University of Illinois at Urbana-Champaign
Math 444

Fall 2006
Group E13

Integration : correction of the exercises.

1. (a) Assume that $f\colon [a,b] \to \mathbb{R}$ is a continuous function such that $f(x) \geq 0$ for all $x \in (a,b)$, and $\int_a^b f(t)dt = 0$. Show that $f(x) = 0$ for all $x \in [a,b]$; can you use the fundamental theorem of calculus to prove this result ?
(b) Use this to show that if f is continuous on $[a,b]$ and $\int_a^b f(t)dt = 0$ then there must exist $t \in (a,b)$ such that $f(t) = 0$.
Correction. (a) First,notice that, since f is continuous, proving that $f(t) = 0$ for all $t \in [a,b]$ is the same as proving that $f(t) = 0$ for all $t \in (a,b)$. Now, let us prove the contraposite of the result we are interested in ; in other words, let us prove that if $f(x) > 0$ for some $x \in (a,b)$, $f(x) \geq 0$ for all $x \in (a,b)$ and f is continuous on $[a,b]$ then $\int_a^b f(t)dt > 0$. To prove this, notice that since f is continuous at x there exists $\delta > 0$ such that $f(y) \geq \dfrac{f(x)}{2}$ for all $y \in [a,b]$ such that $|y - x| \leq \delta$. If δ is small enough, $[x-\delta, x+\delta] \subset [a,b]$; but then

$$\int_{x-\delta}^{x+\delta} f(t)dt \geq 2\delta \frac{f(x)}{2} = \delta f(x) > 0 \ .$$

Since $f(y) \geq 0$ for all $y \in [a,b]$, we know that $\int_a^{x-\delta} f(t)dt \geq 0$ and $\int_{x+\delta}^b f(t)dt \geq 0$; thus the additivity theorem shows that $\int_a^b f(t)dt > 0$, which is what we wanted.
One can indeed prove this result using the fundamental theorem of calculus (and the mean value theorem) : Set $F(x) = \int_a^x f(t)dt$; then F is differentiable and $F'(x) = f(x) \geq 0$, hence F is increasing. We have $F(a) = 0$ by definition, and the assumption that $\int_a^b f(t)dt = 0$ gives $F(b) = 0$. Since F is increasing, this means that actually F is constant on $[a,b]$, thus its derivative is equal to 0 on $[a,b]$, and this gives $f(x) = 0$ for all $x \in [a,b]$.
(b) First, notice that if f doesn't take the value 0 on (a,b) then f is either always > 0 or always < 0 on $[a,b]$ (because of the intermediate value theorem). But then the preceding quesion shows that one cannot have $\int_a^b f(t)dt = 0$. Hence there must exist $t \in (a,b)$ such that $f(t) = 0$.

2. Use the result of the preceding exercise to solve the following questions.
(a) Find all the continuous functions $f\colon [a,b] \to \mathbb{R}$ such that $\displaystyle\int_a^b f(t)dt = (b-a)\sup\{|f(x)|\colon x \in [a,b]\}$.
(b) Assume $f\colon [0,1] \to \mathbb{R}$ is a continuous function such that $\displaystyle\int_0^1 f(t)dt = \frac{1}{2}$; prove that there exists $a \in (0,1)$ such that $f(a) = a$.
(c) Show that if f, g are continuous on $[0,1]$ and $\int_0^1 f(t)dt = \int_0^1 g(t)dt$ then there must exist some $c \in [0,1]$ such that $f(x) = g(c)$.
Correction. (a) The function g defined on $[a,b]$ by $g(x) = \sup\{|f(x)|\colon x \in [a,b]\} - f(x)$ is continuous, and $g(x) \geq 0$ for all $x \in [a,b]$. The assumption $\int_a^b f(t)dt = (b-a)\sup\{|f(x)|\colon x \in [a,b]\}$ is equivalent to $\int_a^b g(t)dt = 0$, which in turn is equivalent to $g(x) = 0$ for all $x \in [a,b]$. This means that the functions that satisfy the equality we are interested in are the functions f which are constant on $[a,b]$, and nonnegative.
(b) Assume that for all $t \in (0,1)$ one has $f(t) > t$; then we know that $\int_0^1 (f(t) - t)dt > 0$, and this is the same as saying that $\int_0^1 f(t)dt > \frac{1}{2}$. Similarly, if $f(t) < t$ for all $t \in (0,1)$ one gets $\int_0^1 f(t)dt > \frac{1}{2}$. Thus, it is only possible that $\int_0^1 f(t)dt = \frac{1}{2}$ if there exist $t, t' \in (0,1)$ such that $f(t) \geq t$, $f(t') \leq t'$. If either $f(t) = t$ or $f(t') = t'$ we are done ; otherwise the function $x \mapsto f(x) - x$ changes sign on $(0,1)$. Since this function is continuous, the mean value theorem ensures that it must have a zero on $(0,1)$, which shows that there exists $a \in (0,1)$ such that $f(a) = a$.
(c) This is a direct consequence of question 1(b) (applied to the continuous function $f - g$).

