University of Illinois at Urbana-Champaign   Fall 2006
Math 444   Group E13

Graded Homework IV
Correction.

1. Compute, if they exist, $\sup(A)$ and $\inf(A)$ in the following cases. In each case, state whether A admits a maximal element, and do the same for minimal elements.
$A = \{\frac{n-\frac{1}{n}}{n+\frac{1}{n}} : n \in \mathbb{N}\}$; $A = \{\frac{p}{pq+1} : q,p \in \mathbb{N}\}$.

Correction. In the first case, we have, for all $x \in A$, that $x = \frac{n-\frac{1}{n}}{n+\frac{1}{n}} = \frac{n^2-1}{n^2+1}$ for some $n \in \mathbb{N}$. From this, we see that $0 \leq x \leq 1$ for all $x \in A$, so A is bounded. Furthermore, $0 \in A$, so since 0 is a lower bound of A we have $0 = \inf(A)$ and 0 is the minimal element of A. Notice that, for all $n \in \mathbb{N}$, one has $\frac{n^2-1}{n^2+1} = 1 - \frac{2}{n^2+1}$. Since $\frac{2}{n^2+1} \leq \frac{2}{n^2} \leq \frac{1}{n}$ for all $n \geq 2$, we know, by the archimedean property of the reals, that for all $\varepsilon > 0$ there is $x \in A$ such that $1 - \varepsilon < x$; therefore, $1 = \sup(A)$. Since $1 \notin A$, A does not have a maximal element.
In the second case, we have, for all $p, q \in \mathbb{N}$, that $0 < \frac{p}{pq+1} < 1$. Thus A is bounded. Furthermore, letting $p = 1$, we see that $\frac{1}{q+1} \in A$ for all $q \in N$, so the archimedean property of the reals implies that for any $\varepsilon > 0$ there exists $a \in A$ such that $a < \varepsilon$; since 0 is a lower bound of A, this proves that $0 = \inf(A)$ and that A doesn't have a minimal element. Similarly, letting $q = 1$, we see that $\frac{p}{p+1} = 1 - \frac{1}{p+1} \in A$ for all $p \in \mathbb{N}$, so the same reasoning as above proves that $1 = \sup(A)$, and A doesn't have a maximal element.

2. Consider the set A of all $x \in \mathbb{R}$ such that there exist two natural integers p, q satisfying $p < q$ and $x = \frac{2p^2-3q}{p^2+q}$.
(a) Prove that -3 is a lower bound of A, and 2 is an upper bound.
(b) Compute $\inf(A)$ and $\sup(A)$
Correction. One has $\frac{2p^2-3q}{p^2+q} = \frac{2(p^2+q)-5q}{p^2+q} = 2 - \frac{5q}{p^2+q}$. This shows that $2 \geq \frac{2p^2-3q}{p^2+q}$ for all $p, q \in \mathbb{N}$, in other words it proves that 2 is an upper bound of A. Similarly, one has $\frac{2p^2-3q}{p^2+q} = -3 + \frac{5p^2}{p^2+q} \geq -3$, so -3 is a lower bound of A.
(b) Letting $p = 1$, we see that, for all $q > 1$, $-3 + \frac{5}{q+1} \in A$. Since the archimedean property of $\mathbb{R}$ implies that for all $\varepsilon > 0$ there exists q such that $\frac{5}{q+1} < \varepsilon$, we see that for all $\varepsilon > 0$ there exists q such that $-3 + \frac{5}{q+1} < -3 + \varepsilon$, in other words $a \in A$ such that $a < -3 + \varepsilon$. Since we already saw that -3 is a lower bound of A, this implies that $-3 = \inf(A)$.
Similarly, set $q = p + 1$; we then see that, for any $p \in \mathbb{N}$, $2 - \frac{5(p+1)}{p^2+p+1} = 2 - 5\frac{1+\frac{1}{p}}{p+1+\frac{1}{p}} \in A$. Since $5\frac{1+\frac{1}{p}}{p+1+\frac{1}{p}} \leq \frac{10}{p}$ for all $p \in \mathbb{N}$, the archimedean property of the reals is again enough to ensure that, for any $\varepsilon > 0$, there exists $a \in A$ such that $a \geq 2 - \varepsilon$; since 2 is an upper bound of A, this means that $2 = \sup(A)$.

