k \in \{1, \dots, m\} \ \sum_{j=1}^n h_{j,k} = g_j \ .$$

Proof. We first pick $\varepsilon > 0$ which is strictly smaller than $\min(f_j)$ and $\min(g_k)$ for all j,k, and such that the constant function equal to ε belongs to F. Then, define $f' = f - \varepsilon$, $f'_j = f_j - \frac{\varepsilon}{n}$ and $g'_k = g_k - \frac{\varepsilon}{m}$. Those maps still belong to F_1^+ and $\sum_j f'_j = f = \sum_k g'_k$; by the finite sum property we may then find $h'_{j,k} \in F^+$ of $A(Q)$ such that $\sum_k h'_{j,k} = f'_j$, $\sum_j h'_{j,k} = g'_k$. Then $h_{j,k} = h'_{j,k} + \frac{\varepsilon}{nm}$ belongs to F_1^+ and these maps have the desired property. $\square$

We now introduce the subclass $\mathcal{K}_Q$ of all finite $A \in \mathcal{B}_Q$ such that for all nonzero $a \in A$ the map $q \mapsto \mu_q(a)$ belongs to $A(Q)_1^+$. The previous lemma yields the following property, which is the key fact in our construction.

Proposition 4.3. *For any Choquet simplex Q, the class $\mathcal{K}_Q$ satisfies the amalgamation property.*

Proof. Let $A, B, C \in \mathcal{K}_Q$ and $\alpha \colon A \to B$, $\beta \colon A \to C$ be embeddings. We list the atoms of A, B, C as $(a_i)_{i \in I}$, $(b_j)_{j \in J}$ and $(c_k)_{k \in K}$. For all $i \in I$ we let J_i (resp. K_i) denote the set of all j such that $b_j \in \alpha(a_i)$ (resp. $c_k \in \beta(a_i)$). We first define the underlying Boolean algebra of our amalgam, using the usual amalgamation procedure for Boolean algebras (as is done for instance in [KR]): the atoms of D

are of the form $b_j \otimes c_k$ for each pair (j,k) which belongs to $J_i \times K_i$ for some i. For all i and all $(j,k) \in J_i \times K_i$ we will set

$$\alpha'(b_j) = \bigvee_{k \in K_i} b_j \otimes c_k \text{ and } \beta'(c_k) = \bigvee_{j \in J_i} b_j \otimes c_k \ .$$

The one thing that requires some care is to define each $\mu_q(b_i \otimes c_j)$; and the constraint we have to satisfy is that, given i, $(j,k) \in J_i \times K_i$ and $q \in Q$ we must have

$$\mu_q(b_j) = \sum_{k \in K_i} \mu_q(b_i \otimes c_j) \text{ and } \mu_q(c_k) = \sum_{j \in J_i} \mu_q(b_i \otimes c_j)$$

The fact that this is possible is guaranteed by the previous lemma and the fact that

$$\forall i \ \sum_{j \in J_i} \mu_q(b_j) = \mu_q(a_i) = \sum_{k \in K_i} \mu_q(c_k) \ .$$

Indeed, since each map $q \mapsto \mu_q(b_j)$, $q \mapsto \mu_q(c_k)$ and $q \mapsto \mu_q(a_i)$ belongs to $A(Q)_1^+$, Lemma 4.2 allows us to find elements $h_{j,k}$ of $A(Q)_1^+$ such that

$$\forall q \in Q \ \forall j \in J_i \sum_{k \in K_i} h_{j,k}(q) = \mu_q(b_j) \text{ and } \forall k \in K_i \sum_{j \in J_i} h_{j,k}(q) = \mu_q(c_k) \ .$$

Setting $\mu_q(b_j \otimes c_k) = h_{j,k}(q)$, we are done. □

Theorem 4.4 (Downarowicz [D]). *Given a metrizable Choquet simplex Q, there exists a minimal homeomorphism of a Cantor space whose set of invariant measures is affinely homeomorphic to Q.*

Note that Downarowicz obtains a more precise result: the minimal homeomorphism that he obtains is a dyadic Toeplitz flow, while here we do not have control over its dynamics (though maybe something could be extracted from the construction in [IM]).

