UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN Fall 2006
Math 380 Group G1

Midterm II.
Monday, October 30.
50 minutes

You are not allowed to use your lecture notes, textbook, or any other kind of documentation. Calculators, mobile phones and other electronic devices are also prohibited.

1.(10 points)
Compute $\int_\gamma (F.\vec{t})ds$ (where $\vec{t}$ is the unit tangent vector along the curve), when F is the vector field defined by $F(x,y) = (x^2-1, y+1)$ and γ is the half-circle of radius 1 joining $(1,0)$ and $(-1,0)$.
Correction We want to compute $I = \int_\gamma (F.\vec{t})ds = \int_\gamma F_x dx + F_y dy$; the first thing to do is to parameterize our path. Here, one can set $x = \cos(t)$, $y = \sin(t)$, with $0 \le t \le \pi$. Thus, our integral is

$$I = \int_0^\pi \big((\cos^2(t)-1)(-\sin(t)) + (\sin(t)+1)\cos(t)\big)dt = \int_0^\pi \big(-\sin(t)\cos^2(t) + \sin(t) + \sin(t)\cos(t) + \cos(t)\big)dt$$

$$= \left[\frac{\cos^3(t)}{3} - \cos(t) + \frac{\sin^2(t)}{2} + \sin(t)\right]_0^\pi = -\frac{2}{3} + 2 + 0 + 0 = \frac{4}{3} \ .$$

2.(20 points)
Use two different methods to compute $\int_\gamma (x+y)dx + (y-x)dy$, where γ is the triangle with edges $(0,2)$, $(1,0)$ and $(0,-1)$, oriented in that order.
Correction. Let us first use the definition of a line integral : as usual, one divides the triangle in three straight line segments. The first one (joining $(0,2)$ to $(1,0)$) has equation $y = 2-2x$, so it may be parameterized by $x = t$, $y = 2-2t$, with $0 \le t \le 1$. Similarly, the second one has equation $y = -1+x$; since x is decreasing from 1 to 0 on that curve, one may set $x = 1-t$, $y = -1+1- = -t$. On the third part of the curve, one has $x = 0$, so one may simply set $y = t$, $-1 \le t \le 2$.
The corresponding line integrals (denoted I_1, I_2, I_3) are :

$$I_1 = \int_{t=0}^1 \big((2-t).1 + (2-3t).(-2)\big)dt = \int_0^1 (5t-2)dt = \frac{5}{2} - 2 = \frac{1}{2} \ ;$$

$$I_2 = \int_{t=0}^1 \big((1-2t)(-1) + (-1)(-1)\big)dt = \int_0^1 2t\,dt = 1 \ ;$$

$$I_3 = \int_{t=-1}^2 \big(0 + t.1\big)dt = \left[\frac{t^2}{2}\right]_{t=-1}^2 = 2 - \frac{1}{2} = \frac{3}{2} \ .$$

Eventually, we obtain $\int_\gamma (x+y)dx + (y-x)dy = I_1 + I_2 + I_3 = 3$.
The second method here is based on Green's theorem ; pay attention to the fact that the curve here is oriented clockwise, so we have

$$I = \int_{x=0}^1 \left(\int_{y=x-1}^{2-2x} (1-(-1))dy\right)dx = 2\int_{x=0}^1 (2-2x-x+1)dx = 2\left[3x - \frac{3x^2}{2}\right]_{x=0}^1 = 3 \ .$$

3.(25 points)
(a) Compute $\iiint_R z\,dxdydz$, where $R=\{(x,y,z)\colon 0\le z\le 1, x^2+y^2+z^2\le 4\}$.
(b) Let $\mathcal{E}$ denote the interior of the ellipse (in the (y,z)-plane) of equation $\dfrac{(y-1)^2}{a^2}+\dfrac{z^2}{b^2}=1$; compute $\iint_{\mathcal{E}} y\,dydz$.
(c) Compute the volume of the torus generated by rotating $\mathcal{E}$ (of question 3(b)) around the z-axis (hint : what is the equation of this torus in cylindrical coordinates ? draw a picture, and use the question above !).
Correction. (a) Using iterated integration, our integral (let's call it I for simplicity) is

$$I=\int_{z=0}^{1}\left(\iint_{x^2+y^2\le 4-z^2} dxdy\right)zdz$$

The double integral inside is just the area of a circle of radius $\sqrt{4-z^2}$, so it is equal to $\pi(4-z^2)$; thus, we obtain

$$I=\pi\int_{z=0}^{1} z(4-z^2)dz=\pi\left[2z^2-\frac{z^4}{4}\right]_{z=0}^{1}=\frac{7\pi}{4}\ .$$

(b) The integral that we are asked to compute is the y- coordinate of the center of gravity of the ellipse $\mathcal{E}$ times the area of the ellipse ; given the symmetries, it has to be the y-coordinate of the center of the ellipse times ab, in other words πab. Of course, we could also compute it using a change of variables, but it's a bit longer. To do it, set $y-1=ar\cos(\theta)$, $z=br\sin(\theta)$; the Jacobian determinant of this transformation is abr, so

$$\iint_{\mathcal{E}} y\,dydz=\int_{r=0}^{1}\left(\int_{\theta=0}^{2\pi}(ar\cos(\theta)+1)abr\,d\theta\right)dr=\int_{r=0}^{1}2\pi abr=\pi ab\ .$$

(c) Denoting the torus by T, our first job is to find a usable equation for it ; going to polar coordinates (r,θ,z), we see that T is just the interior of the domain of equation $\dfrac{(r-1)^2}{a^2}+\dfrac{z^2}{b^2}=1$, so that (denoting by $\mathcal{E}_\theta$ a portion of the torus where θ is fixed)

$$Vol(T)=\iiint_T dxdydz=\int_{\theta=0}^{2\pi}\left(\iint_{\mathcal{E}_\theta} rdrdz\right)d\theta=\int_0^{2\pi}\pi abd\theta=2\pi^2ab\ .$$

(the value of the double integral above is a consequence of question (b).α)