University of Illinois at Urbana-Champaign
Math 444

Fall 2006
Group E13

Graded Homework VII

Due Friday, October 27.

1. Let u_n be the sequence defined by $u_1 = \sqrt{2}$, $u_2 = \sqrt{2+\sqrt{2}}$, $u_n = \sqrt{2+\sqrt{2+\ldots\sqrt{2}}}$.
(a) Give an induction formula $u_{n+1} = f(u_n)$ defining u_{n+1} as a function of u_n.
(b) Prove that (u_n) is convergent and compute its limit (hint : show that (u_n) is increasing and bounded above by 2).

2. Show that the sequences defined by the formulas $u_n = \frac{1}{n} + \cos(\frac{2n\pi}{3})$ and $v_n = \frac{(-1)^n n^2 + n}{3n^2 + n}$ are not convergent.

3. Recall that we saw in class that if (u_n) is a sequence of real numbers such that (u_{2n+1}) and (u_{2n}) converge to the same limit l then (u_n) is convergent and $\lim(u_n) = l$.
(a) Use the same method to show that if (u_n) is a sequence of real numbers such that (u_{3n}), (u_{3n+1}) and (u_{3n+2}) converge to the same limit l, then (u_n) is convergent and $\lim(u_n) = l$.
(b) Let (u_n) be a sequence of real numbers such that (u_{2n}), (u_{2n+1}) and (u_{3n}) are all convergent ; show that (u_n) is convergent (Hint : use the fact that (u_{3n}) converges to prove that (u_{2n}) and (u_{2n+1}) converge to the same limit, then use the result seen in class).

4. Given a sequence (u_n) of reals numbers, define another sequence s_n by the formula $s_n = \frac{u_1 + u_2 + \ldots + u_n}{n}$.
(a) Here we assume that (u_n) is convergent and $\lim(u_n) = 0$. Show that for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ one has

$$|s_n| \leq \frac{|u_1 + u_2 + \ldots u_N|}{n} + \frac{\varepsilon}{2}\,.$$

What can you conclude from this ?
(HInt : for the inequality, pick some suitable N and then cut the sum in two parts ; use the triangle inequality.).
(b) Show that if (u_n) is convergent and $\lim(u_n) = l$ then (s_n) is convergent and $\lim(s_n) = l$.
(c) Show that the converse of this assertion is not true (look at what happens if $(u_n) = (-1)^n$ for instance).

5. Show that a subsequence of a Cauchy sequence is also a Cauchy sequence.