3. Using Riemann sums, compute the limits (when $n \to +\infty$) of the following sequences :

$$\sum_{k=1}^{n} \frac{1}{n+k} \ ; \quad \sum_{k=1}^{n} \frac{n}{n^2+k^2} \ ; \quad \sum_{k=1}^{n} \frac{k^2}{n^3} \ ; \quad \sum_{k=1}^{n} \big(\sin(\frac{k\pi}{2n} - \sin(\frac{(k-1)\pi}{2n}) \ln(1+\sin(\frac{k\pi}{2n})); \quad ; \quad \sum_{k=1}^{n} \frac{(-1)^k}{k}.$$

Correction. Here, the trick is to recognize Riemann sums : the first one is $\displaystyle\sum_{k=1}^{n} \frac{1}{n+k} = \frac{1}{n}\sum_{k=1}^{n} \frac{1}{1+\frac{k}{n}}$, and this is a Riemann sum for the function $x \mapsto \dfrac{1}{1+x}$ for a tagged partition of $[0,1]$ with mesh $1/n$. Thus, we obtain $\displaystyle\lim\big(\sum_{k=1}^{n} \frac{1}{n+k}\big) = \int_0^1 \frac{dt}{1+t} = \ln(2)$.

The second one is similar : $\displaystyle\sum_{k=1}^{n} \frac{n}{n^2+k^2} = \frac{1}{n}\sum_{k=1}^{n} \frac{1}{1+\left(\frac{k}{n}\right)^2}$, hence $\displaystyle\lim(\sum_{k=1}^{n} \frac{n}{n^2+k^2}) = \int_0^1 \frac{dt}{1+t^2} = \arctan(1) - \arctan(0) = \frac{\pi}{4}$.

The third one is more of the same : $\displaystyle\sum_{k=1}^{n} \frac{k^2}{n^3} = \frac{1}{n}\sum_{k=1}^{n} \big(\frac{k}{n}\big)^2$, hence $\displaystyle\lim\big(\sum_{k=1}^{n} \frac{k^2}{n^3}\big) = \int_0^1 t^2 dt = \frac{1}{3}$.

The fourth one looks nasty, but again it is a Riemann sum for the continuous function $t \mapsto \ln(1+t)$ on $[0,1]$, with regard to the tagged partition $\{\big[\sin\big(\frac{(k-1)\pi}{2n}\big), \sin\big(\frac{k\pi}{2n}\big)\big], \sin\big(\frac{k\pi}{2n}\big)\}$, the mesh of which is smaller than $\frac{1}{2n}$ (use the mean value theorem to prove this). Hence when $n \to +\infty$ the sum converges to $\int_0^1 \ln(1+t)dt$, which is computable using integration by parts :

$$\int_0^1 \ln(1+t)dt = \big[(t+1)\ln(t+1)\big]_{t=0}^1 - \int_{t=0}^1 1dt = 2\ln(2) - 1 \ .$$

