Integration: correction of the exercises.

1. (a) Assume that \( f : [a,b] \to \mathbb{R} \) is a continuous function such that \( f(x) \geq 0 \) for all \( x \in (a,b) \), and \( \int_a^b f(t)dt = 0 \). Show that \( f(x) = 0 \) for all \( x \in [a,b] \); can you use the fundamental theorem of calculus to prove this result?

(b) Use this to show that if \( f \) is continuous on \( [a,b] \) and \( \int_a^b f(t)dt = 0 \) then there must exist \( t \in (a,b) \) such that \( f(t) = 0 \).

**Correction.** First, notice that, since \( f \) is continuous, proving that \( f(t) = 0 \) for all \( t \in [a,b] \) is the same as proving that \( f(t) = 0 \) for all \( t \in (a,b) \). Now, let us prove the contrapositive of the result we are interested in; in other words, let us prove that if \( f(x) > 0 \) for some \( x \in (a,b) \), \( f(x) \geq 0 \) for all \( x \in (a,b) \) and \( f \) is continuous on \( [a,b] \), then \( \int_a^b f(t)dt > 0 \). To prove this, notice that since \( f \) is continuous at \( x \) there exists \( \delta > 0 \) such that \( f(y) \geq \frac{f(x)}{2} \) for all \( y \in [a,b] \) such that \( |y - x| \leq \delta \). If \( \delta \) is small enough, \( |x - \delta, x + \delta| \subset [a,b] \); but then

\[
\int_{x-\delta}^{x+\delta} f(t)dt \geq 2\delta \frac{f(x)}{2} = \delta f(x) > 0.
\]

Since \( f(y) \geq 0 \) for all \( y \in [a,b] \), we know that \( \int_a^{x-\delta} f(t)dt \geq 0 \) and \( \int_{x+\delta}^b f(t)dt \geq 0 \); thus the additivity theorem shows that \( \int_a^b f(t)dt > 0 \), which is what we wanted.

One can indeed prove this result using the fundamental theorem of calculus (and the mean value theorem):
Set \( F(x) = \int_a^x f(t)dt \); then \( F \) is differentiable and \( F'(x) = f(x) \geq 0 \), hence \( F \) is increasing. We have \( F(a) = 0 \) by definition, and the assumption that \( \int_a^b f(t)dt = 0 \) gives \( F(b) = 0 \). Since \( F \) is increasing, this means that actually \( F \) is constant on \( [a,b] \), thus its derivative is equal to 0 on \( [a,b] \), and this gives \( f(x) = 0 \) for all \( x \in [a,b] \).

(b) First, notice that if \( f \) doesn’t take the value 0 on \( (a,b) \) then \( f \) is either always > 0 or always < 0 on \( [a,b] \) (because of the intermediate value theorem). But then the preceding question shows that one cannot have \( \int_a^b f(t)dt = 0 \). Hence there must exist \( t \in (a,b) \) such that \( f(t) = 0 \).

2. Use the result of the preceding exercise to solve the following questions.

(a) Find all the continuous functions \( f : [a,b] \to \mathbb{R} \) such that \( \int_a^b f(t)dt = (b-a) \sup\{|f(x)| : x \in [a,b]| \} \).

(b) Assume \( f : [0,1] \to \mathbb{R} \) is a continuous function such that \( \int_0^1 f(t)dt = \frac{1}{2} \); prove that there exists \( a \in (0,1) \) such that \( f(a) = a \).

(c) Show that if \( f, g \) are continuous on \([0,1]\) and \( \int_0^1 f(t)dt = \int_0^1 g(t)dt \) then there exists some \( c \in [0,1] \) such that \( f(x) = g(c) \).

