University of Illinois at Urbana-Champaign — Fall 2006

Math 444 — Group E13

Graded Homework IX

Due Monday, November 13.

1. Let (u_n) be a sequence of real numbers. We say that $a \in \mathbb{R}$ is an *accumulation point* of (u_n) if there exists a subsequence of (u_n) which converges to a.
(a) What are the accumulation points of a convergent sequence ?
(b) What are the accumulation points of the sequence $u_n = \cos(n\frac{\pi}{3})$?
(c) Let (u_n) be a bounded, divergent sequence. Prove that it has at least two (distinct) accumulation points (Hint : why does there exist one accumulation point ? Can you use the fact that this point is not the limit of (u_n) ?)

2. We define a sequence by setting $u_1 = \frac{1}{2}$, $u_{n+1} = 1 - u_n^2$. Show that (u_{2n}) is increasing, (u_{2n+1}) is decreasing and both sequences are convergent. Show that (u_n) is not convergent.

3. Let a, b be two reals different from 0. We define a sequence (u_n) by setting $u_1 = u \neq 0$, $u_{n+1} = a + \frac{b}{u_n}$. We assume that u is chosen in such a way that $u_n \neq 0$ for all $n \in \mathbb{N}$.
(a) What are the possible limits for (u_n) ?
(b) We suppose that the equation $x^2 = ax + b$ has two distinct solutions $\alpha, \beta \in \mathbb{R}$ and that $\alpha < \beta$. Prove that the sequence defined by $v_n = \frac{u_n - \alpha}{u_n - \beta}$ is geometric (i.e $\frac{v_{n+1}}{v_n}$ is constant) and use this to determine the limit of (u_n) (depending on u).

4. Pick $0 < x_1 < y_1$ and define two sequences $(x_n), (y_n)$ by setting $\begin{cases} x_{n+1} = \frac{x_n^2}{x_n + y_n} \\ y_{n+1} = \frac{y_n^2}{x_n + y_n} \end{cases}$. Show that these sequences are convergent and compute their limit (this is harder than it looks ; justify carefully all your assertions).