The last one doesn't look like a Riemann sum ; there is some work to be done before one can see a Riemann sum appear. Assume first that $n = 2p$; one has

$$u_{2p} = \sum_{k=1}^{n} \frac{(-1)^k}{k} = -\sum_{k=1}^{n} \frac{1}{k} + 2\sum_{k=1}^{p} \frac{1}{2k} = -\sum_{k=1}^{n} \frac{1}{k} + \sum_{k=1}^{p} \frac{1}{k}$$

Hence, when $n = 2p$, one has $\displaystyle\sum_{k=1}^{n} \frac{(-1)^k}{k} = -\sum_{k=p+1}^{2p} \frac{1}{k} = -\sum_{k=1}^{p} \frac{1}{p+k}$. The sum on the right is actually a Riemann sum for the continuous function $t \mapsto \frac{1}{1+x}$ on $[0,1]$ and the tagged partition $\{[\frac{k-1}{p}, \frac{k}{p}], \frac{k}{p}\}$, the mesh of which is $\frac{1}{p}$ (can you see why ?). So, we see that u_{2p} converges to $-\int_0^1 \frac{dt}{1+t} = \ln(2)$. Since $u_{2p+1} - u_{2p}$ converges to 0, we see that one also has $\lim(u_{2p+1}) = -\ln(2)$. A theorem we saw in class ensures that (u_n) is convergent and $\lim(u_n) = -\ln(2)$.

4. Let $f, g \colon [0,1] \to \mathbb{R}$ be continuous functions. Show that $\displaystyle\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f\big(\frac{k}{n}\big)g\big(\frac{k-1}{n}\big) = \int_0^1 f(t)g(t)\,dt.$

Correction. This is trickier than it looks : if we had $f\big(\dfrac{k}{n}\big)g\big(\dfrac{k}{n}\big)$ in the sum, then it would just be a usual Riemann sum and we could apply the results seen in class. Unfortunately, this is not what we have ; how can we deal with this ? One can proceed as follows : first, write that

$$\frac{1}{n}\sum_{k=1}^{n} f\big(\frac{k}{n}\big)g\big(\frac{k-1}{n}\big) = \frac{1}{n}\sum_{k=1}^{n} f\big(\frac{k}{n}\big)g\big(\frac{k}{n}\big) + \frac{1}{n}\sum_{k=1}^{n} f\big(\frac{k}{n}\big)\big(g\big(\frac{k-1}{n}\big) - g\big(\frac{k}{n}\big)\big) \ .$$

The first term converges to $\int_0^1 f(t)g(t)dt$, so we want to prove that the second term converges to 0. For that, we use the fact that g is uniformly continuous on $[0,1]$; given $\varepsilon > 0$, there exists δ_ε such that $|x-y| \le \delta_\varepsilon \Rightarrow$

$|f(x)-f(y)| \le \varepsilon$ for all $x, y \in [a,b]$. Hence if n is big enough one has $|g\big(\frac{k-1}{n}\big) - g\big(\frac{k-1}{n}\big)| \le \varepsilon$, so that

$$|\frac{1}{n}\sum_{k=1}^{n} f\big(\frac{k}{n}\big)\big(g\big(\frac{k-1}{n}\big) - g\big(\frac{k}{n}\big)| \le \varepsilon \frac{1}{n}\sum_{k=1}^{n} |f\big(\frac{k}{n}\big)| \ .$$