Proof. We need to build a dynamical simplex which is affinely homeomorphic to Q. A simple closure argument allows us to find a countable dense $\mathbf{Q}$-vector subspace F of $A(Q)$ which contains the constant function 1 and satisfies the finite sum property: start from a countable dense $\mathbf{Q}$-vector subspace of $A(X)$ F_0 containing 1, then add witnesses to the finite sum property as needed to form a countable $F_1 \subset A(Q)$ containing F_0, let F_2 be the $\mathbf{Q}$-vector subspace generated by F_1, and keep going : $\bigcup_{i \in \mathbf{N}} F_i$ will have the desired property.

Then we consider the class $\mathcal{L}$ consisting of all $A \in \mathcal{K}_Q$ such that for all nonzero $a \in A$ the map $q \mapsto \mu_q(a)$ belongs to F_1^+. By the same argument as above, the finite sum property of F ensures that $\mathcal{L}$ has the amalgamation property; clearly $\mathcal{L}$ also satisfies the hereditary property, and has only countably many elements up to isomorphism because F is countable, hence $\mathcal{L}$ is a Fraïssé class.

The fact that $\mathcal{L}$ satisfies condition (1) of Proposition 3.9 follows from the fact that elements of $A(Q)_1^+$ are continuous and take values in $]0,1]$, hence are bounded below by some strictly positive constant; condition (2) is asily deduced from the fact that for any $f \in F$ and any $N > 0$ the function $\frac{1}{N}f$ also belongs to f. Finally, condition (3) is deduced by noticing that if $f, g \in F_1^+$ and $f(q) < g(q)$ for all $q \in Q$ then $g - f$ also belongs to F_1^+.

To sum up, we just built a dynamical simplex $S(\mathcal{L})$. We now claim that the map $\Phi \colon q \mapsto \mu_q$ is a affine homeomorphism from Q to $S(\mathcal{L})$.

Continuity of Φ is equivalent to the fact that for each $U \in \mathrm{Clopen}(X_{\mathcal{L}})$ the map $q \mapsto \mu_q(U)$ is continuous, which is built into our definition of $\mathcal{L}$ since this map belongs to $A(Q)$. Similarly, Φ is affine because each map $q \mapsto \mu_q(U)$ is affine. This implies that $\Phi(Q) = \{\mu_q \colon q \in Q\}$ is compact and convex, so its closed convex hull $S(\mathcal{L})$ is equal to it and Φ is onto. Finally, injectivity of Φ is ensured by the density of F in $A(X)$: if q, q' are distinct elements of Q, there exists $f \in F_1^+$ such that $f(q) \neq f(q')$, and both $f(q), f(q')$ are different from 1. Then by definition of $\mathcal{L}$ this yields $U \in \mathrm{Clopen}(X_{\mathcal{L}})$ such that $\mu_q(U) = f(q) \neq f(q') = \mu_{q'}(U)$. Thus $\Phi \colon Q \to S(\mathcal{L})$ is an affine homeomorphism and we are done. $\square$

5. Speedup equivalence vs orbit equivalence

Definition 5.1. Recall that two minimal homeomorphisms φ, ψ of a Cantor space X are *orbit equivalent* if there is $g \in \mathrm{Homeo}(X)$ such that g maps the φ-orbit of x onto the ψ-orbit of $g(x)$ for all $x \in X$.

Definition 5.2. For $\varphi \in \mathrm{Homeo}(X)$, we denote by $S(\varphi)$ the simplex of all φ-invariant Borel probability measures on X.

The following theorem is a cornerstone of the study of orbit equivalence of minimal homeomorphisms of Cantor spaces.

Theorem 5.3 (Giordano–Putnam–Skau [GPS]). *Whenever φ, ψ are two minimal homeomorphisms of a Cantor space X, the following conditions are equivalent.*

1. *φ and ψ are orbit equivalent.*
2. *There exists $g \in \mathrm{Homeo}(X)$ such that $g_*S(\varphi) = S(\psi)$.*

We recall that for $\mu \in P(X)$ and $g \in \mathrm{Homeo}(X)$ one has $g_*\mu(A) = \mu(g^{-1}A)$; and for $K \subset P(X)$ and $g \in \mathrm{Homeo}(X)$ we set $g_*K = \{g_*\mu \colon \mu \in K\}$.

