1. Compute, if they exist, sup(A) and inf(A) in the following cases. In each case, state whether A admits a maximal element, and do the same for minimal elements.

\[ A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}; \quad A = \left\{ x \in \mathbb{Q} : x^2 < 2 \right\}; \quad A = \left\{ (-1)^n + \frac{1}{n} : n \in \mathbb{N} \right\}. \]

**Correction.** Let \( A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}. \) Then for all \( a \in A \) we have \( a = \frac{1}{n} \) for some \( n \in \mathbb{N} \), so that \( 0 \leq a \leq 1 \). This shows that 0 is a lower bound of \( A \) and that 1 is a upper bound of \( A \). So both sup(\( A \)) and inf(\( A \)) exist and, since \( 1 \in A \), this immediately implies that sup(\( A \)) = 1. Furthermore, by the archimedean property of \( \mathbb{R} \), we know that for any \( \varepsilon > 0 \) there exists \( n \in \mathbb{N} \) such that \( \frac{1}{n} \leq \varepsilon \), so that for any \( \varepsilon > 0 \) there exists some \( a \in A \) such that \( a \leq 0 + \varepsilon \). Since 0 is a lower bound of \( A \) this proves that \( 0 = \inf(A) \). Thus we see that \( A \) admits a maximal element (because sup(\( A \)) \( \in A \)) but no minimal element (because inf(\( A \)) \( \notin A \)).

Let now \( A = \left\{ x \in \mathbb{Q} : x^2 < 2 \right\}. \) We have \( a^2 < 2 \) for all \( a \in A \), so that \( -\sqrt{2} \leq a \leq \sqrt{2} \) for all \( a \in A \), and this proves that both sup(\( A \)) and inf(\( A \)) exist.

To compute sup(\( A \)), recall that we saw in class that for any \( x \in \mathbb{R} \) and any \( \varepsilon > 0 \) there exists \( q \in \mathbb{Q} \) such that \( x - \varepsilon \leq q \leq x \). Applying this to \( x = \sqrt{2} \), we see that for any \( \varepsilon > 0 \) there exists \( a \in A \) such that \( a \geq x - \varepsilon \); since we saw that \( \sqrt{2} \) is a upper bound of \( A \) this is enough to prove that sup(\( A \)) = \( \sqrt{2} \). The same idea works to prove that inf(\( A \)) = \( -\sqrt{2} \); notice that one could also use the fact that \( A = \{ -a : a \in A \} \) (why?), which shows inf(\( A \)) = \( -\sqrt{2} \). We saw in class that \( \sqrt{2} \notin \mathbb{Q} \), so that sup(\( A \)) \( \notin A \), inf(\( A \)) \( \notin A \), and this proves that \( A \) has neither a maximal element nor a minimal element.

Let this time \( A = \left\{ (-1)^n + \frac{1}{n} : n \in \mathbb{N} \right\}. \) We have \( -1 \leq (-1)^n + \frac{1}{n} \leq 1 \) for all \( n \in \mathbb{N} \), so that \( -1 \) is a lower bound of \( A \), 2 is a upper bound of \( A \), and both sup(\( A \)) and inf(\( A \)) exist. Let us now compute inf(\( A \)):

\[ \text{for all } \varepsilon > 0, \text{ we have } \frac{1}{n} \leq \varepsilon \text{ for all big enough } n. \] Therefore, if \( n \) is odd, we have \( (-1)^n + \frac{1}{n} \leq -1 + \varepsilon \) and this, added to the fact that \( -1 \) is a lower bound of \( A \), yields \( -1 = \inf(A) \). Therefore \( A \) doesn’t have a minimal element \( (-1) \notin A \). To compute sup(\( A \)), notice that for all \( n \geq 2 \) we have \( (-1)^n + \frac{1}{n} \leq 1 = \frac{1}{2} = \frac{3}{2} \). Since the element obtained for \( n = 1 \) is 0, this proves that \( a \leq \frac{3}{2} \) for all \( a \in A \); and since \( \frac{3}{2} \in A \), this is enough to conclude that sup(\( A \)) = \( \frac{3}{2} \) and that \( A \) has a maximal element.

2. Let \( A = \{ x^2 + y^2 : x, y \in \mathbb{R} \text{ and } xy = 1 \} \). Prove that \( A \) is bounded below, but not bounded above. Compute inf(\( A \)).

