University of Illinois at Urbana-Champaign — Fall 2006
Math 444 — Group E13

Graded Homework II
Due Friday, September 15 .

1. Let $f\colon \mathbb{N} \to \mathbb{R}$ be defined by : $f(1) = 1$, $f(2) = 2$ and, for all $n \geq 2$, $f(n+1) = \begin{cases} 2f(\frac{n}{2}) + 1 & \text{if } n \text{ is even} \\ f(n-1) + 2 & \text{if } n \text{ is odd} \end{cases}$.
Prove by induction that $f(n) = n$ for all n.
Correction. Let $P(n)$ denote the property "$f(n) = n$". Then $P(1)$ is true, and $P(2)$ is also true.
Assume now that $n \geq 2$ is such that $P(n)$ is true for all $k \leq n$ (strong induction). Then, if n is even, we have that $f(\frac{n}{2}) = \frac{n}{2}$ by our strong induction hypothesis, so that $f(n+1) = 2f(\frac{n}{2}) + 1 = n+1$. If n is odd, then the fact that $f(n-1) = n-1$ implies that $f(n+1) = f(n-1) + 2 = n+1$. In both cases, we see that $f(n+1) = n+1$, so that $P(n+1)$ is true. The strong induction theorem enables us to conclude that $P(n)$ is true for all $n \in \mathbb{N}$, i.e $f(n) = n$ for all $n \in \mathbb{N}$.
(Remark : as usual, if you want to prove this using classic induction rather than strong induction, you may simply introduce the following property $P'(n)$:"$f(k) = k$ for all $k \leq n$").

2. Let $f\colon E \to F$ be a function ; show that :
(a) $\big(f$ is one-to-one$\big) \Leftrightarrow \big($for all $A, A' \subset E,\ f(A) \cap f(A') = f(A \cap A')\big)$;
(b) $\big(f$ is onto$\big) \Leftrightarrow \big($ for all $B \subset F$ $B = f(f^{-1}(B))\big)$.
Correction. (a) Assume that f is one-to-one ; by definition, we have, for all $A, A' \subset E$, that $f(A \cap A') = \{f(x)\colon x \in A \cap A'\} \subset f(A) = \{f(x)\colon x \in A\}$ (because $A \cap A' \subset A$). Similarly, we have $f(A \cap A') \subset f(A')$, so that it is always true that $f(A \cap A') \subset f(A) \cap f(A')$. Thus, we simply have to prove that $f(A) \cap f(A') \subset f(A \cap A')$. Let now $x \in f(A) \cap f(A')$: by definition, there exist $a \in A$ and $a' \in A'$ such that $x = f(a)$ and $x = f(a')$. Since f is one-to-one, we necessarily have $a = a'$, so that a belongs to both A and A', and $x \in f(A \cap A')$. Therefore, $f(A) \cap f(A') \subset f(A \cap A')$, and this is enough to finish the proof of the left-to-right sense of the implication. To go from right to left, assume that $f(A) \cap f(A') = f(A \cap A')$ for all $A, A' \subset E$, and pick $x \neq y \in E$. Then $\{x\} \cap \{y\} = \emptyset$, so our hypothesis implies that $f(\{x\}) \cap f(\{y\}) = \emptyset$. Since $f(\{x\}) = \{f(x)\}$, and $f(\{y\}) = \{f(y)\}$, this means that $f(x) \neq f(y)$, so that f is one-to-one. This concludes the proof of the first equivalence.
(b) Notice that $f(f^{-1}(B)) = \{f(x)\colon x \in E \text{ and } f(x) \in B\}$, so one always has $f(f^{-1}(B)) \subset B$. Assuming now that f is onto, we need to prove that $B \subset f(f^{-1}(B))$. For that, pick $b \in B$; since f is onto, there exists $z \in E$ such that $f(z) = b$. In particular $f(z) \in B$, so $z \in f^{-1}(B)$ and $b \in f(f^{-1}(B))$. We finally obtain $b \in f(f^{-1}(B))$ for all $b \in B$, which yields $B \subset f(f^{-1}(B))$. This is enough to prove that $B = f(f^{-1}(B))$ for all $B \subset F$ if f is onto.
To prove the converse, assume that $B = f(f^{-1}(B))$ for all $B \subset F$, and pick $z \in F$. Then $\{z\} = f(f^{-1}(\{z\}))$, so in particular $f^{-1}(z) = \{x\colon f(x) = z\}$ is nonempty (otherwise its image by f would be the empty set) ; in other words, there exists x such that $f(x) = z$, which proves that g is onto. This concludes the proof of the second equivalence.