Note that any g witnessing that φ and ψ are orbit equivalent must be such that $g_*S(\varphi) = S(\psi)$; the converse is false, and the dynamical content of the above theorem is still somewhat mysterious to the author. The Giordano–Putnam–Skau theorem admits a one-sided analogue, due to Ash [A], which we proceed to describe.

Definition 5.4. Let φ, ψ be minimal homeomorphisms. Say that φ is a *speedup* of ψ if φ is conjugate to a map of the form $x \mapsto \psi^{n_x}(x)$, with $n_x > 0$ for all x.

Theorem 5.5 (Ash [A]). *Whenever φ, ψ are two minimal homeomorphisms of a Cantor space X, the following conditions are equivalent.*

1. *φ is a speedup of ψ.*
2. *There exists $g \in \mathrm{Homeo}(X)$ such that $S(\psi) \subset g_*S(\varphi)$.*

Again, one of the two implications above is easy; the hard part is building the speedup from the assumption that one set of invariant measures is contained in the other. Ash showed that this can be done using Kakutani–Rokhlin partitions.

Note that, if φ, ψ are minimal homeomorphisms such that each φ-orbit is contained in a ψ-orbit, then the set of φ-invariant measures contains the set of ψ-invariant measures; conversely, if the set of φ-invariant measures contains the set of ψ-invariant measures then it follows from Ash's result that one may conjugate φ so that each φ-orbit is contained in a ψ-orbit (this is weaker than Ash's theorem cited above and admits a shorter proof). It is natural to wonder, as Ash did

[A], whether speedup equivalence is actually the same notion as orbit equivalence, which also corresponds to asking whether there is a Schröder-Bernstein property for equivalence relations induced by minimal homeomorphisms. As pointed out by Ash, one easily sees that such is the case when one of the homeomorphisms has a finite-dimensional simplex of invariant measures.

Proposition 5.6 (Ash [A]). *Assume that φ, ψ are minimal homeomorphisms and that $S(\varphi)$ has finitely many extreme points. If φ and ψ are speedup equivalent then they are orbit equivalent.*

The situation turns out to be very different for infinite-dimensional simplices.

Definition 5.7. We say that two dynamical simplices K, L on a Cantor space X are *biembeddable* if there exists $g, h \in \mathrm{Homeo}(X)$ such that $g_*K \subseteq L$ and $h_*L \subseteq K$.

Thus, biembeddability is the translation of speedup equivalence at the level of dynamical simplices.

Theorem 5.8. *Assume that Q, R are two metrizable Choquet simplices such that Q affinely continuously embeds as a face of R and R affinely continuously embeds as a face of Q. Then there exist dynamical simplices $S(Q)$, $S(R)$ on a Cantor space X such that $S(Q)$ is affinely homeomorphic to Q; $S(R)$ is affinely homeomorphich to R; $S(Q)$ and $S(R)$ are biembeddable.*

In particular, given any Q, R as above which are not affinely homeomorphic, and minimal homeomorphisms φ_Q, φ_R with sets of invariant measures equal to $S(Q)$, $S(R)$ respectively, φ_Q and φ_R are speedup equivalent but not orbit equivalent. Such examples exist: for instance, let Q be the Poulsen simplex, and R the tensor product of Q and $[0,1]$ (see [NP]). Then Q embeds as a closed face of R, and since any Choquet simplex is affinely homeomorphic to a closed face of the Poulsen simplex this is in particular true for R. But Q and R are not homeomorphic since the set of extreme points of R is not dense in R.

There is some work to do before we can prove Theorem 5.8; we need an additional property of Choquet simplices.

Proposition 5.9 (Edwards [E]). *Let Q be a Choquet simplex, R be a closed face of Q and $f \in A(R)$. Assume that $f_1, f_2 \in A(Q)$ are such that $f_{1|R} \leq f \leq f_{2|R}$. Then there exists $g \in A(Q)$ extending f and such that $f_1 \leq g \leq f_2$.*

Definition 5.10. Assume that Q is a Choquet simplex, R is a closed face of Q and F is a subset of $A(Q)$. We say that F has the *(R, Q)-extension property* if for any $f, f_1, \dots, f_n \in F_1^+$ such that $f_{|R} = \sum_{i=1}^n f_{i|R}$ there exist $g_1, \dots, g_n \in F_1^+$ such that $g_{i|R} = f_{i|R}$ for all $i \in \{1, \dots, n\}$, and $f = \sum_{i=1}^n g_i$.

