University of Illinois at Urbana-Champaign | Fall 2006
Math 444 | Group E13

Final Exam.

Wednesday, December 13.
3 hours.

You are allowed to use your textbook, but no other kind of documentation.
Calculators, mobile phones and other electronic devices are prohibited.

NAME ____

SIGNATURE ____

1. (20 points)
Define a function $f\colon [0,+\infty) \to \mathbb{R}$ by setting $f(x) = \sin(\sqrt{x})$. Show that f is continuous on $[0,+\infty)$ and differentiable on $(0,+\infty)$. Is f differentiable at 0?

2. (30 points)
Let $0 < \alpha < 1$.
(a) Show that for all $x > 0$ one has

$$\frac{\alpha}{(x+1)^{1-\alpha}} \le (x+1)^{\alpha} - x^{\alpha} \le \frac{\alpha}{x^{1-\alpha}} \ .$$

(b) Define a sequence (u_n) by the formula $u_n = \sum_{k=1}^{n} \frac{1}{k^{\alpha}} = 1 + \frac{1}{2^{\alpha}} + \ldots \frac{1}{n^{\alpha}}$.
Use the inequalities above (applied to $\alpha' = 1 - \alpha$) to prove that for all $n \in \mathbb{N}$ one has

$$(1-\alpha)(u_n - 1) \le n^{1-\alpha} - 1 \le (1-\alpha)u_{n-1} \le (1-\alpha)u_n \ .$$

Prove that (u_n) is not convergent but $\left(n^{\alpha-1}u_n\right)$ is, and compute $\lim \left(n^{\alpha-1}u_n\right)$.

3. (30 points)

Let f be continuous on $[0, +\infty)$; for all $x > 0$, set $g(x) = \dfrac{1}{x}\displaystyle\int_0^x f(t)dt$.

(a) Show that g is continuous on $(0, +\infty)$, and that g has a limit at 0; compute this limit.

(b) Show that g is differentiable on $(0, +\infty)$ and that for all $x > 0$ one has

$$g'(x) = \frac{f(x) - g(x)}{x} \ .$$

4. (30 points)

Pick two real numbers a, b such that $a < b$ and let $f\colon [a,b] \to \mathbb{R}$ be continuous. We want to show that

$$\sup\{f(x)\colon x \in (a,b)\} = \sup\{f(x)\colon x \in [a,b]\} \ .$$

(a) Explain why $\sup\{f(x)\colon x \in (a,b)\}$ and $\sup\{f(x)\colon x \in [a,b]\}$ exist.

(b) Show that $\sup\{f(x)\colon x \in (a,b)\} \leq \sup\{f(x)\colon x \in [a,b]\}$.

(c) Assume $f(a) = \sup\{f(x)\colon x \in [a,b]\}$. Show that one also has $f(a) = \sup\{f(x)\colon x \in (a,b)\}$. Can you prove a similar result when $f(b) = \sup\{f(x)\colon x \in [a,b]\}$?

(d) Prove the equality $\sup\{f(x)\colon x \in (a,b)\} = \sup\{f(x)\colon x \in [a,b]\}$.

5. (30 points)
Let $0 < \lambda < 1$ and $f\colon \mathbb{R} \to \mathbb{R}$ be such that $f(\lambda x) = \lambda f(x)$ for all $x \in \mathbb{R}$.
(a) Prove that $f(0) = 0$.
(b) Assume that f is differentiable at 0. Show that there exists $a \in \mathbb{R}$ such that $f(x) = ax$ for all $x \in \mathbb{R}$.
Hint. What can you say of the sequence $\left(\dfrac{f(\lambda^n x)}{\lambda^n x}\right)$? Show that $a = f'(0)$ works .
(c) Is the result above still true if one no longer assumes that f is differentiable at 0?

6. (30 points)
Let $f\colon [0,1] \to [0,1]$ be an increasing function (not necessarily continuous). Show that there exists $x \in [0,1]$ such that $f(x) = x$.
Hint. Consider the set $E = \{x \in [0,1] \colon f(x) > x\}$; show that one can assume that $0 \in E$. Show that $x = \sup(E)$ works.

7. (30 points)
Recall that if X is a set, one denotes by $\mathcal{P}(X)$ the set whose elements are the subsets of X; in other words, $\mathcal{P}(X) = \{A \colon A \subset X\}$. Let now X, Y be sets and $f \colon X \to Y$ be a function.
(a) Define a function $\hat{f} \colon \mathcal{P}(X) \to \mathcal{P}(Y)$ by setting $\hat{f}(A) = f(A)$ for all $A \subset X$.
Show that $\hat{f}$ is injective if, and only if, f is injective.
(b) Similarly, define a function $\tilde{f} \colon \mathcal{P}(Y) \to \mathcal{P}(X)$ by setting $\tilde{f}(B) = f^{-1}(B)$ for all $B \subset Y$. Compute $\tilde{f}(\emptyset)$.
Show that $\tilde{f}$ is injective if, and only if, f is surjective.

Note. To solve this exercise, you need to remember the following principle : to show that two subsets A, B of a set X are equal, one has to prove that $A \subset B$ and $B \subset A$; in other words, one must show that for all $x \in X$ $x \in A \Rightarrow x \in B$, and $x \in B \Rightarrow x \in A$.