**Correction.** For all \( a \in A \), we have \( a = x^2 + y^2 \) for some \( x, y \in \mathbb{R} \). It implies that \( a \geq 0 \), so 0 is a lower bound of \( A \), which proves that \( A \) is bounded below.

To prove that \( A \) is not bounded above, notice that \( n^2 + \frac{1}{n^2} \in A \) for all \( n \in \mathbb{N} \) (because \( n, \frac{1}{n} = 1 \) for all \( n \)), so any upper bound \( u \) of \( A \) would have to satisfy \( u \geq n^2 \) for all \( n \in \mathbb{N} \). We saw in class that \( \mathbb{N} \) is not bounded above, so this is impossible, and this proves that \( A \) has no upper bound.

To compute inf(\( A \)), let \( x \) and \( y \) be such that \( xy = 1 \). Then \( x^2 + y^2 = x^2 + \frac{1}{x^2} \). Looking at a picture of a circle and a hyperbola (what is the link here?), one can guess that inf(\( A \)) is attained for \( x = y = 1 \). To prove this, notice that since the equation for \( A \) depends only on \( |x| \) and \( |y| \) one may assume that both are positive; furthermore, one of \( |x| \) and \( |y| \) has to be bigger than 1 (why?), so one may assume without loss of generality...
that $x \geq 1$. We have then $x = 1 + \varepsilon$, with $\varepsilon > 0$. Then one can write
\[
x^2 + \frac{1}{x^2} = \frac{(1+\varepsilon)^2 + 1}{(1+\varepsilon)^2} = 1 + 2\varepsilon + \varepsilon^2 + 1 = 2 + \frac{(2\varepsilon + \varepsilon^2)(1 + 2\varepsilon + \varepsilon^2) + 1}{1 + 2\varepsilon + \varepsilon^2} = 2 + \frac{2\varepsilon^3 + \varepsilon^4}{1 + 2\varepsilon + \varepsilon^2} \geq 2.
\]

Therefore 2 is a lower bound of $A$; since one has $2 \in A$, this proves that $\inf(A) = 2$.

3. Let $A, B \subset \mathbb{R}$ be bounded subsets of $\mathbb{R}$. We define $A + B = \{a + b : a \in A, b \in B\}$.
Show that $\sup(A)$, $\sup(B)$, $\sup(A + B)$ exist and that $\sup(A + B) = \sup(A) + \sup(B)$.

**Correction.**

By definition of a bounded set, there exist $M, N$ such that $a \leq M$ for all $a \in A$, and $b \leq N$ for all $b \in B$. This implies that $a + b \leq M + N$ for all $(a, b) \in A \times B$; in other words, $x \leq M + N$ for all $x \in A + B$, which proves that $A + B$ is bounded above, so that $\sup(A + B)$ exists. The fact that $\sup(A)$, $\sup(B)$ exist is a direct consequence of the fact that $A, B$ are bounded.

Notice that above we could have taken $M = \sup(A)$, $N = \sup(B)$, so that the preceding inequality implies that $x \leq \sup(A) + \sup(B)$ for all $x \in A + B$; in other words, $\sup(A) + \sup(B)$ is an upper bound of $A + B$, so that $\sup(A + B) \leq \sup(A) + \sup(B)$.

To show the converse inequality, we need to find, for all $\varepsilon > 0$, some $x \in A + B$ such that $x \geq \sup(A) + \sup(B) - \varepsilon$.

We know that, for all $\delta > 0$, there exists $a \in A$ such that $a \geq \sup(A) - \delta$, and $b \geq \sup(B) - \delta$; this implies $a + b \geq (\sup(A) + \sup(B)) - 2\delta$. Thus, if we now let $\delta = \frac{\varepsilon}{2}$, the above inequality becomes $a + b \geq (\sup(A) + \sup(B)) - \varepsilon$.

Therefore there does exist, for all $\varepsilon > 0$, some $x \in A + B$ such that $x \geq \sup(A) + \sup(B) - \varepsilon$. This concludes the proof of the fact that $\sup(A + B) = \sup(A) + \sup(B)$.