Since f is Riemann-integrable $|f|$ also is Riemann-integrable, hence $\frac{1}{n}\sum_{k=1}^{n}|f\big(\frac{k}{n}\big)|$ converges to $\int_a^b |f(t)|dt$. So if n is big enough one has $\frac{1}{n}\sum_{k=1}^{n}|f\big(\frac{k}{n}\big)| \le \int_a^b |f(t)|dt + 1$. Putting all this together, we get that for any ε there exists $K \in \mathbb{N}$ such that $|\frac{1}{n}\sum_{k=1}^{n} f\big(\frac{k}{n}\big)\big(g\big(\frac{k-1}{n}\big) - g\big(\frac{k}{n}\big)\big)| \le \varepsilon(\int_a^b |f(t)|dt + 1)$ for all $n \ge K$. This proves that $\frac{1}{n}\sum_{k=1}^{n} f\big(\frac{k}{n}\big)\big(g\big(\frac{k-1}{n}\big) - g\big(\frac{k}{n}\big)\big)$ converges to 0 (when $n \to +\infty$), which is what we needed to prove.

5. Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function such that $\displaystyle\int_0^1 f(u)u^k du = 0$ for all $k \in \{0, \dots, n\}$. Show that f has at least $n+1$ distinct zeros in $(0,1)$.
Hint : prove the result by induction using integration by parts and Rolle's theorem.
Correction. Following the hint, let us prove the result by induction. For $n = 0$ the result is a direct consequence of exercise 1 ; assume the result is true for n. Then pick a continuous function f such that $\displaystyle\int_0^1 f(u)u^k du = 0$ for all $k \in \{0, \dots, n+1\}$, and set $F(x) = \displaystyle\int_0^x f(t)dt$. The assumption on f for $k = 0$ yields $F(0) = F(1) = 0$. Also, for any $k = 1, \dots, n$, one has

$$\int_0^1 u^k du = \big[ku^{k-1}\big]_0^1 - k\int_0^1 u^{k-1}F(u)du$$

Thus we obtain $\displaystyle\int_0^1 u^{k-1}F(u)du = 0$ for all $k = 1, \dots, n$, which yields (because of our induction hypothesis) that F has at least n distinct zeros in $(0,1)$. Since $F(0) = F(1) = 0$, F must have at least $n+2$ distinct zeros on $[0,1]$. And $F' = f$ has a zero between any two zeros of F, which shows that f has at least $n+1$ distinct zeros on $(0,1)$.

7.1.13. We need to use the definition of a Riemann integral ; assume the points $c_1, \dots, c_n$ are indexed in such a way that $c_1 < c_2 < \dots < c_{n-1} < c_n$, and set $M = \max\{|f(c_i)| \colon i = 1, \dots, n\}$. Then pick a tagged partition $\dot{\mathcal{P}} = \{[x_{i-1}, x_i], t_i\}_{i=1,\dots m}$ of $[a,b]$. One has

$$|S(f,\dot{\mathcal{P}})| = |\sum_{i=1}^{m}(x_i - x_{i-1})f(t_i)| \le \sum_{i=1}^{m}|(x_i - x_{i-1})f(t_i)| \le \|\dot{\mathcal{P}}\|\sum_{i=1}^{m}|f(t_i)| \ .$$

Since there are only n points in the interval at which $f(x) \neq 0$, and at each of these points one has $|f(x)| \le M$, we see that $|S(f,\dot{\mathcal{P}})| \le \|\dot{\mathcal{P}}\|.2n.M$ (because there can be at most two t_i with the same value, and at most n points at which f is nonzero, so at most $2n$ of them can appear in the sum). But then (since n, M are constant) we are done : if one sets $\delta_\varepsilon = \frac{\varepsilon}{2nM}$, what we have proved implies that for any partition $\dot{\mathcal{P}}$ with mesh less than δ_ε one has $|S(f,\dot{\mathcal{P}})| \le \varepsilon$. This is exactly what we needed to prove that $f \in \mathcal{R}([a,b])$ and $\int_a^b f(x)dx = 0$.

