1. Prove by induction that for all $n \geq 1$ one has

$$\frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{n^2} < 2 \; .$$

Correction.
First, following the hint, let us call $P(n)$ the property $\frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$. Then $P(1)$ is the assertion "$1 \leq 1$", so $P(1)$ is true.
Now, assume that $n \in \mathbb{N}$ is such that $P(n)$ is true. We wish to deduce from this that $P(n+1)$ is also true. We have

$$\frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{(n+1)^2} = \frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{n^2} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n} + \frac{1}{(n+1)^2} (*)$$

by the induction hypothesis. To obtain what we want, it is therefore enough to prove that $-\frac{1}{n} + \frac{1}{(n+1)^2} \leq -\frac{1}{n+1}$, which is equivalent to $\frac{1}{(n+1)^2} \leq \frac{1}{n} - \frac{1}{n+1}$.
Since $\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$, we see that indeed $\frac{1}{n} - \frac{1}{n+1} \geq \frac{1}{(n+1)^2}$; putting this back into (*), we obtain that

$$\frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{(n+1)^2} \leq \frac{1}{n+1} \; .$$

This proves that $P(n+1)$ is true as soon as $P(n)$ is ; since $P(1)$ is also true, the induction theorem enables us to conclude that $P(n)$ is true foe all $n \in \mathbb{N}$.

2. Let us prove by induction that all that the pencils in the world are the same color : denote by $P(n)$ the property "in any group of n pencils, they are all the same color". Then $P(1)$ est vraie.
Let us now assume that $P(n)$ is true, and try to prove that $P(n+1)$ is also true. Given a group of $n+1$ pencils, take one of them away : by the induction hypothesis, the n pencils remaining are all the same color. Put that pencil back, and take away another one : the n pencils remaining are again the same color ; consequently, all the $n+1$ pencils are the same color, so $P(n+1)$ is true.
Therefore $P(n)$ is true for all n, and we are done with the proof.
What is the problem with the reasoning above ?

Correction.
The fact that $P(1)$ is true is clear, so the problem must be somewhere else ; look at how the above reasoning shows that, if $P(1)$ is true, then $P(2)$ is true : one takes away one of the pencils, and says that all the others (i.e, the other one) are all the same color ; this is true. Taking away the other pencil, one again says that the others (i.e, the first one) are the same color (again, true). Then one says that this implies that all the pencils are the same color : this is obviously false, since in that case the first group of pencils and the second group have empty intersection, so knowing that all the pencils in the first group are the same color and that all pencils in the second group are the same color does not imply that all pencils in the reunion of the groups are the same color. Notice, however, that for $n \geq 2$ the proof of the fact that $P(n) \Rightarrow P(n+1)$ is perfectly correct ; it just happens to be useless, since $P(n)$ is false for all $n \geq 2$.

3. Let X, Y, Z be three sets and $f \colon Y \to Z$, $g \colon X \to Y$ two bijective maps.
- What is the domain of $f \circ g$? what is its range ?
- Same questions for $(f \circ g)^{-1}$.
- Prove that $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

Correction.

• By definition, the domain of $f \circ g$ is the set of points x which belong to the domain of g and are such that $g(x)$ belongs to the domain of f ; therefore, in that case, the domain of $f \circ g$ is simply the domain of g, i.e X. Since both maps are bijective, we know that the range of $f \circ g$ is simply Z ; to prove it, let $z \in Z$. Then, since f is onto, there is some $y \in Y$ such that $z = f(y)$; since g is onto, there exists $x \in X$ such that $y = g(x)$. The two equalities together give us $z = f(g(x)) = (f \circ g)(x)$, so z is in the range of $f \circ g$.
This proves that $Z \subset R(f \circ g)$, and the converse inclusion is a consequence of the definition of f, g. Therefore, $R(f \circ g) = Z$.

• By definition of an inverse mapping, the domain of $(f \circ g)^{-1}$ is the range of $f \circ g$, so it is Z ; similarly, the range of $(f \circ g)^{-1}$ is the domain of $f \circ g$, so it is X.

• The map $(f \circ g)^{-1}$ is defined by the following relation, for $z \in Z$ and $x \in X$:
$x = (f \circ g)^{-1}(z) \Leftrightarrow (f \circ g)(x) = z$ (The fact that this defines a function is due to our assumption on f and g, which implies that $f \circ g$ is bijective). Therefore, to check that $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$, it is enough to prove that, for all $z \in Z$, one has $(f \circ g)(g^{-1} \circ f^{-1}(z)) = z$. One has $(f \circ g)(g^{-1} \circ f^{-1}(z)) = f(g(g^{-1}(f^{-1}(z))))$; by definition of an inverse mapping, $g(g^{-1}(y)) = y$ for all $y \in Y$, so that $g(g^{-1}(f^{-1}(z))) = f^{-1}(z)$; this yields $f(g(g^{-1}(f^{-1}(z)))) = f(f^{-1}(z)) = z$, which finishes the proof of the fact that $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

4. Let X, Y, Z be three sets and $f\colon X \to Y$, $g\colon Y \to Z$ be functions . Prove the following assertions :
• $(g \circ f$ one-to-one$) \Rightarrow (f$ one-to-one$)$
• $(g \circ f$ onto$) \Rightarrow (g$ onto$)$.
Are the converse assertions true in general ?

Correction.