3 (a). Prove that, for any $x \in \mathbb{R}$, $E(x) = \sup(\{n \in \mathbb{Z} \colon n \leq x\})$ exists, and that it is the unique integer n such

that $n \leq x < n+1$ (we more or less saw this in class). Use this characterization of $E(x)$ to solve the questions (b), (c), (d) and (e) below.
(b) Show that, for all $x \in \mathbb{R}$ and all $n \in \mathbb{Z}$, one has $E(x+n) = E(x) + n$.
(c) Prove that, for all $x, y \in \mathbb{R}$ one has $E(x) + E(y) \leq E(x+y) \leq E(x) + E(y) + 1$.
(d) Given $x \in \mathbb{R}$, what is the value of $E(x) + E(-x)$? (Hint : distinguish the cases $x \in \mathbb{Z}$ and $x \notin \mathbb{Z}$).
(e) Show that, for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$, one has $E(x) = E(\frac{E(nx)}{n})$.
Correction. (a) The set $\{n \in \mathbb{Z} \colon n \leq x\}$ is bounded above by x, so it admits a least upper bound, which, as indicated, we denote by $E(x)$. Furthermore, by definition of the least upper bound we know that for any $\varepsilon > 0$ there is $n_\varepsilon \in \mathbb{Z}$ such that $n_\varepsilon \leq x$ and $E(x) - \varepsilon \leq n_\varepsilon \leq E(x)$ (1). This means that, for any $\varepsilon, \varepsilon' > 0$ we have $|n_\varepsilon - n_{\varepsilon'}| \leq \varepsilon + \varepsilon'$. Thus, as soon as $\varepsilon < \frac{1}{2}$ and $\varepsilon' < \frac{1}{2}$, one has $|n_\varepsilon - n_{\varepsilon'}| < 1$; since $n_\varepsilon - n_{\varepsilon'} \in \mathbb{Z}$, this means that $n_\varepsilon - n_{\varepsilon'} = 0$, in other words that $n_\varepsilon = n_{\varepsilon'} = n$. Then (1) yields $|n - E(x)| \leq \varepsilon$ for all $\varepsilon > 0$, so $E(x) = n$. We finally proved that $E(x) \in \mathbb{Z}$. Since x is an upper bound of the set $\{n \in \mathbb{Z} \colon n \leq x\}$, we have $E(x) \leq x$ by definition of the least upper bound. Also, if we had $E(x) + 1 \leq x$, then we would have $E(x) + 1 \in \{n \in Z \colon n \leq x\}$; since $E(x) = \sup\{n \in Z \colon n \leq x\}$, and $E(x) + 1 > E(x)$, this is impossible. This means that we indeed have $E(x) \leq x < E(x) + 1$. Assume that another integer n has this property ; then we have both $n \leq x < E(x) + 1$, and $E(x) \leq x < n + 1$, so that $n < E(x) + 1$ and $E(x) < n + 1$, in other words $|n - E(x)| < 1$, so that $n = E(x)$: $E(x)$ indeed is the unique integer such that $E(x) \leq x < x + 1$.
(b) Let $n \in \mathbb{Z}$; by definition of $E(x)$, one has $E(x) \leq x < E(x)+1$, so that $E(x)+n \leq x+n < (E(x)+n)+1$; since $E(x) + n$ is an integer, the characterization obtained in question (a) enables us to conclude that $E(x+n) = E(x) + n$.
(c) One has $E(x) \leq x < E(x) + 1$ and $E(y) \leq y < E(y) + 1$. This implies that $E(x) + E(y) \leq x + y < E(x) + E(y) + 2$. The left-hand part of the inequality implies that $E(x) + E(y)$ is an integer which is smaller than $x+y$, so $E(x)+E(y) \leq E(x+y)$; the right-hand part of the inequality means that $E(x)+E(y)+2 > x+y$, so that $E(x+y) < E(x)+E(y)+2$; since these are integers, this may be rewritten as $E(x+y) \leq E(x)+E(y)+1$.
(d) If $x \in \mathbb{Z}$, then one has $E(x) = x$, because x is such that $x \leq x < x + 1$. For the same reason, $E(-x) = -x$, so in that case we have $E(x) + E(-x) = 0$. If $x \notin \mathbb{Z}$, then one has $E(x) < x < E(x) + 1$, so $-E(x) - 1 < -x < -E(-x)$; this implies that $E(-x) = -E(x) - 1$, so in that case we get $E(x) + E(-x) = -1$.
(e) For all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$, one has $nE(x) \leq nx$, which implies that $nE(x) \leq E(nx)$ (because $E(nx)$ is the largest integer smaller that x). So, we have $nE(x) \leq E(nx) \leq nx$. Dividing by n, we obtain that $E(x) \leq \frac{E(nx)}{n} \leq x$. Since $x < E(x) + 1$, we finally obtain $E(x) \leq \frac{E(nx)}{n} < E(x) + 1$, and this means that $E(\frac{E(nx)}{n}) = E(x)$.