7.1.14. This is a consequence of the preceding exercise : indeed, the function $f - g$ satisfies the condition of exercise 7.1.13, hence $f - g \in \mathcal{R}[a,b]$ and $\int_a^b (f(t) - g(t))dt = 0$. But then $f = (f-g) + g$ is the sum of two Riemann-integrable functions, so $f \in \mathcal{R}([a,b])$ and $\int_a^b f(t)dt = \int_a^b (f(t) - g(t))dt + \int_a^b g(t)dt = \int_a^b g(t)dt$.

7.1.15. Let us follow the hint : pick $\varepsilon > 0$, set $\delta_\varepsilon = \frac{\varepsilon}{4\alpha}$ and pick a tagged partition $\dot{\mathcal{P}} = \{[x_{i-1}, x_i], t_i\}_{i=1,\dots,n}$ with mesh $\le \delta_\varepsilon$. Then by definition one has $S(\varphi,\dot{\mathcal{P}}) = \sum_{i=1}^{n}(x_i - x_{i_1})\varphi(t_i)$. There are two possibilities for $\varphi(t_i)$: either it is equal to 0, or it is equal to α. Only the t_i's that belong to $[c,d]$ contribute to the sum. Let $I = \{i \colon t_i \in [c,d]\}$. Then $S(\varphi,\dot{\mathcal{P}}) = \alpha\sum_{i\in I}(x_i - x_{i-1})$. Since $t_i \in [x_{i-1}, x_i]$, t_i can only be in $[c,d]$ if ($x_{i-1} < a$ and $x_i \ge a$), or (x_{i-1}, x_i are both in $[c,d]$), or ($x_{i-1} \le d$ and $x_i > d$). The first and third condition can each be satisfied at most by one index, and the remaining $[x_{i-1}, x_i]$ from a partition of a subinterval of $[c, d]$, so that $S(\varphi, \dot{\mathcal{P}}) \leq \alpha(d-c) + 2\delta_\varepsilon \alpha$. Similarly, the "chunk" of $[x, d]$ that can be missed by the t_i's is at most $2\delta_\varepsilon$ long, hence $S(\varphi, \dot{\mathcal{P}}) \geq \alpha(d-c) - 2\delta_\varepsilon \alpha$. This shows that whenever $\dot{\mathcal{P}}$ is a tagged partition with mesh less than $\delta_\varepsilon = \frac{\varepsilon}{4\alpha}$ one has

$$\alpha(d-c) - \frac{\varepsilon}{2} \leq S(\varphi, \dot{\mathcal{P}}) \leq \alpha(d-c) + \frac{\varepsilon}{2} \ .$$

This is enough to show that $\varphi \in \mathcal{R}([a, b]$ and $\int_a^b \varphi(t)dt = \alpha(d-c)$.

7.2.11. Let's follow the hint and define (given $\varepsilon > 0$) $\alpha_\varepsilon, \omega_\varepsilon$ by $\alpha_\varepsilon(x) = \begin{cases} -M & \text{if } x \in [a, c) \\ f(x) & \text{if } x \in [c, b] \end{cases}$ and $\omega_\varepsilon(x) = \begin{cases} M & \text{if } x \in [a, c) \\ f(x) & \text{if } x \in [c, b] \end{cases}$ (c is to be specified later). Then one has $\alpha_\varepsilon(x) \leq f(x) \leq \omega_\varepsilon(x)$ for all $x \in [a, b]$. Also, α_ε, ω_ε are both Riemann-integrable on $[a, b]$ because of the Additivity theorem. Finally, one has $\int_a^b (\omega_\varepsilon(t) - \alpha_\varepsilon(t))dt = \int_a^c 2Mdt = 2M(c-a)$ by definition of α_ε, ω_ε. Hence if one sets $c = a + \frac{\varepsilon}{2M}$ we get that $\int_a^b (\omega_\varepsilon(t) - \alpha_\varepsilon(t)) \leq \varepsilon$. So we managed to prove that the assumptions of the Squeeze Theorem are satisfied, hence $f \in \mathcal{R}([a, b])$. Then since $|f(x) \leq M$ for all $x \in [a, b]$ we see that $\left|\int_a^c f(t)dt\right| \leq M(c-a)$, so $\lim_{c \to a} \int_a^c f(t)dt = 0$. Thus the additivity theorem gives $\lim_{c \to a} \int_c^b f(t)dt = \int_a^b f(t)dt$.

