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Existentially Closed Groups and Determinacy

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Martin Ziegler [7] and others have described games G for constructing groups. If G is such a game and ϕ is a property, then we write $G(\phi)$ for the form of G in which the second player wins if and only if the constructed group has property ϕ. Let us say that the property ϕ is *determined* (with respect to G) if one of the two players has a winning strategy for $G(\phi)$; let us say that G is *wholly determined* if every property ϕ is determined. In their elegant recent book [3], Graham Higman and Elizabeth Scott consider two games G for constructing groups, and they suggest [3, p. 85, line 10] that both games should be wholly determined.

By Corollary 5.1 below, neither game is wholly determined. Both games can be generalised from groups to other types of structure; I give necessary and sufficient conditions for each game to be wholly determined, depending on the type of structure. It is known that groups fail to meet the conditions for either game.

1. Preliminaries

A *pre-game* is a set of instructions for playing a game, but without the criterion for deciding which player wins. One makes a pre-game into a *game* by adding such a criterion.

We consider some pre-games of the following form. There are two players, a male player ∀(belard) and a female player ∃(loise). There is a non-empty set of *conditions*, which are partial descriptions of structures. This set is partially ordered by inclusion: $p \subseteq q$ iff the condition q says everything that the condition p says, and perhaps more besides. (For games of this form I generally follow the notation of Chapter 2 of Hodges [4].)

The players play the pre-game G as follows. They choose conditions in turn, so as to form an increasing chain $(p_n : n < \omega)$ of conditions; p_n is

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chosen by player $\forall$ if n is even, and by player $\exists$ if n is odd. The player who chooses p_n is allowed to know what $p_0, \ldots, p_{n-1}$ were. At the end of the play, the set $p_\omega = \bigcup_{n<\omega} p_n$ determines a structure called the *compiled structure*. Henceforth the letter A is reserved for the compiled structure.

By a *property* we mean a class ϕ of structures which is closed under isomorphism. We say that a structure B *has* (or *satisfies*) ϕ if B lies in the class. This notion extends in a natural way to sentences ϕ: the structure B satisfies ϕ if B is in the class of models of ϕ. In fact it will often be convenient to state properties as sentences about the compiled structure: for example "the property that A is an abelian group" is the class of abelian groups.

Let ϕ be a property and G a pre-game. Then the game $G(\phi)$ is played as G. Player $\exists$ wins a play of $G(\phi)$ if the compiled structure A has property ϕ; otherwise player $\forall$ wins.

A *strategy* σ for player $\forall$ in a game is a set of rules which tell player $\forall$ how to choose, depending on what conditions were chosen earlier in the play. More formally, σ is a family of functions σ^i (i even), such that for each i, σ^i is an i-ary function from the set of conditions to the set of conditions. (In particular σ^0 is a condition.) If $p_0, \ldots, p_{i-1}$ were the conditions chosen at the first i steps, then σ requires player $\forall$ to choose $p_i = \sigma^i(p_0, \ldots, p_{i-1})$. Likewise a strategy for player $\exists$ is a family σ^i (i odd).

A strategy in a game $G(\phi)$ is *winning* if the player who uses it will always win, regardless of what the other player does. A property ϕ is said to be *enforceable* (with respect to the pre-game G) if player $\exists$ has a winning strategy for the game $G(\phi)$. The property ϕ is *coenforceable* if player $\forall$ has a winning strategy for $G(\text{not-}\phi)$. We say that a property p *forces* ϕ if player $\exists$ has a strategy which enables her to win whenever player $\forall$ chooses p_0 so that $p \subseteq p_0$.

LEMMA 1.1. (a) *A property ϕ is enforceable if and only if every condition forces ϕ.*

(b) *A property ϕ is coenforceable if and only if some condition forces ϕ.*

(c) *Every enforceable property is coenforceable.*

(d) *The conjunction of countably many enforceable properties is enforceable.*

Proof. (a) is immediate from the definition: a strategy for player $\exists$ is winning if it enables her to win regardless of how player $\forall$ makes his first move. For (b), suppose first that ϕ is coenforceable, so that player $\forall$ has a winning strategy σ for $G(\text{not-}\phi)$. Let p be the condition σ^0. Then p forces

ϕ. For suppose player $\forall$ in a play of $G(\phi)$ chooses $p_0 \supseteq p$. Then player $\exists$ can choose p_{2n} to be $\sigma^{2n+1}(p, p_0, p_1, ..., p_{2n-1})$, for each $n < \omega$; this way she will win. The converse is similar. Then (c) is immediate from (a) and (b).