4. Let $A \subset \mathbb{R}$ be a bounded set containing at least two elements, and $x \in A$.

(a) Prove that $\sup(A \setminus \{x\})$ exists (remember that $A \setminus \{x\} = \{a \in A : a \neq x\}$).

(b) Prove that if $x < \sup(A \setminus \{x\})$ then $\sup(A \setminus \{x\}) = \sup(A)$.

(c) Prove that if $\sup(A \setminus \{x\}) < \sup(A)$ then $x = \sup(A)$.

**Correction.**

(a) $A \setminus \{x\} \subset A$, so any upper bound of $A$ is also an upper bound of $A \setminus \{x\}$; since $A$ is bounded, this proves that the set of upper bounds of $A \setminus \{x\}$ is nonempty, so that $\sup(A \setminus \{x\})$ exists (and is $\leq \sup(A)$).

(b) The fact that $A \setminus \{x\} \subset A$ implies that $\sup(A \setminus \{x\}) \leq \sup(A)$. To see that the converse inequality is true in our case, let $\varepsilon > 0$ be small enough that $\sup(A) - \varepsilon > x$. By definition of the sup, there exists $a \in A$ such that $a > \sup(A) - \varepsilon$. This implies that $a \neq x$, so that we actually proved that for all $\varepsilon > 0$ there is $a \in A \setminus \{x\}$ such that $a > \sup(A) - \varepsilon$. This shows that $\sup(A) - \varepsilon$ is not an upper bound of $A \setminus \{x\}$, so $\sup(A) - \varepsilon \leq \sup(A \setminus \{x\})$, for all $\varepsilon > 0$, so that $\sup(A) \leq \sup(A \setminus \{x\})$. This concludes the proof of the fact that $\sup(A) = \sup(A \setminus \{x\})$.

(c) Assume that $\sup(A \setminus \{x\}) < \sup(A)$, and pick $\varepsilon > 0$ small enough that $\sup(A) - \varepsilon > \sup(A \setminus \{x\})$. By definition of the sup, there exists $a \in A$ such that $a \geq \sup(A) - \varepsilon$; in particular $a > \sup(A \setminus \{x\})$, so $a \notin A \setminus \{x\}$. Since $a \in A$, this implies that $a = x$. Thus we obtained $x = \sup(A) - \varepsilon$ for all $\varepsilon$. This implies that $x \geq \sup(A)$, and $x \leq \sup(A)$ is also true because $x \in A$. We finally obtained $x = \sup(A)$.

5. Let $A, B$ be bounded subsets of $\mathbb{R}$. Prove that $A \cup B$ is also bounded and that $\sup(A \cup B) = \max(\sup(A), \sup(B))$.

**Correction.** Let $M$ (resp. $M'$) be an upper bound of $A$ (resp $B$), and $m$ (resp. $m'$) be a lower bound of $A$ (resp. $M'$). Then, for all $x \in A$ we have $m \leq x \leq M$, and for all $x \in B$ we have $m' \leq x \leq M'$. Thus, for all $x \in A \cup B$ we have $m \leq x \leq M'$.

This shows that $m$ is a lower bound for $A \cup B$, and $M'$ is an upper bound for $A \cup B$. Thus, $A \cup B$ is bounded. Notice that we could have taken $M = \sup(A)$, $M' = \sup(B)$, $m = \inf(A)$, $m' = \inf(B)$; thus the above reasoning implies that $\min(\inf(A), \inf(B))$ is a lower bound of $A \cup B$, and $\max(\sup(A), \sup(B))$ is an upper bound of $A \cup B$.

For any $\varepsilon > 0$ there exist $a \in A$ and $b \in B$ such that $a \leq \inf(A) + \varepsilon$ and $b \leq \inf(B) + \varepsilon$; therefore, $a \leq \min(\inf(A), \inf(B)) + \varepsilon$ (if $\inf(A) \leq \inf(B)$) or $a \leq \min(\inf(A), \inf(B)) + \varepsilon$ (if $\inf(B) \leq \inf(A)$). This means that for any $\varepsilon > 0$ there exists $x \in A \cup B$ such that $x \leq \min(\inf(A), \inf(B)) + \varepsilon$. This, added to the fact that $\min(\inf(A), \inf(B))$ is a lower bound of $A \cup B$, implies that $\min(\inf(A), \inf(B)) = \inf(A \cup B)$.

The proof for the least upper bound is similar.