University of Illinois at Urbana-Champaign
Math 444

Fall 2006
Group E13

Integration : training exercises.

1. (a) Assume that $f\colon [a,b] \to \mathbb{R}$ is a continuous function such that $f(x) \geq 0$ for all $x \in (a,b)$, and $\int_a^b f(t)dt = 0$. Show that $f(x) = 0$ for all $x \in [a,b]$; can you use the fundamental theorem of calculus to prove this result ?
(b) Use this to show that if f is continuous on $[a,b]$ and $\int_a^b f(t)dt = 0$ then there must exist $t \in (a,b)$ such that $f(t) = 0$.

2. Use the result of the preceding exercise to solve the following questions.
(a) Find all the continuous functions $f\colon [a,b] \to \mathbb{R}$ such that $\int_a^b f(t)dt = (b-a)\sup\{|f(x)|\colon x \in [a,b]\}$.
(b) Assume $f\colon [0,1] \to \mathbb{R}$ is a continuous function such that $\int_0^1 f(t)dt = \frac{1}{2}$; prove that there exists $a \in (0,1)$ such that $f(a) = a$.
(c) Show that if f, g are continuous on $[0,1]$ and $\int_0^1 f(t)dt = \int_0^1 g(t)dt$ then there must exist some $c \in [0,1]$ such that $f(x) = g(c)$.

3. Using Riemann sums, compute the limits (when $n \to +\infty$) of the following sequences :

$$\sum_{k=1}^{n} \frac{1}{n+k} \ ; \quad \sum_{k=1}^{n} \frac{n}{n^2+k^2} \ ; \quad \sum_{k=1}^{n} \frac{k^2}{n^3} \ ; \quad \sum_{k=1}^{n} \big(\sin(\frac{k\pi}{n}) - \sin(\frac{(k-1)\pi}{n}) \ln(1+\sin(\frac{k\pi}{n})); \quad ; \quad \sum_{k=1}^{n} \frac{(-1)^k}{k}.$$

4. Let $f, g\colon [0,1] \to \mathbb{R}$ be continuous functions. Show that $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(\frac{k}{n}) g(\frac{k-1}{n}) = \int_0^1 f(t)g(t)\, dt$.

5. Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function such that $\int_0^1 f(u)u^k du = 0$ for all $k \in \{0, \dots, n\}$. Show that f has at least $n+1$ distinct zeros in $(0,1)$.
Hint : prove the result by induction using integration by parts and Rolle's theorem.

The following exercises from the textbook should help you master the material : 7.1 : 13,14,15 ; 7.2 : 11,12,16,17 ; 7.3 : 11,13,14 ;21. As an application of 7.3.21, prove that whenever $0 < a \leq b$ one has $\int_a^b \frac{dt}{t} \leq \frac{(b-a)}{\sqrt{ab}}$.