(d) Suppose each property ϕ_i $(i < \omega)$ is enforceable. Player $\exists$ can partition her infinitely many moves into countably many countable sets X_i $(i < \omega)$. She can use the moves in set X_i to make sure that the compiled structure has property ϕ_i. ∎

2. The Ziegler and Fraïssé Pre-games

To define a particular pre-game, we need to say first what the conditions are and how they are partially ordered, and second how the compiled structure is built out of the set p_ω. For example, the two pre-games which Higman and Scott [13, p. 82f] discuss are as follows. In both of them the players build a group.

G_I. In *Ziegler's pre-game* a condition is a finite set of equations and inequations (using a language with symbols for multiplication, inverse and the identity element, together with countably many constants) which is satisfiable in some group. The partial ordering "$p \subseteq q$" is simply inclusion. When $G(\phi)$ is played, the compiled group is the group which is presented by the set of all equations in p_ω. (These games were introduced by Ziegler [7] as a way of handling Abraham Robinson's finite forcing construction. Higman and Scott call the pre-game the "finite code of rules.")

G_{II}. In *Fraïssé's pre-game* the conditions are finitely generated groups; $p \subseteq q$ if and only if p is a subgroup of q. The compiled structure is the group p_ω. (This is essentially the "stable code of rules" of Higman and Scott. Read on below for the connection with Fraïssé.)

Both these pre-games can be widely generalised with the help of some model theory. Let L be a first-order language, always assumed to be countable. A structure with functions, relations, etc. to match the symbols of L is called an *L-structure*; structures are assumed to have at least one element. A *theory in L* is a set of sentences of L.

First we generalise Ziegler's pre-game (Hodges [4] sections 2.3, 3.4). Let T be a consistent theory in the language L. We say that T is *inductive* if every union of a chain of models of T is again a model of T. Let $L(W)$ be the language which results when we add to L a countable set W of new constants known as *witnesses*. Let a *condition* be a finite set of atomic and negated atomic sentences of $L(W)$ which is consistent with T. We partially order the set of conditions by inclusion. At the end of a play, there is (up to isomorphism) a unique $L(W)$-structure A' whose positive diagram is the set of all atomic sentences deducible from $T \cup p_\omega$; we define the *compiled structure* A to be A' considered as an L-structure.

We shall call this the *Ziegler pre-game on* T. If T is the first-order theory which axiomatises the class of groups (or for short, if T is the *theory of groups*), then the Ziegler pre-game on T is exactly the pre-game G_I described above.

Next we generalise the Fraïssé pre-game. The generalisation is implicit already in a construction of Fraïssé [2]. Let L be a countable first-order language, and suppose $\mathbf{K}$ is a class of L-structures. We write $\mathbf{K}_{\mathrm{fg}}$ for the class of all finitely generated structures in $\mathbf{K}$. We say that $\mathbf{K}$ is a *Fraïssé class* if $\mathbf{K}$ has the following properties: (1) If $B \in \mathbf{K}$ and C is embeddable in B then $C \in \mathbf{K}$. (2) $\mathbf{K}$ is closed under unions of countable chains. (3) If C_1, $C_2 \in \mathbf{K}_{\mathrm{fg}}$ then there is a structure D in $\mathbf{K}_{\mathrm{fg}}$ such that both C_1 and C_2 are embeddable in D. (4) If B, C_1, $C_2 \in \mathbf{K}_{\mathrm{fg}}$ and B is a substructure of both C_1 and C_2, then there are a structure $D \in \mathbf{K}_{\mathrm{fg}}$ and embeddings $e_i: C_i \to D$ $(i = 1, 2)$ which agree on B. In this setting a *condition* is a structure in $\mathbf{K}_{\mathrm{fg}}$, and $p \subseteq q$ iff p is a substructure of q. Thus a play $(p_n : n < \omega)$ will be a chain of structures, and we can define the *compiled structure* to be the union p_ω.

We shall call this the *Fraïssé pre-game on* $\mathbf{K}$. If $\mathbf{K}$ is the class of groups, then $\mathbf{K}$ is a Fraïssé class and the Fraïssé pre-game on $\mathbf{K}$ is the pre-game G_{II} described above.