The proof of the following lemma from Proposition 5.9 is straightforward.

Lemma 5.11. *Assume that Q is a Choquet simplex and R is a closed face of Q. Then $A(Q)$ has the (R, Q)-extension property.*

Proof. By induction, it is enough to prove that the property holds for $n = 2$. So, take $f, f_1, f_2 \in F_1^+$ such that $f_{|R} = f_{1|R} + f_{2|R}$. By compactness we have some $\varepsilon > 0$ such that $\varepsilon \leq f_{1|R} \leq f_{|R} - \varepsilon$, so we can apply Lemma 5.9 and extend $f_{1|R}$ to $g_1 \in A(Q)$ such that $\varepsilon \leq g_1 \leq f - \varepsilon$. These inequalities imply both that $g_1 \in A(Q)_1^+$ and that $g_2 = f - g_1 \in A(Q)_1^+$. □

We move on towards proving Theorem 5.8. Starting from Q, R as in the statement of the theorem, we first assume that R is a closed face of Q and pick an affine continuous injection g of Q into itself such that $g(Q)$ is a closed face of R.

Note that, given any $f \in A(Q)_1^+$, we may define $h \in A(g(Q))_1^+$ by setting $h(g(q)) = f(q)$. Then, using Proposition 5.9 we can extend h to $A(Q)_1^+$. We just proved that, for any $f \in A(Q)_1^+$, there exists $h \in A(Q)_1^+$ such that $h \circ g = f$. Granting that, we may again apply a closure argument to find $F \subset A(Q)$ such that:

- F is a countable dense $\mathbf{Q}$-vector subspace which contains the constant function 1.
- F satisfies the finite sum property.
- F satisfies both the (R, Q)-extension property and the $(g(Q), Q)$-extension property.
- $\{f \circ g \colon f \in F_1^+\} = F_1^+$.

We then run the same construction as in the proof of Theorem 4.4, considering again the class $\mathcal{L}$ consisting of all $A \in \mathcal{K}_Q$ such that for all nonzero $a \in A$ the map $q \mapsto \mu_q(a)$ belongs to F_1^+. We denote by M_Q the Fraïssé limit of this class, view its underlying Boolean algebra as the clopen algebra of a Cantor space $X_{\mathcal{L}}$ and again denote by $\Phi \colon Q \to S(\mathcal{L})$ the affine homeomorphism $q \mapsto \mu_q$.

We consider the classes $\mathcal{L}_R$ (resp. $\mathcal{L}_{g(Q)}$) obtained by taking restrictions of elements of $\mathcal{L}$ to R (resp. $g(Q)$); that is, whenever $A_Q = (A, (\mu_q)_{q \in Q})$ is an element of $\mathcal{L}$, we obtain an element $A_R = (A, (\mu_r))_{r \in R}$ of $\mathcal{L}_R$, and all elements of $\mathcal{L}_R$ are obtained in this way (and, of course, similarly for $\mathcal{L}_{g(Q)}$).

Lemma 5.12. *The classes $\mathcal{L}_R$ and $\mathcal{L}_{g(Q)}$ have the amalgamation property.*

Proof. We only write down the argument for $\mathcal{L}_R$, the other one is identical. As in the proof of Proposition 4.3, showing that the amalgamation property holds reduces to proving that, whenever $(A, (\mu_r))_{r \in R}$ belongs to $\mathcal{L}_R$, $a \in A$ is a nonzero atom and $f_1, \dots, f_n, g_1, \dots, g_m \in F_1^+$ are such that

$$\forall r \in R \; \mu_r(a) = \sum_{i=1}^{n} f_i(r) = \sum_{j=1}^{m} g_j(r)$$

there exists $h_{i,j} \in F_1^+$ such that

$$\forall i \; \sum_{j=1}^{m} h_{i,j|R} = f_{i|R} \text{ and } \forall j \; \sum_{i=1}^{n} h_{i,j|R} = g_{j|R} \, .$$