**Correction.** The function \( g \) defined on \( [a,b] \) by \( g(x) = \sup\{|f(x)| : x \in [a,b]| \} - f(x) \) is continuous, and \( g(x) \geq 0 \) for all \( x \in [a,b] \). The assumption \( \int_0^1 f(t)dt = (b-a) \sup\{|f(x)| : x \in [a,b]| \} \) is equivalent to \( \int_0^1 g(t)dt = 0 \), which in turn is equivalent to \( g(x) = 0 \) for all \( x \in [a,b] \). This means that the functions that satisfy the equality we are interested in are the functions \( f \) which are constant on \([a,b]\) and nonnegative.

(b) Assume that for all \( t \in (0,1) \) one has \( f(t) > t \); then we know that \( \int_0^1 (f(t) - t)dt > 0 \), and this is the same as saying that \( \int_0^1 f(t)dt > \frac{1}{2} \). Similarly, if \( f(t) < t \) for all \( t \in (0,1) \) one gets \( \int_0^1 f(t)dt > \frac{1}{2} \). Thus, it is only possible that \( \int_0^1 f(t)dt = \frac{1}{2} \) if there exist \( t,t' \in (0,1) \) such that \( f(t) \geq t \), \( f(t') \leq t' \). If either \( f(t) = t \) or \( f(t') = t' \) we are done; otherwise the function \( x \mapsto f(x) - x \) changes sign on \((0,1)\). Since this function is continuous, the mean value theorem ensures that it must have a zero on \((0,1)\), which shows that there exists \( a \in (0,1) \) such that \( f(a) = a \).

(c) This is a direct consequence of question 1(b) (applied to the continuous function \( f - g \)).
3. Using Riemann sums, compute the limits (when \( n \to +\infty \)) of the following sequences:

\[
\sum_{k=1}^{n} \frac{1}{n+k} : \sum_{k=1}^{n} \frac{n}{n^2+k^2} : \sum_{k=1}^{n} \frac{k^2}{n^2} : \sum_{k=1}^{n} \left( \sin\left(\frac{k\pi}{2n} - \sin\left(\frac{(k-1)\pi}{2n}\right) \right) \ln(1 + \sin(\frac{k\pi}{2n})); \right) : \sum_{k=1}^{n} \frac{(-1)^k}{k}.
\]

**Correction.** Here, the trick is to recognize Riemann sums: the first one is \( \sum_{k=1}^{n} \frac{1}{n+k} = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1 + \frac{k}{n}} \), and this is a Riemann sum for the function \( x \mapsto \frac{1}{1+x} \) for a tagged partition of \([0,1]\) with mesh \(1/n\). Thus, we obtain

\[
\lim \left( \frac{n}{k=1} \frac{1}{n+k} \right) \int_{0}^{1} \frac{dt}{1+t} = \ln(2).
\]

The second one is similar: \( \sum_{k=1}^{n} \frac{n}{n^2+k^2} = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1 + \left(\frac{k}{n}\right)^2} \), hence \( \lim \left( \frac{n}{k=1} \frac{n^2+k^2}{n^2} \right) \int_{0}^{1} \frac{dt}{1+t^2} = \frac{\pi}{4} \).

The third one is more of the same: \( \sum_{k=1}^{n} \frac{k^2}{n^2} = \frac{1}{n} \sum_{k=1}^{n} \frac{k^2}{n^2} \), hence \( \lim \left( \frac{n}{k=1} \frac{k^2}{n^2} \right) \int_{0}^{1} \frac{dt}{1+t^2} = \frac{1}{3} \).

The fourth one looks nasty, but again it is a Riemann sum for the continuous function \( t \mapsto \ln(1+t) \) on \([0,1]\), with regard to the tagged partition \{ \sin\left(\frac{(k-1)\pi}{2n}\right), \sin\left(\frac{k\pi}{2n}\right), \sin\left(\frac{k\pi}{2n}\right) \}, \) the mesh of which is smaller than \( \frac{1}{2n} \) (use the mean value theorem to prove this). Hence when \( n \to +\infty \) the sum converges to \( \int_{0}^{1} \ln(1+t) \, dt \), which is computable using integration by parts:

\[
\int_{0}^{1} \ln(1+t) \, dt = \left[ (t+1) \ln(t+1) \right]_{t=0}^{t=1} - \int_{0}^{1} 1 \, dt = 2 \ln(2) - 1.
\]