Since a Fraïssé class $\mathbf{K}$ and the class of models of an inductive theory T are both closed under unions of chains, it makes sense to speak of an existentially closed (e.c.) structure in $\mathbf{K}$ or model of T. The next lemma states the main reason why people have studied this type of construction. (See, e.g., Hodges [4, Sect. 3.2] for the definition of e.c. in general; for groups it reduces to the notion studied by Higman and Scott.)

LEMMA 2.1. *Let the setting be either an inductive theory* T *or a Fraïssé class* $\mathbf{K}$. *Then the property "The compiled structure is an e.c. model of* T *(resp. an e.c. structure in* $\mathbf{K}$*)" is enforceable.*

Proof. For the Ziegler case see Hodges [4, Corollary 3.4.3]. For the Fraïssé case the argument of Higman and Scott [3, Lemma 7.1(i)] generalises. ∎

A theory T is said to have the *joint embedding property* (JEP) if: for any two models B, C, of T there is a model D of T in which both B and C are embeddable. (The theory of groups has the JEP, with $D = B \times C$.)

LEMMA 2.2. *Let* L *be a first-order language and* ϕ *a property of* L*-structures.*

(a) *Suppose G is the Fraïssé pre-game on a class* **K**. *Then ϕ is enforceable if and only if it is coenforceable.*

(b) *The same holds for the Ziegler pre-game on an inductive theory T, provided that T has the JEP.*

Proof. Lemma 1.1 shows that enforceable implies coenforceable. For the converse in (a), suppose that σ is a winning strategy for player $\forall$ in $G(\text{not-}\phi)$. Then player $\exists$ can win $G(\phi)$ as follows. When player $\forall$ has chosen a condition p_0, she uses property (3) of Fraïssé classes to find a condition $p_1 \supseteq p_0$ in which σ^0 is embeddable. Thereafter she chooses p_{2n+1} to be $\sigma^{2n}(p_1, ..., p_{2n})$.

For the converse in (b), suppose ϕ is coenforceable. Then some condition q forces ϕ. To win $G(\phi)$, player $\exists$ proceeds as follows. When player $\forall$ has chosen p_0, she notes that since ϕ is a property of L-structures, it is not affected by permutations of the set of witnesses; so there is a condition q' which forces ϕ and has no witnesses in common with p_0. Since p_0 and q' are both satisfiable in models of T and T has the joint embedding property, there is a model of T in which p_0 and q' are satisfiable, and hence (since p_0 and q' have no witnesses in common) the set $p_0 \cup q'$ is satisfiable. Then player $\exists$ chooses p_1 to be $p_0 \cup q'$, and thereafter she chooses as in case (a). ∎

3. When Is the Ziegler Pre-game Wholly Determined?

Throughout this section, T is a fixed inductive theory and we play the Ziegler pre-game with respect to T. By the dichotomy theorem (Theorem 4.2.6 in Hodges [4]), exactly one of the following is true:

(a) There is an at most countable set X of structures, such that (i) it is enforceable that the compiled structure A is isomorphic to some structure in X, and (ii) if B is any structure in X, then it is coenforceable that A is isomorphic to B.

(b) For every enforceable property ϕ there are continuum many non-isomorphic finite or countable structures which have ϕ.

Let us say that the theory T is *good* if (a) holds, and *bad* if (b) holds.

THEOREM 3.1. *The Ziegler pre-game on a theory T is wholly determined if and only if T is good.*

Proof. Suppose first that T is good. If some structure B in the set X (as in (a) above) fails to have property ϕ, then player $\forall$ can win $G(\phi)$ by choosing B and playing so that A is isomorphic to B. If every structure in

X has property ϕ, then player $\exists$ wins by playing so that A is isomorphic to some structure in X.

Henceforth suppose that T is bad. By Lemma 1 it follows that there is a family of 2^ω non-isomorphic finite or countable models of T; list them without repetition as $(C_\alpha : \alpha < 2^\omega)$. We say that an ordinal $\alpha < 2^\omega$ is *constrainable* if there is a condition which forces the property "A is isomorphic to C_α." At most countably many ordinals are constrainable, since there are only countably many conditions.

