Midterm I Correction.

1.(15 points)
Compute the gradient $\nabla f(x,y,z)$ of the function $f\colon \mathbb{R}^3 \to \mathbb{R}$ defined by $f(x,y,z) = \cos(xyz)$. Use this to compute the directional derivative of f at the point $(1, \pi, \frac{1}{2})$ in the direction of $u = (3, 0, -4)$.

Correction. By definition, $\nabla f(x,y,z) = (\frac{\partial f}{\partial x}(x,y,z), \frac{\partial f}{\partial y}(x,y,z), \frac{\partial f}{\partial z}(x,y,z))$. Here, this yields $\nabla f(x,y,z) = (-yz\sin(xyz), -xz\sin(xyz), -xy\sin(xyz))$. In particular, we have $\nabla f(1,\pi,\frac{1}{2}) = (-\frac{\pi}{2}, -\frac{1}{2}, -\pi)$. Hence the directional derivative of f at $(1,\pi,\frac{1}{2})$ in the direction u is

$$\nabla_u f(x,y,z) = \nabla f(x,y,z).\frac{u}{||u||} = (-\frac{\pi}{2}, -\frac{1}{2}, -\pi).(\frac{3}{5}, 0, -\frac{4}{5}) = -\frac{3\pi}{10} + \frac{4\pi}{5} = \frac{\pi}{2} \ .$$

2.(15 points)
Suppose $g\colon \mathbb{R}^3 \to \mathbb{R}^2$ is a differentiable function, such that $g(1,-1,2) = (1,5)$ and $Jg(1,-1,2) = \begin{pmatrix} 1 & -1 & 0 \\ 4 & 0 & 2 \end{pmatrix}$ (where $Jg(x,y,z)$ is the Jacobian matrix of g at the point (x,y,z).)
Let then $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the function defined by $f(x,y) = (xy, 3x^2 - 2y + 3)$. Find the Jacobian matrix of $f \circ g$ at the point $(1,-1,2)$
Correction. The chain rule tells us that $J(f\circ g)(1,-1,2) = Jf(g(1,-1,2))Jg(1,-1,2)$. Since we are given the value of $Jg(1,-1,2)$, we have to compute $Jf(g(1,-1,2)) = Jf(1,5)$. By definition of a Jacobian matrix, we have $Jf(x,y) = \begin{pmatrix} y & x \\ 6x & -2 \end{pmatrix}$, so $Jf(1,5) = \begin{pmatrix} 5 & 1 \\ 6 & -2 \end{pmatrix}$. The only thing remaining is to compute the product of the two matrices, which yields

$$J(f\circ g)(1,-1,2) = \begin{pmatrix} 5 & 1 \\ 6 & -2 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 4 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 9 & -5 & 2 \\ -2 & -6 & -4 \end{pmatrix} \ .$$

3.(25 points)
Consider the function $F\colon \mathbb{R}^4 \to \mathbb{R}^2$ defined by

$$F(x_1,x_2,y_1,y_2) = (x_2y_2 - x_1\cos(y_1), x_2\sin(y_1) + x_1y_2 - 1) \ .$$

Does the equation $F(x_1,x_2,y_1,y_2) = (\frac{\pi}{4}, \frac{\pi}{4})$ define implicitly y_1, y_2 as continuously differentiable functions of x_1, x_2 near $(1,1,\frac{\pi}{2},\frac{\pi}{4})$? If so, compute $\frac{\partial y_1}{\partial x_1}(1,1)$ and $\frac{\partial y_1}{\partial x_2}(1,1)$, and use this to compute the equation of the tangent plane to the surface of equation $y_1 = y_1(x_1,x_2)$ at the point $(x_1,x_2,y_1) = (1,1,\frac{\pi}{2})$ (See it as an equation in the three-dimensional space where the variables are x_1, x_2, y_1 and $y_1(x_1,x_2)$ is the function induced by applying the Implicit Function Theorem at the point $(1,1,\frac{\pi}{2},\frac{\pi}{4})$).
What about the same questions for the equation $F(x_1,x_2,y_1,y_2) = (5,1)$ near the point $(x_1,x_2,y_1,y_2) = (0,2,\frac{\pi}{2},\frac{5}{2})$?

Correction. Since we want to apply the Implicit Function Theorem, we first need to compute the Jacobian matrix of F, which is

$$JF(x,y,z) = \begin{pmatrix} -\cos(y_1) & y_2 & x_1\sin(y_1) & x_2 \\ y_2 & \sin(y_1) & x_2\cos y_1 & x_1 \end{pmatrix} .$$

For (y_1, y_2) to be implicitly defined by F as functions of (x_1, x_2) near some point (x_1, x_2, y_1, y_2), it is a necessary and sufficient condition that the matrix $\begin{pmatrix} x_1\sin(y_1) & x_2 \\ x_2\cos(y_1) & x_1 \end{pmatrix}$ be invertible, i.e that its determinant $x_1^2\sin(y_1) - x_2^2\cos(y_1)$ be different from 0.
At the point $(1, 1, \frac{\pi}{2}, \frac{\pi}{4})$, the determinant is $1 \neq 0$, so (y_1, y_2) are defined implicitly as functions of (x_1, x_2) near that point.
To compute the derivative that we are asked for, we use implicit differentiation to obtain

$$\begin{cases} y_2dx_2 + x_2dy_2 - \cos(y_1)dx_1 + x_1\sin(y_1)dy_1 = 0 & (1) \\ \sin(y_1)dx_2 + x_2\cos(y_1)dy_1 + y_2dx_1 + x_1dy_2 = 0 & (2) \end{cases}$$

Taking $x_1(1) - x_2(2)$ yields that

$$(x_1y_2 - x_2\sin(y_1))dx_2 - (\cos(y_1)x_1 + y_2x_2)dx_1 + (x_1^2\sin(y_1) - x_2^2\cos(y_1))dy_1 = 0 \ ,$$

so that $dy_1 = \dfrac{1}{x_1^2\sin(y_1) - x_2^2\cos(y_1)}((\cos(y_1)x_1 + y_2x_2)dx_1 - (x_1y_2 - x_2\sin(y_1))dx_2)$.
At the point $(1, 1, \frac{\pi}{2}, \frac{\pi}{4})$ this becomes $dy_1 = \frac{\pi}{4}dx_1 - (\frac{\pi}{4} - 1)dx_2$.
Thus, we finally obtain $\dfrac{\partial y_1}{\partial x_1}(1,1) = \dfrac{\pi}{4}$, and $\dfrac{\partial y_1}{\partial x_2}(1,1) = 1 - \dfrac{\pi}{4}$.
To obtain the equation of the tangent plane to the surface of equation $y_1 = y_1(x_1, x_2)$ at the point $(1, 1, \frac{\pi}{2})$, one simply writes it as $(x_1 - 1, x_2 - 1, y_1 - \frac{\pi}{2}).(\frac{\pi}{4}, 1 - \frac{\pi}{4}, -1) = 0$, which yields $\frac{\pi}{4}x_1 + (1 - \frac{\pi}{4})x_2 - y_1 = 1 - \frac{\pi}{2}$.
At the point $(x_1, x_2, y_1, y_2) = (0, 2, \frac{\pi}{2}, \frac{5}{2})$ the determinant that appears when one wants to check the hypothesis of the Implicit Function Theorem is 0, so (y_1, y_2) are not defined as implicit functions of (x_1, x_2) near that point.