The last one doesn't look like a Riemann sum; there is some work to be done before one can see a Riemann sum appear. Assume first that \( n = 2p \); one has

\[
u_{2p} = \sum_{k=1}^{n} \frac{(-1)^k}{k} = -\sum_{k=1}^{n} \frac{1}{k} + 2 \sum_{k=1}^{p} \frac{1}{2k} = -\sum_{k=1}^{n} \frac{1}{k} + \sum_{k=1}^{p} \frac{1}{k}.
\]

Hence, when \( n = 2p \), one has \( \sum_{k=1}^{n} \frac{(-1)^k}{k} = -\sum_{k=p+1}^{2p} \frac{1}{k} = -\sum_{k=p+1}^{2p} \frac{1}{k} \). The sum on the right is actually a Riemann sum for the continuous function \( t \mapsto -\frac{1}{t^2} \) on \([0,1]\) and the tagged partition \{ \frac{1}{2p}, \frac{1}{2p+1}, \frac{1}{2p} \}, \) the mesh of which is \( \frac{1}{2p} \) (can you see why?). So, we see that \( u_{2p} \) converges to \( -\int_{0}^{1} \frac{dt}{1+t^2} = \ln(2) \). Since \( u_{2p+1} \) converges to \( 0 \), we see that one also has \( \lim(u_{2p+1}) = -\ln(2) \). A theorem we saw in class ensures that \( (u_n) \) is convergent and \( \lim(u_n) = -\ln(2) \).

4. Let \( f, g: [0,1] \to \mathbb{R} \) be continuous functions. Show that \( \lim \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\left(\frac{k}{n} - 1\right) \right) = \int_{0}^{1} f(t)g(t) \, dt \).

**Correction.** This is trickier than it looks: if we had \( f\left(\frac{k}{n}\right)g\left(\frac{k}{n}\right) \) in the sum, then it would just be a usual Riemann sum and we could apply the results seen in class. Unfortunately, this is not what we have; how can we deal with this? One can proceed as follows: first, write that

\[
\frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right)g\left(\frac{k}{n} - 1\right) = \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right)g\left(\frac{k}{n}\right) + \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\left(g\left(\frac{k}{n} - 1\right) - g\left(\frac{k}{n}\right)\right).
\]

The first term converges to \( \int_{0}^{1} f(t)g(t) \, dt \), so we want to prove that the second term converges to 0. For that, we use the fact that \( g \) is uniformly continuous on \([0,1]\); given \( \varepsilon > 0 \), there exists \( \delta_\varepsilon \) such that \( |x - y| \leq \delta_\varepsilon \Rightarrow \)
\[ |f(x) - f(y)| \leq \varepsilon \text{ for all } x, y \in [a, b]. \] Hence if \( n \) is big enough one has \( |g\left(\frac{k-1}{n}\right) - g\left(\frac{k+1}{n}\right)| \leq \varepsilon \), so that
\[
\frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\left(g\left(\frac{k-1}{n}\right) - g\left(\frac{k+1}{n}\right)\right) \leq \varepsilon \frac{1}{n} \sum_{k=1}^{n} |f\left(\frac{k}{n}\right)|.
\]

Since \( f \) is Riemann-integrable \( |f| \) also is Riemann-integrable, hence \( \frac{1}{n} \sum_{k=1}^{n} |f\left(\frac{k}{n}\right)| \) converges to \( \int_{a}^{b} |f(t)| dt \).

So if \( n \) is big enough one has \( \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\left(g\left(\frac{k-1}{n}\right) - g\left(\frac{k+1}{n}\right)\right) \leq \varepsilon \left(\int_{a}^{b} |f(t)| dt + 1\right) \) for all \( n \geq K \). This proves that \( \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\left(g\left(\frac{k-1}{n}\right) - g\left(\frac{k+1}{n}\right)\right) \) converges to 0 (when \( n \to +\infty \)), which is what we needed to prove.