Let us call a strategy for player $\forall$ (resp. $\exists$) a $\forall$-*strategy* (resp. $\exists$-*strategy*). Since there are countably many conditions to choose from, there are 2^ω possible $\forall$-strategies. Likewise there are 2^ω possible $\exists$-strategies. List all the possible $\forall$-strategies and $\exists$-strategies together as $(\sigma_\beta : \beta < 2^\omega)$. We shall choose the property ϕ so that ϕ defeats every σ_β.

By induction on β we build up chains $(X_\beta : \beta < 2^\omega)$ and $(Y_\beta : \beta < 2^\omega)$ of subsets of 2^ω, so that (i) for each β, X_β and Y_β are disjoint and have cardinality $< 2^\omega$, and (ii) for each β, $X_{\beta+1} \setminus X_\beta$ and $Y_{\beta+1} \setminus Y_\beta$ both have cardinality at most 1.

To begin the chains, we put $X_0 = \emptyset$ and we take Y_0 to be the set of constrainable ordinals. We take unions at limit ordinals.

When X_β and Y_β have been chosen, we consider the strategy σ_β; suppose it is a $\forall$-strategy. Define Z to be $\{\alpha < 2^\omega :$ when player $\forall$ follows σ_β, it is possible for A to be isomorphic to $C_\alpha\}$. We shall ensure that Z meets $Y_{\beta+1}$. There are two cases to consider.

Case One. Z has cardinality 2^ω. In this case we take some number y_β in $Z \setminus (X_\beta \cup Y_\beta)$, and we put $X_{\beta+1} = X_\beta$ and $Y_{\beta+1} = Y_\beta \cup \{y_\beta\}$.

Case Two. Z has cardinality $< 2^\omega$. Now the condition σ_β^0 forces the property "The compiled structure is isomorphic to some C_α with $\alpha \in Z$." Let T' be the theory $T \cup \sigma_\beta^0$; then this same property is enforceable with respect to T'. So T' fails (b) above, and hence it is a good theory. Therefore there are an ordinal γ and a condition p for the pre-game on T' such that p forces "The compiled structure is isomorphic to C_γ"—call this property ψ. So $\sigma_\beta^0 \cup p$ forces the same for the pre-game on T, and it follows that γ is constrainable, so that $\gamma \in Y_0 \subseteq Y_{\beta+1}$. But also player $\exists$ can play against σ_β (taking $\sigma_\beta^0 \cup p$ as her first move) in such a way that ψ holds; so $\gamma \in Z$. We set $X_{\beta+1} = X_\beta$ and $Y_{\beta+1} = Y_\beta$.

On the other hand suppose σ_β is an $\exists$-strategy; again put $Z = \{\beta < 2^\omega :$ when player $\exists$ follows σ_β, it is possible for A to be isomorphic to $C_\beta\}$. This time Z must have cardinality 2^ω, since T is bad and the property "A is isomorphic to some C_β with $\beta \in Z$" is enforceable. We choose some $x_\beta \in Z \setminus (X_\beta \cup Y_\beta)$, and we put $X_{\beta+1} = X_\beta \cup \{x_\beta\}$, $Y_{\beta+1} = Y_\beta$.

Finally take ϕ to be the property "A is isomorphic to C_α for some

$\alpha \in \bigcup_{\beta < 2^\omega} Y_\beta$." We assert that ϕ is not determined. First, player $\forall$ has no winning strategy for $G(\phi)$. For suppose σ_β is a strategy for him. By our choice of $Y_{\beta+1}$ we made sure that player $\exists$ can play against σ_β in such a way that A is isomorphic to some C_γ with $\gamma \in Y_{\beta+1}$. Hence σ_β is not a winning strategy for player $\forall$. A similar argument shows that player $\exists$ has no winning strategy. ∎

4. When Is the Fraïssé Pre-game Wholly Determined?

Throughout this section, **K** is a fixed Fraïssé class and we play the Fraïssé pre-game on **K**.

THEOREM 4.1. *The Fraïssé pre-game on* **K** *is wholly determined if and only if there do not exist uncountably many non-isomorphic structures in* $\mathbf{K}_{fg}$.

Proof. The proof needs some notions based on Fraïssé [2]. Fraïssé considered only structures without function or constant symbols, but many of his arguments adapt at once to our setting.