To prove that this is true, first apply the (R, Q)-extension property of F to $f, f_1, \dots f_n$ and $f, g_1, \dots, g_m$, then use the fact that F has the finite sum property. □

Let $M_R, M_{g(Q)}$ denote the structures $(\mathrm{Clopen}(X_{\mathcal{L}}), (\mu_r)_{r \in R})$ and $(\mathrm{Clopen}(X_{\mathcal{L}}), (\mu_{g(q)})_{q \in Q})$.

Lemma 5.13. *M_R (resp. $M_{g(Q)}$) is the Fraïssé limit of $\mathcal{L}_R$ (resp. $\mathcal{L}_{g(Q)}$).*

Proof. The (R, Q)-extension property of F and the Fraïssé property of M_Q combine to show that M_R has the Fraïssé property. Indeed, let A be a finite algebra of clopen subsets of $X_{\mathcal{L}}$, and pick an embedding $\alpha \colon A \to B \in \mathcal{L}_R$. For each atom a of A, we let $(b_i^a)_{i \in I_a}$ denote the atoms of B which are contained in $\alpha(a)$. The measures of each b_i^a correspond to maps $f_i^a \in F_1^+$ such that $\sum_{i \in I_a} f_i^a(r) = \mu_a(r)$ for all $r \in R$. Then, the (R, Q)-extension property lets us pick maps $g_i^a \in F_1^+$ such that $g^a_{i\,|R} = f^a_{i\,|R}$ and $\sum_{i \in I_a} g_i^a(q) = \mu_q(a)$ for all $q \in Q$.

Using the Fraïssé property of M_Q, we can then write each atom a of A as a union of clopen sets U_i^a such that $\mu_q(U_i^a) = g_i^a(q)$ for all $q \in Q$; the map $\beta \colon b_i^a \mapsto U_i^a$ is then an embedding of B inside M_R such that $\beta \circ \alpha(a) = a$ for all $a \in A$.

The argument for $M_{g(Q)}$ is similar.

□

Since $\mathcal{L}_R$, $\mathcal{L}_{g(Q)}$ are both suitable and satisfy the conditions of 3.9, we now have three dynamical simplices $S(\mathcal{L}_{g(Q)}) \subset S(\mathcal{L}_R) \subset S(\mathcal{L}_Q)$. Our last remaining task is to prove that there exists $h \in \mathrm{Homeo}(X)$ such that $h_* S(\mathcal{L}_Q) = S(\mathcal{L}_{g(Q)})$. Indeed, then the dynamical simplices $S(R) = S(\mathcal{L}_R)$, $S(Q) = S(\mathcal{L}_Q)$ satisfy the desired conditions.

End of the proof of Theorem 5.8. We define $N_Q = (\mathrm{Clopen}(X_{\mathcal{L}}), (\nu_q)_{q \in Q}))$ by setting $\nu_q = \mu_{g(q)}$; N_Q is a homogeneous Q-structure since $M_{g(Q)}$ is homogeneous, hence N_Q is the Fraïssé limit of its age. The condition $\{f \circ g \colon f \in F_1^+\} = F_1^+$, combined with the $(g(Q), Q)$ extension property of F, says that the ages of N_Q and M_Q are the same; thus, M_Q and N_Q are isomorphic. Hence there is an automorphism h of $\mathrm{Clopen}(X_{\mathcal{L}})$, equivalently a homeomorphism h of $X_{\mathcal{L}}$, such that

$$\forall A \in \mathrm{Clopen}(X_{\mathcal{L}})\ \nu_q(h(A)) = \mu_q(A)\ .$$

This amounts to stating that $h_* \mu_q = \nu_{g(q)}$ for all $q \in Q$, and in particular $h_*(S(\mathcal{L}_Q)) = S(\mathcal{L}_{g(Q)})$. □

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Université Claude Bernard – Lyon 1, Institut Camille Jordan, CNRS UMR 5208, 43 boulevard du 11 novembre 1918, 69622 Villeurbanne Cedex, France