5. Let \( f : [0, 1] \to \mathbb{R} \) be a continuous function such that \( \int_{0}^{1} f(u)u^k du = 0 \) for all \( k \in \{0, \ldots, n\} \). Show that \( f \) has at least \( n+1 \) distinct zeros in \((0, 1)\).

Hint: prove the result by induction using integration by parts and Rolle’s theorem.

Correction. Following the hint, let us prove the result by induction. For \( n = 0 \) the result is a direct consequence of exercise 1; assume the result is true for \( n \). Then pick a continuous function \( f \) such that \( \int_{0}^{1} f(u)u^k du = 0 \) for all \( k \in \{0, \ldots, n+1\} \), and set \( F(x) = \int_{0}^{x} f(t) dt \). The assumption on \( f \) for \( k = 0 \) yields \( F(0) = F(1) = 0 \).

Also, for any \( k = 1, \ldots, n \), one has
\[
\int_{0}^{1} u^k du = \left[ku^{k-1}\right]_{0}^{1} = k \int_{0}^{1} u^{k-1}f(u) du.
\]

Thus we obtain \( \int_{0}^{1} u^{k-1}F(u) du = 0 \) for all \( k = 1, \ldots, n \), which yields (because of our induction hypothesis) that \( F \) has at least \( n \) distinct zeros in \((0, 1)\). Since \( F(0) = F(1) = 0 \), \( F \) must have at least \( n+1 \) distinct zeros on \([0, 1]\). And \( F' = f \) has a zero between any two zeros of \( F \), which shows that \( f \) has at least \( n+1 \) distinct zeros on \((0, 1)\).

7.1.13. We need to use the definition of a Riemann integral; assume the points \( c_1, \ldots, c_n \) are indexed in such a way that \( c_1 < c_2 < \ldots < c_{n-1} < c_n \), and set \( M = \max\{|f(c_i)| : i = 1, \ldots, n\} \). Then pick a tagged partition \( P = \{[x_{i-1}, x_i], t_i\}_{i=1}^{n} \) of \([a, b]\). One has
\[
|S(f, P)| = \sum_{i=1}^{m} (x_i - x_{i-1})|f(t_i)| \leq \sum_{i=1}^{m} |x_i - x_{i-1}||f(t_i)| \leq |P| \sum_{i=1}^{m} |f(t_i)|.
\]

Since there are only \( n \) points in the interval at which \( f(x) \neq 0 \), and at each of these points one has \(|f(x)| \leq M\), we see that \( |S(f, P)| \leq |P|2nM \) (because there can be at most two \( t_i \) with the same value, and at most \( n \) points at which \( f \) is nonzero, so at most \( 2n \) of them can appear in the sum). But then (since \( n, M \) are constant) we are done: if one sets \( \varepsilon = \frac{\varepsilon}{2nM} \), what we have proved implies that for any partition \( P \) with mesh less than \( \delta \) one has \( |S(f, P)| \leq \varepsilon \). This is exactly what we needed to prove that \( f \in \mathcal{R}(\mathbb{R}[a, b]) \) and \( \int_{a}^{b} f(x) dx = 0 \).

7.1.14. This is a consequence of the preceding exercise: indeed, the function \( f - g \) satisfies the condition of exercise 7.1.13, hence \( f - g \in \mathcal{R}(\mathbb{R}[a, b]) \) and \( \int_{a}^{b} (f - g)(t) dt = 0 \). But then \( f = (f - g) + g \) is the sum of two Riemann-integrable functions, so \( f \in \mathcal{R}(\mathbb{R}[a, b]) \) and \( \int_{a}^{b} f(t) dt = \int_{a}^{b} (f(t) - g(t)) dt + \int_{a}^{b} g(t) dt = \int_{a}^{b} g(t) dt \).