7.2.12. This is a consequence of the preceding exercise : $|g(x)| \leq 1$ for all $x \in [0, 1]$, and g is continuous on $[c, 1]$ for all $c \in (0, 1)$. Hence it is Riemann-integrable on $[0, 1]$.

7.2.16. Set $F(x) = \int_a^x f(t)dt$; since f is continuous on $[a, b]$, the fundamental theorem of calculus ensures that F is differentiable on $[a, b]$, hence it satisfies the assumptions of the Mean Value theorem on this interval, so there exists $c \in (a, b)$ such that $F(b) - F(a) = F'(c)(b-a)$. This is the same as saying that there exists $c \in (a, b)$ such that $\int_a^b f(t)dt = f(c)(b-a)$.
One can also solve this exercise differently : one has $f([a, b]) = [m, M]$ by the theorems about continuous functions, from which we get $m(b-a) \leq \int_a^b f(t)dt \leq M(b-a)$ But then $m \leq \frac{\int_a^b f(t)dt}{b-a} \leq M$, hence there exists c such that $f(c) = \frac{\int_a^b f(t)dt}{b-a}$, which is the same as saying that $(b-a)f(x) = \int_a^b f(t)dt$.

7.2.17. We can apply a similar method to the one in the exercise above : denote again $f([a, b])$ by $[m, M]$. Then one has $\int_a^b f(t)g(t) - m \int_a^b g(t)dt = \int_a^b (f(t) - m)g(t)dt \geq 0$ (because $f(t) \geq m$ and $g(t) \geq 0$ for all $t \in [a, b]$. Similarly, one finds that $\int_a^b f(t)g(t)dt \leq M \int_a^b g(t)dt$. Put together, this yields

$$m \leq \frac{1}{\int_a^b g(t)dt} \int_a^b f(t)g(t)dt \leq M \ .$$

Thanks to the intermediate value theorem, we can now conclude : there exists $c \in [a, b]$ such that $f(c) = \frac{1}{\int_a^b g(t)dt} \int_a^b f(t)g(t)dt$, which is the same as $\int_a^b f(t)g(t)dt = f(c) \int_a^b g(t)dt$. This result is clearly false if one no longer assumes that g takes nonnegative values; for instance, let $a = -1$, $b = 1$, $f(t) = t$ and $g(t) = t$. Then one has $\int_a^b f(t)g(t)dt = 1$ but $f(c) \int_a^b g(t)dt = 0$ for all $c \in [0, 1]$.

7.3.11. Here one needs to apply the Chain Rule (and the fundamental theorem of calculus), which yields :
(a) In this case $F(x) = G(x^2)$, where $G'(x) = \dfrac{1}{1+x^3}$; hence $F'(x) = \dfrac{2x}{1+(x^2)^3} = \dfrac{2x}{1+x^6}$.
(b) This time $F(x) = G(x) - G(x^2)$, where $G'(x) = \sqrt{1+x^2}$. Hence $F'(x) = G'(x) - 2xG'(x^2) = \sqrt{1+x^2} - 2x\sqrt{1+x^4}$.

7.3.13. Set first $F(x) = \int_0^x f(t)dt$. Then we know that F is differentiable and $F'(x) = f(x)$. By definition, we have $g(x) = F(x+c) - F(x-c)$, hence g is a composition of differentiable functions. Thus g is differentiable on $\mathbb{R}$, and the Chain Rule yields $g'(x) = F'(x+c) - F'(x-c) = f(x+c) - f(x-c)$.