First, an *isomorphism type* is an equivalence class of structures under the relation of isomorphism. We write **F** for the set of isomorphism types of structures in $\mathbf{K}_{fg}$. Since the language is at most countable, the cardinality of **F** is at most 2^ω.

Next, if B is any structure, the *age* of B, in symbols age(B), is the set of all elements of **F** which are the isomorphism types of substructures of B. Thus if B is countable, its age is countable.

Thirdly we say that a structure B in **K** is *weakly homogeneous* if for every pair of structures C, D whose types are in the age of B, with $C \subseteq D$ and $C \subseteq B$, there is an embedding $e: D \to B$ which is the identity on C. By 5.4.2 in Fraïssé [2], any two countable weakly homogeneous structures of the same age are isomorphic.

We prove right to left in Theorem 4.1. Suppose that up to isomorphism, $\mathbf{K}_{fg}$ is countable. Then the argument of 8.3 in Fraïssé [2] shows that there is a unique countable weakly homogeneous structure B of age $\mathbf{K}_{fg}$, and that the property "A is isomorphic to B" is enforceable (and hence also coenforceable). So player $\exists$ has a winning strategy for $G(\phi)$ if B satisfies ϕ, and otherwise player $\forall$ has a winning strategy for $G(\phi)$.

Thus right to left holds in the theorem. For the converse, we suppose henceforth that $\mathbf{K}_{fg}$ is uncountable (counting up to isomorphism).

LEMMA 4.2. (a) *The property "A is weakly homogeneous" is enforceable.*

(b) *If for each $n < \omega$, B_n is a finite or countable structure in* **K**, *then* *there is a structure C in* **K** *which is weakly homogeneous and at most countable, such that each* B_n *is embeddable in C.*

(c) *The class of weakly homogeneous structures is closed under unions of chains.*

Proof. This is all essentially in Fraïssé [2, Sects. 5, 8]. ▌ Lemma.

For any set X, we write $\mathscr{P}_\omega(X)$ for the set of all finite or countable subsets of X. We say that a subset W of $\mathscr{P}_\omega(X)$ is *closed unbounded* in $\mathscr{P}_\omega(X)$ if (1) W is closed under unions of countable chains, and (2) every countable subset of X is a subset of some element of W. A subset of $\mathscr{P}_\omega(X)$ is *fat* if it contains some closed unbounded set. We need the following facts:

LEMMA 4.3. *For any uncountable set* X:

(a) *The intersection of countably many closed unbounded subsets of* $\mathscr{P}_\omega(X)$ *is a closed unbounded subset of* $\mathscr{P}_\omega(X)$.

(b) *The set of all fat subsets of* $\mathscr{P}_\omega(X)$ *is a countably complete non-principal filter over* $\mathscr{P}_\omega(X)$.

Proof. Cf. Kueker [6, Proposition 2.1]. For completeness, here is a sketch. For (a), suppose s_n $(n<\omega)$ are closed unbounded sets and $s=\bigcap_{n<\omega} s_n$. Then s is clearly closed under unions of countable chains. If $Y\in\mathscr{P}_\omega(X)$, choose by induction on i an increasing chain $(Y_i : i<\omega)$ of sets in $\mathscr{P}_\omega(X)$, with $Y\subseteq Y_0$, such that for each $n<\omega$ there are infinitely many i with $Y_i\in s_n$. Then $Y=\bigcup_{i<\omega} Y_i$ is in s_n for each n, and hence in s.

For (b), the set is certainly a countably complete filter $\mathscr{F}$. If $\mathscr{F}$ was principal, there would be a closed unbounded set $s\in\mathscr{F}$ such that $\mathscr{F}=\{t\subseteq\mathscr{P}_\omega(X) : s\subseteq t\}$. Choose some $Y\in s$. Since X is uncountable, there is $x\in X\setminus Y$. Let t be $\{Z\cup\{x\} : Z\in s\}$. Then $s\nsubseteq t$, but t is closed unbounded and so $t\in\mathscr{F}$. ▌ Lemma

Let S be the set of all subsets of **F** which are of the form age(B) for a finite or countable weakly homogeneous structure B in **K**. Thus $S\subseteq\mathscr{P}_\omega(\mathbf{F})$.

LEMMA 4.4. (a) *S is closed unbounded in* $\mathscr{P}_\omega(\mathbf{F})$.