7.1.15. Let us follow the hint: pick \( \varepsilon > 0 \), set \( \delta = \frac{\varepsilon}{\max_{t \in [c, d]} f(t)} \) and pick a tagged partition \( P = \{[x_{i-1}, x_i], t_i\}_{i=1}^{n} \) with mesh \( \leq \delta \). Then by definition one has \( S(\varphi, P) = \sum_{i=1}^{n} (x_i - x_{i-1})\varphi(t_i) \). There are two possibilities for \( \varphi(t_i) \): either it is equal to 0, or it is equal to \( \alpha \). Only the \( t_i \)'s that belong to \([c, d]\) contribute to the sum. Let \( I = \{i : t_i \in [c, d]\} \). Then \( S(\varphi, P) = \alpha \sum_{i \in I} (x_i - x_{i-1}) \). Since \( t_i \in [x_{i-1}, x_i] \), \( t_i \) can only be in \([c, d]\) if \( x_{i-1} < a \) and \( x_i \geq a \), or \( x_{i-1}, x_i \) are both in \([c, d]\), or \( x_{i-1} \leq d \) and \( x_i > d \). The first and third condition can each
be satisfied at most by one index, and the remaining $[x_{i-1}, x_i]$ from a partition of a subinterval of $[c, d]$, so that $S(\varphi, \mathcal{P}_i) \leq \alpha(d-c) + 2\delta_c \alpha$. Similarly, the "chunk" of $[x, d]$ that can be missed by the $t_i$'s is at most $2\delta_c$ long, hence $S(\varphi, \mathcal{P}_i) \geq \alpha(d-c) - 2\delta_c \alpha$. This shows that whenever $\mathcal{P}_i$ is a tagged partition with mesh less than $\delta_c = \frac{\varepsilon}{2}$, one has
\[
\alpha(d-c) - \frac{\varepsilon}{2} \leq S(\varphi, \mathcal{P}_i) \leq \alpha(d-c) + \frac{\varepsilon}{2}.
\]
This is enough to show that $\varphi \in \mathcal{R}([a, b])$ and $\int_a^b \varphi(t)dt = \alpha(d-c)$.

7.2.11. Let's follow the hint and define (given $\varepsilon > 0$) $\alpha_\varepsilon, \omega_\varepsilon$ by $\alpha_\varepsilon(x) = \begin{cases} -M & \text{if } x \in [a, c] \\ f(x) & \text{if } x \in [c, b] \end{cases}$ and $\omega_\varepsilon(x) = \begin{cases} M & \text{if } x \in [a, c] \\ f(x) & \text{if } x \in [c, b] \end{cases}$ (is to be specified later). Then one has $\alpha_\varepsilon(x) \leq f(x) \leq \omega_\varepsilon(x)$ for all $x \in [a, b]$. Also, $\alpha_\varepsilon, \omega_\varepsilon$ are both Riemann-integrable on $[a, b]$ because of the Additivity theorem. Finally, one has $\int_a^b (\omega_\varepsilon(t) - \alpha_\varepsilon(t))dt = \int_a^c 2M dt = 2M(c-a)$ by definition of $\alpha_\varepsilon, \omega_\varepsilon$. Hence if one sets $c = a + \frac{\varepsilon}{2M}$ we get that $\int_a^b (\omega_\varepsilon(t) - \alpha_\varepsilon(t)) \leq \varepsilon$.

So we managed to prove that the assumptions of the Squeeze Theorem are satisfied, hence $f \in \mathcal{R}([a, b])$. Then since $|f(x)| \leq M$ for all $x \in [a, b]$ we see that $\left| \int_a^c f(t)dt \right| \leq M(c-a)$, and the Chain Rule yields $\left| \int_a^c f(t)dt \right| = 0$. Thus the additivity theorem gives $\lim_{c \to a} \int_a^b f(t)dt = \int_a^b f(t)dt$.