7.3.14. First notice that the assumption on f implies that $\int_0^1 f(t)dt = 0$ (take $x = 0$). Set $F(x) = \int_0^x f(t)dt$. Then the assumption on F become $F(x) = F(1) - F(x)$ for all $x \in [0,1]$, and since $F(1) = 0$ this yields $F(x) = 0$ for all $x \in [0,1]$. Since f is continuous the fundamental theorem of calculus gives $F' = f$, hence $f(x) = 0$ for all $x \in [0,1]$.

7.3.21. (a) The functions $x \mapsto (tf(x) + g(x))^2$ and $x \mapsto (tf(x) - g(x))^2$ are both Riemann-integrable on $[a,b]$ and take nonnegative values, hence $\int_a^b (tf(u) \pm g(u))^2 dt \geq 0$.
(b) We have :

$$\int_a^b (tf(u)+g(u))^2 du = \int_a^b (t^2 f^2(u)+2tf(u)g(u)+g(u)^2)du = t^2 \int_a^b f(u)^2 du + 2t \int_a^b f(u)g(u)du + \int_a^b g(u)^2 du \ .$$

Since the quantity on the left is positive, we obtain $-2t \int_a^b f(u)g(u)du \leq t^2 \int_a^b f(u)^2 du + \int_a^b g(u)^2 du$. Hence for any $t > 0$ we have $-2 \int_a^b f(u)g(u)du \leq t \int_a^b f(u)^2 du + \frac{1}{t} \int_a^b g(u)^2 du$. Similarly, using the fact that $\int_a^b (tf(u) - g(u))^2 du \geq 0$, one obtains $2 \int_a^b f(u)g(u)du \leq t \int_a^b f(u)^2 du + \frac{1}{t} \int_a^b g(u)^2 du$. The two inequalities together yield

$$2|\int_a^b f(u)g(u)du| \leq t \int_a^b f(u)^2 du + \frac{1}{t} \int_a^b g(u)^2 du \ .$$

(c) If $\int_a^b f^2(u)du = 0$ then the result above implies that $2|\int_a^b f(u)g(u)du| \leq \frac{1}{t} \int_a^b g(u)^2 du$ for all $t > 0$. This is only possible if $\int f(u)g(u)du = 0$.
(d) Since one has both $fg \leq |fg|$ and $-fg \leq |fg|$, it is true that both $\int_a^b f(u)g(u)du \leq \int_a^b |f(u)g(u)|du$ and $-\int_a^b f(u)g(u)du \leq \int_a^b |f(u)g(u)|du$. This means that $|\int_a^b f(u)g(u)du| \leq \int_a^b |f(u)g(u)|du$, which is equivalent to the inequality on the left.
To prove the inequality on the right,recall that we know from (b) (applied to $|f|, |g|$) that $t^2 \int_a^b f^2(u)du + 2t \int_a^b |f(u)g(u)|du + \int_a^b g - u)^2 du \geq 0$ for all $t \in \mathbb{R}$. This means that the polynomial function $t \mapsto t^2 \int_a^b f^2(u)du + 2t \int_a^b |f(u)g(u)|du + \int_a^b g - u)^2 du$ keeps a constant sign on $\mathbb{R}$, and this is possible only if its discriminant $4\big(\int_a^b |f(u)g(u)|u\big)^2 - 4 \int_a^b f^2(u)du \int_a^b g(u)^2 du$ is ≤ 0. In other words, one must have

$$\left(\int_a^b |f(u)g(u)|du \right)^2 \leq \int_a^b f(u)^2 du \int_a^b g(u)^2 du \ .$$

To get the inequality we are asked to prove, apply this inequality to the functions $f(t) = 1/t$ and $g(t) = 1$: this yields $\left(\int_a^b \frac{dt}{t} \right)^2 \leq \int_a^b \frac{dt}{t^2} \int_a^b dt = (\frac{1}{a} - \frac{1}{b})(b-a) = \frac{(b-a)^2}{ab}$. Taking the square root, one has

$$\int_a^b \frac{dt}{t} \leq \frac{(b-a)}{\sqrt{ab}}$$