3. Let $f\colon E \to F$ be a function. Given $A \subset E$, $B \subset F$, are the following assertions true in general ? You have to either prove the result or provide a counterexample, and explain your assertions in detail (using if necessary the definition of a finite set given in class).
(a) A is finite $\Rightarrow f(A)$ is finite .
(b) $f(A)$ is finite $\Rightarrow A$ is finite.
(c) B is finite $\Rightarrow f^{-1}(B)$ is finite.
(d) $f^{-1}(B)$ is finite $\Rightarrow B$ is finite .
Correction. (a) is true : stating that A is finite means that there is a bijection φ from $\{1, \ldots, m\}$ onto A for

some $m \in \mathbb{N}$. Then $f \circ \varphi$ is a surjection from $\{1, \dots, m\}$ onto $f(A)$, which means that $f(A)$ is finite.
(b) is false : simply consider $E = F = \mathbb{N}$, $A = \mathbb{N}$, and $f(x) = 1$ for all x. Then $f(\mathbb{N}) = \{1\}$ is finite, but we saw in class that $\mathbb{N}$ is infinite.
(c) is false, as is shown by the same counterexample as for (b) : with the notations of (b), we have $f^{-1}(\{1\}) = \mathbb{N}$.
(d) is again false : consider the same function f as above, and $B = \{n \in \mathbb{N} \colon n \geq 2\}$. Then B is infinite, but $f^{-1}(B) = \emptyset$, so $f^{-1}(B)$ is finite.

4. Let $x, y \in \mathbb{R}$. Prove that $\max(x, y) = \dfrac{x + y + |x - y|}{2}$, and $\min(x, y) = \dfrac{x + y - |x - y|}{2}$.
Correction. If $x \geq y$, then $\max(x, y) = x$ and $\dfrac{x + y + |x - y|}{2} = \dfrac{x + y + x - y}{2} = x$, so in that case we do have $\max(x, y) = \dfrac{x + y + |x - y|}{2}$. If $x < y$, then $\max(x, y) = y$, and $\dfrac{x + y + |x - y|}{2} = \dfrac{x + y - x + y}{2} = y$, so again we obtain $\max(x, y) = \dfrac{x + y + |x - y|}{2}$. Since necessarily one of the assertions "$x \geq y$" and "$x < y$" holds, this is enough to prove the result.
Similarly, if $x \geq y$ then $\dfrac{x + y - |x - y|}{2} = y = \min(x, y)$, and if $x < y$ then $\dfrac{x + y - |x - y|}{2} = x = \min(x, y)$, so we see that the second statement is also true.

5. Prove that, for all $a, b \in \mathbb{R}$, one has $|a - b| + |a + b| \geq |a| + |b|$.
Correction. To prove this, as usual with statements involving the absolute value, we break the proof into subcases :
(a) $a - b \geq 0$ and $a + b \geq 0$. Then $|a - b| + |a + b| = a - b + a + b = 2a$; but we know that we have both $a \geq b$ (because $a - b \geq 0$) and $a \geq -b$ (because $a + b \geq 0$), so that $a \geq |b|$. This gives us that $a = |a|$ (because $a \geq |b| \geq 0$), and $2a = a + a \geq |a| + |b|$, so finally we obtain $|a - b| + |a + b| \geq |a| + |b|$.
(b) $a - b \geq 0$ and $a + b < 0$. Then $|a - b| + |a + b| = a - b - a - b = -2b$. We have this time $a < -b$, so knowing that $a \geq b$ yields $-b > b$, in other words $b < 0$, so$-b = |b|$. Notice that we have $b \leq a < -b$, so that $|a| \leq |b|$, and thus $|a - b| + |a + b| = 2|b| \geq |b| + |a|$.
(c) $a - b < 0$ and $a + b \leq 0$. Setting $a' = -a$, $b' = -b$, one can see, since $|x| = |-x|$ for any $x \in \mathbb{R}$, that this case may be reduced to case (a) (why ?).
(d) $a - b < 0$ and $a + b > 0$. For the same reason, this may be reduced to case (b).
Notice that there are several possible ways of breaking the proof into subcases ; for instance, here, one could have used the cases "a and b are ≥ 0"," $a \geq 0$ and $b < 0$", "$a < 0$ and $b \geq 0$", and "$a < 0$ and $b < 0$". As an exercise, you may try to write down the proof using these subcases, noticing that here again the two last cases may be reduced to the first two.

6. Using only the axioms seen in class (or those in the textbook), prove that, for all reals a, b, c, d, the following assertions are true :
• $(a + b) + (c + d) = (a + d) + (c + b)$.
• $ab = 0 \Rightarrow a = 0$ or $b = 0$).
Correction. To prove the first one, we may write that
$(a + b) + (c + d) = a + (b + (c + d))$ (associativity) $= a + ((b + c) + d)$ (associativity again)
$= a + (d + (c + b))$ (commutativity, twice) $= (a + d) + (c + b)$ (associativity).
For the second one, we use a proof by contradiction : assume that a, b are not 0 and $ab = 0$. Then first recall that $0.x = x.0 = 0$ for all $x \in \mathbb{R}$ (this was done in class) ; therefore, $\dfrac{1}{a}(ab) = 0$, but by associativity of multiplication, and by definition of the inverse, $\dfrac{1}{a}(ab) = (\dfrac{1}{a}a)b = 1.b = b$. Thus this gives $b = 0$, which contradicts our assumption.
(Notice that we could have avoided a proof by contradiction : we could have proved directly that ($a \neq 0$ and $b \neq 0$) $\Rightarrow ab \neq 0$, which is equivalent to the property of the reals we proved above).