7.2.12. This is a consequence of the preceding exercise : $|g(x)| \leq 1$ for all $x \in [0, 1]$, and $g$ is continuous on $[c, 1]$ for all $c \in (0, 1)$. Hence it is Riemann-integrable on $[0, 1]$.

7.2.16. Set $F(x) = \int_a^x f(t)dt$; since $f$ is continuous on $[a, b]$, the fundamental theorem of calculus ensures that $F$ is differentiable on $[a, b]$, hence it satisfies the assumptions of the Mean Value theorem on this interval, so there exists $c \in (a, b)$ such that $F(b) - F(a) = F'(c)(b-a)$. This is the same as saying that there exists $c \in (a, b)$ such that $\int_a^b f(t)dt = f(c)(b-a)$.

One can also solve this exercise differently : one has $f([a, b]) = [m, M]$ by the theorems about continuous functions, from which we get $m(b-a) \leq \int_a^b f(t)dt \leq M(b-a)$ But then $m \leq \frac{\int_a^b f(t)dt}{b-a} \leq M$, hence there exists $c$ such that $f(c) = \frac{\int_a^b f(t)dt}{b-a}$, which is the same as saying that $(b-a)f(x) = \int_a^b f(t)dt$.

7.2.17. We can apply a similar method to the one in the exercise above : denote again $f([a, b])$ by $[m, M]$.

Then one has $\int_a^b f(t)g(t)dt - m \int_a^b g(t)dt = \int_a^b (f(t) - m)g(t)dt \geq 0$ (because $f(t) \geq m$ and $g(t) \geq 0$ for all $t \in [a, b]$). Similarly, one finds that $\int_a^b f(t)g(t)dt \leq M \int_a^b g(t)dt$. Put together, this yields
\[
m \leq \frac{1}{\int_a^b g(t)dt} \int_a^b f(t)g(t)dt \leq M.
\]

Thanks to the intermediate value theorem, we can now conclude : there exists $c \in [a, b]$ such that $f(c) = \frac{1}{\int_a^b g(t)dt} \int_a^b f(t)g(t)dt$, which is the same as $\int_a^b f(t)g(t)dt = f(c) \int_a^b g(t)dt$. This result is clearly false if one no longer assumes that $g$ takes nonnegative values ; for instance, let $a = -1$, $b = 1$, $f(t) = t$ and $g(t) = t$. Then one has $\int_a^b f(t)g(t)dt = 1$ but $f(c) \int_a^b g(t)dt = 0$ for all $c \in [0, 1]$.

7.3.11. Here one needs to apply the Chain Rule (and the fundamental theorem of calculus), which yields :

(a) In this case $F(x) = G(x^2)$, where $G'(x) = \frac{1}{1+x^2}$; hence $F'(x) = \frac{2x}{1+(x^2)^2} = \frac{2x}{1+x^6}$.

(b) This time $F(x) = G(x^2) - G(x^2)$, where $G'(x) = \sqrt{1+x^2}$. Hence $F'(x) = G'(x) - 2xG'(x^2) = \sqrt{1+x^2} - 2x\sqrt{1+x^4}$.

7.3.13. Set first $F(x) = \int_x^0 f(t)dt$. Then we know that $F$ is differentiable and $F'(x) = f(x)$. By definition, we have $g(x) = F(x+c) - F(x-c)$, hence $g$ is a composition of differentiable functions. Thus $g$ is differentiable on $\mathbb{R}$, and the Chain Rule yields $g'(x) = F'(x+c) - F'(x-c) = f(x+c) - f(x-c)$.
7.3.14. First notice that the assumption on \( f \) implies that \( \int_a^b f(t)dt = 0 \) (take \( x = 0 \)). Set \( F(x) = \int_0^x f(t)dt \). Then the assumption on \( F \) become \( F(x) = F(1) - F(x) \) for all \( x \in [0,1] \), and since \( F(1) = 0 \) this yields \( F(x) = 0 \) for all \( x \in [0,1] \). Since \( f \) is continuous the fundamental theorem of calculus gives \( F' = f \), hence \( f(x) = 0 \) for all \( x \in [0,1] \).