(b) *If* $s\in S$, *then up to isomorphism there is a unique finite or countable weakly homogeneous structure of age s.*

Proof. (a) follows from (b) and (c) of Lemma 4.2. (b) is 5.4.2 of Fraïssé [2]. ▌ Lemma

For each set $W\subseteq\mathscr{P}_\omega(\mathbf{F})$, let ϕ_W be the property which a structure B has if age$(B)\in W$.

LEMMA 4.5. *ϕ_W is enforceable if and only if W is fat.*

Proof. The argument adapts Proposition 2.1(c) of Kueker [6]. Suppose first that ϕ_W is enforceable, and let σ be a winning strategy for player $\exists$ in the game $G(\phi_W)$. Let Y be the set of all those finite or countable subsets of **F** which are closed under all the functions of σ (in the sense that if i is odd and the isomorphism types of $p_0, ..., p_{i-1}$ are in Y, then the isomorphism type of $\sigma^i(p_0, ..., p_{i-1})$ is in Y too). Then Y is closed unbounded in $\mathscr{P}_\omega(\mathbf{F})$, and so by Lemmas 4.3(a) and 4.4(a), $Y \cap S$ is closed unbounded in $\mathscr{P}_\omega(\mathbf{F})$. To prove that W is fat, we show that $Y \cap S \subseteq W$.

Suppose $s \in Y \cap S$. Let **J** be the class of structures with age $\subseteq s$. Then since $s \in S$, **J** is a Fraïssé class by the proof of Fraïssé [2, Theorem VI]. Let G' be the Fraïssé pre-game on **J**. Because s is countable, we are in the first case of Theorem 4.1, and it is enforceable with respect to G' that the compiled structure A will be weakly homogeneous with age s. But since s is closed under the functions of σ, player $\exists$ can use σ in the pre-game G'. Imagine a play of G' in which player $\exists$ uses σ and player $\forall$ ensures that A is weakly homogeneous with age s. We can also regard this play as a play of the game $G(\phi_W)$ on **K** rather than **J**, and here player $\exists$ is using a winning strategy. So the age of A must lie in W, and hence $s \in W$ as required.

For the converse it suffices to show that if W is closed unbounded then ϕ_W is enforceable. Player $\exists$ should play as follows. Whenever player $\forall$ chooses a condition p_{2i}, she should use the unboundedness of W to find a set $s_{2i} \in W$ with $p_{2i} \in s_{2i}$ and (when $i > 0$) $s_{2i-2} \subseteq s_{2i}$. Since s_{2i} is countable, she can use a countable subset of her moves (and property (3) of Fraïssé classes) to ensure that each type in s_{2i} appears in the compiled structure. The result will be that $\text{age}(A) = \bigcup_{i<\omega} s_{2i}$, which is in W since W is closed under unions of countable chains. So player $\exists$ wins $G(\phi_W)$. ▌ Lemma

Now we can prove the remainder of Theorem 4.1. Let $\mathscr{F}$ be the filter of fat subsets of $\mathscr{P}_\omega(\mathbf{F})$. By Lemma 4.3(b), $\mathscr{F}$ is a countably complete non-principal filter over $\mathscr{P}_\omega(\mathbf{F})$. But $\mathscr{P}_\omega(\mathbf{F})$ has cardinality at most 2^ω, and (by a theorem of Ulam, cf. Jech [5, p. 297]) there is no measurable cardinal $\leqslant 2^\omega$. It follows that $\mathscr{F}$ is not an ultrafilter. Hence there is some subset W of $\mathscr{P}_\omega(\mathbf{F})$ such that neither W nor its complement in $\mathscr{P}_\omega(\mathbf{F})$ is in $\mathscr{F}$. Then neither ϕ_W nor $\phi_{\text{not-}W}$ is enforceable. So by Lemma 2.2(a), ϕ_W is not determined. ▌

Condition (3) in the definition of a Fraïssé class **K** is not essential. If we drop it, we can define an equivalence relation $\sim$ on $\mathbf{K}_{\text{fg}}$ by $B \sim C$ iff there is D in $\mathbf{K}_{\text{fg}}$ such that both B and C are embeddable in D. (Transitivity is by (4) in the definition of a Fraïssé class.) Call the equivalence classes the *components* of $\mathbf{K}_{\text{fg}}$. In the Fraïssé pre-game, player $\forall$'s first move decides what component the conditions come from.