• Assume that $g \circ f$ is one-to-one, and let $x, x' \in X$ be such that $f(x) = f(x')$. Then we have $g(f(x)) = g(f(x'))$, in other words $(g \circ f)(x) = (g \circ f)(x')$. Our assumption on $g \circ f$ implies that $x = x'$, and this is enough to prove that f is one-to-one.

• Assume now that $g \circ f$ is onto, and pick $z \in Z$. We want to find some $y \in Y$ such that $z = g(y)$; by the hypothesis on $g \circ f$, we know that there exists some $x \in X$ such that $z = (g \circ f)(x) = g(f(x))$. Letting $y = f(x)$, we obtain $z = g(y)$, and this proves that g is onto.
To see that the converse assertions are not true in general, let $f(x) = x$ for all $x \in \mathbb{R}$ and $g(x) = 0$ for all $x \in \mathbb{R}$. Then f is one-to-one but $g \circ f$ is not. Similarly, if one sets $f(x) = 0$ for all $x \in \mathbb{R}$ and $g(x) = x$ for all $x \in \mathbb{R}$, then g is onto but $g \circ f$ is not.

5. Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be the functions defined by $f(x) = x^2$, $g(x) = x + 2$. Let $h = f \circ g$.
• Determine $h(\mathbb{R})$ and $h(E)$, where $E = \{x \in \mathbb{R}\colon 0 \leq x \leq 1\}$.
• Determine $h^{-1}(F)$ and $h^{-1}(G)$, where $F = (1, +\infty)$ and $G = \{x \in \mathbb{R}\colon -2 \leq x \leq 4\}$.

Correction.

• We have, for all $x \in \mathbb{R}$, $h(x) = (x+2)^2$. Therefore, $h(x) \geq 0$ for all x, and this proves that $h(\mathbb{R})$ is a subset of $[0, +\infty)$. Conversely, pick some $y \geq 0$; we want to find x such that $y = (x+2)^2$; it is clear that $x = \sqrt{y} - 2$ is a solution of that equation, so that $y = h(x)$ and $y \in h(\mathbb{R})$. This means that $h(\mathbb{R}) = [0, \infty)$.
Similarly, if $x \in E$ then we have $0 \leq x \leq 1$, so $2 \leq x \leq 3$ and $4 \leq h(x) = (x+2)^2 \leq 9$. Therefore $h(E) \subset [4, 9]$. Conversely, if $y \in [4, 9]$ then $x = \sqrt{y} - 2 \in [0, 1] = E$, and $y = h(x)$. This implies that $h(E) = [4, 9]$.

• $x \in h^{-1}(F) \Leftrightarrow h(x) \in F \Leftrightarrow (x+2)^2 > 1 \Leftrightarrow |x+2| > 1 \Leftrightarrow x + 2 > 1$ or $x + 2 < -1$.
Therefore, $x \in h^{-1}(F) \Leftrightarrow x > -1$ or $x < -3$; this means that $h^{-1}(F) = (-\infty, -3) \cup (-1, +\infty)$.
The same method yields that

$$x \in h^{-1}(G) \Leftrightarrow -2 \leq (x+2)^2 \leq 4 \Leftrightarrow |x+2| \leq 2 \Leftrightarrow x \in [-4, 0] \ .$$

Therefore, $h^{-1}(G) = [-4, 0]$.

6. Let $f\colon \mathbb{N} \to \mathbb{N}$ and $g\colon \mathbb{N} \to \mathbb{N}$ be the functions defined by

$$\forall k \in \mathbb{N}\ f(k) = 2k \text{ and } g(k) = \begin{cases} \frac{k}{2} & \text{if } k \text{ is even} \\ \frac{k+1}{2} & \text{if } k \text{ is odd} \end{cases} \ .$$

• Determine if f is one-to-one, onto ; same question for g.
• Compute $f \circ g$ and $g \circ f$; determine whether they are one-to-one, onto.

Correction.
• It is clear that f is not onto because $f(k)$ is even for all k, so that the range of f does not contain any odd integer. On the other hand, f is one-to-one because for all $k, k' \in \mathbb{N}$ one has $f(k) = f(k') \Leftrightarrow 2k = 2k' \Leftrightarrow k = k'$. The function g is not one-to-one because (for instance) $g(1) = g(2) = 1$. On the other hand, it is onto, because for all $k \in \mathbb{N}$ one has $g(2k) = k$, so that $k \in R(g)$.
• To compute $f \circ g$, we need to distinguish two cases (because of the way g is defined) : if k is even, then $g(k) = \dfrac{k}{2}$, so $f \circ g(k) = f(g(k)) = 2\dfrac{k}{2} = k$; if k is odd, then $g(k) = \dfrac{k+1}{2}$, so $f \circ g(k) = k + 1$. This means that $f \circ g(k)$ is even for all k (we already knew that ; why ?), so that $f \circ g$ is not onto ; furthermore, $f \circ g(1) = 2 = f \circ g(2)$ so $f \circ g$ is not one-to-one either.
Computing $g \circ f$ is easier : $f(k) = 2k$ is even for alll k, so that $g(f(k)) = \dfrac{f(k)}{2} = k$; thus it is obvious that $g \circ f$ is both one-to-one and onto.
Notice that here we have $g \circ f(n) = n$ for all n, yet neither g nor f is a bijection. Compare this with the characterization of bijections as the functions admitting an inverse function. What's the difference here ?