University of Illinois at Urbana-Champaign — Fall 2006
Math 380 — Group G1

Graded Homework IV

Due Friday, October 6.

1. Let $F\colon \mathbb{R}^3\backslash\{(0,0,0)\} \to \mathbb{R}^3$ be the mapping defined by $F(x,y,z) = (\frac{x}{x^2+y^2+z^2}, \frac{y}{x^2+y^2+z^2}, \frac{z}{x^2+y^2+z^2})$.
Let (x,y,z) be on the sphere of center 0 and radius 1 ; show in two different ways that the Jacobian matrix of F at (x,y,z) is equal to its inverse matrix.
(Hint : compute $F \circ F(x,y,z)$ and use the Chain Rule)

2. Assume $F\colon (u,v) \mapsto F(u,v)$ is a continuously differentiable function from $\mathbb{R}^2$ to $\mathbb{R}$ such that $F(0,0) = 0$ and $\frac{\partial F}{\partial v}(0,0) \neq 0$. Let $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $\varphi(x,y,z) = (xy, x^2 - y^2 - z)$, and define $f = F \circ \varphi$.
Show that the equation $f(x,y,z) = 0$ implicitly defines z as a function of (x,y) near $(0,0,0)$, and that one has $x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 2(x^2+y^2)$.

3. Recall that we saw in class that, if a system of two equations $F(x,y,z) = 0$ and $G(x,y,z) = 0$ defines implicitly two of the variables as a function of the third one near a point $P \in \mathbb{R}^3$, then that system of equations defines a curve in the neighborhood of P.
Prove that the system of equations $\begin{cases} 4xy + 2xz + 4y - z = 0 \\ xy + xz + yz + 2x + zy - z = 0 \end{cases}$ defines a curve near $(0,0,0)$. What is the tangent line to this curve at that point ?

4. Consider the application from $\mathbb{R}^3 \times \mathbb{R}^3$ to $\mathbb{R}$ that maps (u,v) to $u.v$. Identifying $\mathbb{R}^3 \times \mathbb{R}^3$ with $\mathbb{R}^6$ (the first three variables giving the coordinates of u, and the last three giving the coordinates of v), compute the Jacobian matrix of this application. Use this, and the Chain Rule, to show that, if $u = u(t)$ and $v = v(t)$, then $(u.v)'(t) = u'(t)v(t) + u(t)v'(t)$.
Similarly, one may consider the cross product $(u,v) \mapsto u \times v$ as a function from $\mathbb{R}^6 \to \mathbb{R}^3$. Write the Jacobian matrix of this application. Use it to show that again $(u \times v)'(t) = u'(t) \times v(t) + u(t) \times v'(t)$.