7.3.21. (a) The functions \( x \mapsto (tf(x) + g(x))^2 \) and \( x \mapsto (tf(x) - g(x))^2 \) are both Riemann-integrable on \([a,b]\) and take nonnegative values, hence \( \int_a^b (tf(u) \pm g(u))^2 dt \geq 0 \).

(b) We have:
\[
\int_a^b (tf(u) + g(u))^2 du = \int_a^b (t^2f^2(u) + 2tf(u)g(u) + g^2(u)) du = t^2 \int_a^b f(u)^2 du + 2t \int_a^b f(u)g(u) du + \int_a^b g(u)^2 du.
\]
Since the quantity on the left is positive, we obtain \(-2t \int_a^b f(u)g(u) du \leq t^2 \int_a^b f(u)^2 du + \int_a^b g(u)^2 du \). Hence for any \( t > 0 \) we have \(-2 \int_a^b f(u)g(u) du \leq t \int_a^b f(u)^2 du + \frac{1}{t} \int_a^b g(u)^2 du \). Similarly, using the fact that \( \int_a^b (tf(u) - g(u))^2 du \geq 0 \), one obtains \( 2 \int_a^b f(u)g(u) du \leq t \int_a^b f(u)^2 du + \frac{1}{t} \int_a^b g(u)^2 du \). The two inequalities together yield
\[
2|\int_a^b f(u)g(u) du| \leq t \int_a^b f(u)^2 du + \frac{1}{t} \int_a^b g(u)^2 du.
\]
(c) If \( \int_a^b f^2(u) du = 0 \) then the result above implies that \( 2|\int_a^b f(u)g(u) du| \leq \frac{1}{2} \int_a^b g(u)^2 du \) for all \( t > 0 \). This is only possible if \( \int f(u)g(u) du = 0 \).

(d) Since one has both \( fg \leq |fg| \) and \(-fg \leq |fg| \), it is true that both \( \int_a^b f(u)g(u) du \leq \int_a^b |f(u)g(u)| du \) and \(-\int_a^b f(u)g(u) du \leq \int_a^b |f(u)g(u)| du \). This means that \( |\int_a^b f(u)g(u) du| \leq \int_a^b |f(u)g(u)| du \), which is equivalent to the inequality on the left.

To prove the inequality on the right, recall that we know from (b) (applied to \(|f|, |g|\)) that \( t^2 \int_a^b f^2(u) du + 2t \int_a^b f(u)g(u) du + \int_a^b g(u)^2 du \geq 0 \) for all \( t \in \mathbb{R} \). This means that the polynomial function \( t \mapsto t^2 \int_a^b f^2(u) du + 2t \int_a^b f(u)g(u) du + \int_a^b g(u)^2 du \) keeps a constant sign on \( \mathbb{R} \), and this is possible only if its discriminant \( 4(\int_a^b |f(u)g(u)| u)^2 - 4 \int_a^b f^2(u) du \int_a^b g^2(u) du \) is \( \leq 0 \). In other words, one must have
\[
\left( \int_a^b |f(u)g(u)| du \right)^2 \leq \int_a^b f^2(u) du \int_a^b g^2(u) du.
\]
To get the inequality we are asked to prove, apply this inequality to the functions \( f(t) = 1/t \) and \( g(t) = 1 \):
\[
\left( \int_a^b \frac{dt}{t} \right)^2 \leq \int_a^b \frac{dt}{t^2} \int_a^b dt = \left( \frac{1}{a} - \frac{1}{b} \right)(b - a) = \frac{(b-a)^2}{ab}.
\]
Taking the square root, one has
\[
\int_a^b \frac{dt}{t} \leq \frac{(b-a)}{\sqrt{ab}}.
\]