4. Let $\{a_i \colon i \in \mathbb{N}\}$ and $\{b_i \colon i \in \mathbb{N}\}$ be two bounded countable subsets of $\mathbb{R}$.
Prove that $\{|a_i - b_i| \colon i \in \mathbb{N}\}$ is bounded, and that $|\sup(a_i) - \sup(b_i)| \leq \sup(|a_i - b_i|)$.
Correction. For all $i \in \mathbb{N}$ one has $0 \leq |a_i - b_i| \leq |a_i| + |b_i| \leq \sup |a_i| \colon i \in \mathbb{N} + sup\{|b_i| \colon i \in \mathbb{N}\}$ (the two suprema on the right exist because of the assumption stating that $\{a_i \colon i \in \mathbb{N}\}$ and $\{b_i \colon i \in \mathbb{N}\}$ are bounded). Thus 0 is a lower bound of $\{|a_i - b_i| \colon i \in \mathbb{N}\}$, and $\sup |a_i| \colon i \in \mathbb{N} + sup\{|b_i| \colon i \in \mathbb{N}\}$ is an upper bound, which shows that this set is bounded.
To prove the inequality, pick some $\varepsilon > 0$; there exists $j \in \mathbb{N}$ such that $a_j \geq \sup(a_i) - \varepsilon$. We then have $\sup(a_i) - \sup(b_i) \leq a_j - \sup(b_i) + \varepsilon$; since $\sup(b_i) \geq b_j$, we see that $\sup(a_i) - \sup(b_i) \leq a_j - b_j + \varepsilon \leq |a_j - b_j| + \varepsilon$. This implies that $\sup(a_i) - \sup(b_i) \leq \sup(|a_i - b_i|) + \varepsilon$; since this is true for all $\varepsilon > 0$, we see that $\sup(a_i) - \sup(b_i) \leq \sup(|a_i - b_i|)$. Thus, we have proved that, for any two bounded countable subsets $A = \{a_i \colon i \in \mathbb{N}\}$ and $B = \{b_i \colon i \in \mathbb{N}\}$, one has $\sup(a_i) - \sup(b_i) \leq \sup(|a_i - b_i|)$. Applying this result to $A' = B$ and $B' = A$, we get $\sup(b_i) - \sup(a_i) \leq \sup(|b_i - a_i|)$. Putting these two inequalities together, we finally obtain that $|\sup(a_i) - \sup(b_i)| \leq \sup(|a_i - b_i|)$.