Corollary 4.6. *In this broader setting, the Fraïssé pre-game on* $\mathbf{K}$ *is wholly determined if and only if each of the components of* $\mathbf{K}_{\text{fg}}$ *is at most countable (up to isomorphism).*

Proof. Suppose first that each of the components is countable. Then for each component c there is a structure B_c whose age lies in c, such that it is coenforceable that the compiled structure A is isomorphic to B_c. It is enforceable that A is isomorphic to at least one of the B_c. Now the same argument applies as for good theories in Theorem 3.1.

Second, suppose that some component c is uncountable. The proof of Theorem 4.1 shows that if the players are compelled to play within c, there is an undermined property ϕ. Now let ψ be the property "Either the age of A lies outside c, or ϕ holds." If player $\forall$ puts the play of $G(\psi)$ into some other component than c, then he loses. It follows that ψ is undetermined with respect to G. ▮

5. Conclusions

Corollary 5.1. *Let G be either the Ziegler or the Fraïssé pre-game for groups. Then G is not wholly determined.*

Proof. For the Ziegler pre-game, compare Theorem 3.1 with the fact (Hodges [4, Theorem 4.1.6]) that the theory of groups is bad. For the Fraïssé pre-game, note that there are uncountably many non-isomorphic finitely generated groups and use Theorem 4.1. ▮

There are useful classes of properties which are always determined. If L is a first-order language, we write $L_{\omega_1\omega}$ for the language which is like L except that conjunctions and disjunctions of countable sets of formulas are allowed.

Proposition 5.2. (a) *Let T be an inductive theory in a first-order language L, and G the Ziegler pre-game on T. Then every sentence of L is determined. More generally, so is every sentence of $L_{\omega_1\omega}$.*

(b) *Let* $\mathbf{K}$ *be a Fraïssé class of structures for the first-order language L, and G the corresponding pre-game. Then the same conclusion holds.*

Proof. For (a) see Hodges [4, Theorem 2.3.4]. The proof of (b) is similar; here is a sketch for the case when $\mathbf{F}$ is uncountable. To get the effect of witnesses, we first rephrase the Fraïssé game so that the domain of each condition p is a proper initial segment of the ordinal ω^2. Then we add to L a constant c_i for each ordinal $i < \omega^2$, so that c_i names the element i in p. Quantifiers can be replaced by countable conjunctions, writing

$\bigwedge_{i<\omega^2} \phi(c_i)$ in place of $\forall x \phi(x)$. For each $s \in S$ (as in Lemma 4.4) there is up to isomorphism a unique finite or countable weakly homogeneous structure B_s of age s; if ψ is a sentence, write W_ψ for the set of $s \in S$ such that ψ is true in B_s. Then we can show, by induction on the complexity of ψ, that for each quantifier-free sentence ψ, either W_ψ or $W_{\text{not-}\psi}$ is fat. Lemma 4.5 concludes the argument. ∎

Problem. Generalise Theorem 4.1 to the situation where clauses (3) and (4) are dropped from the definition of Fraïssé classes.

For example, is the class of commutative rings wholly determined for Fraïssé games? One can ask the same question for Ziegler games; in this case it is probably relevant that finitely generated commutative rings are residually finite (Baumslag [1, p. 64]), so that we can assume we are dealing only with locally finite rings.

Note added in proof. Saharon Shelah has indicated an answer.

References

1. G. Baumslag, "Lecture Notes on Nilpotent Groups," Amer. Math. Soc., Providence, RI, 1971.
2. R. Fraïssé, Sur l'extension aux relations de quelques propriétés des ordres, *Ann. Sci. École Norm. Sup.* **71** (1954), 363–388.
3. G. Higman and E. Scott, "Existentially Closed Groups," Clarendon, Oxford, 1988.
4. W. Hodges, "Building Models by Games," London Mathematical Society Student Texts 2, Cambridge Univ. Press, London/New York, 1985.
5. T. Jech, "Set Theory," Academic Press, New York, 1978.
6. D. W. Kueker, Countable approximations and Löwenheim–Skolem theorems, *Ann. of Math. Logic* **11** (1977), 57–103.
7. M. Ziegler, Algebraisch abgeschlossene Gruppen, *in* "Word Problems II, The Oxford Book" (S. I. Adian *et al.*, Eds.), pp. 449–576, North-Holland, Amsterdam, 1980.

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