University of Illinois at Urbana-Champaign — Fall 2006
Math 444 — Group E13

Training exercises : Correction.

5.2.7 : We more or less saw that example in class : define a function $f\colon [0,1] \to [0,1]$ by setting, for all $x \in [0,1]$, $f(x) = \begin{cases} -1 & \text{if } x \in \mathbb{Q} \\ 1 & \text{else} \end{cases}$. Pick a number $x \in [0,1]$, and recall that there exist a sequence (q_n) of rational numbers in $[0,1$ and a sequence (α_n) of irrational numbers in $[0,1]$ such that $\lim(q_n) = \lim(\alpha_n) = x$. One also has $f(q_n) = 0$ for all $n \in \mathbb{N}$, and $f(\alpha_n) = 1$ for all $n\ inN$; this proves that f doesn't have a limit at x, so it cannot be continuous at that point. Thus f is discontinuous at every point of $[0,1]$, yet $|f|$ is constant (equal to 1), so it is a continuous function.

5.2.8 : Recall again that any real number is the limit of a sequence of rational numbers ; pick $x \in \mathbb{R}$, and a sequence of rational numbers (q_n) that converges to x. Then one has $\lim f(q_n) = f(x)$ since f is continuous at x, and for the same reason $\lim g(q_n) = g(x)$. Since by assumption $f(q_n) = g(q_n)$, we obtain that $f(x) = g(x)$. The same argument is sufficient to prove that, given two continuous function f, g, it is enough that they coincide on a dense subset of the real line to ensure that they are equal everywhere.

5.2.9 : First, notice that for any $x \in \mathbb{R}$ and any $\varepsilon > 0$ there exist $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that $x - \varepsilon \leq \frac{m}{2^n} \leq x$. Indeed, pick $n \in \mathbb{N}$ such that $\frac{1}{2^n} \leq \varepsilon$. Then there exists $m \in \mathbb{Z}$ such that $\frac{m}{2^n} < x$ and the set A of such $m's$ is bounded above. Let m denote the supremum of this set ; as usual, using the definition of a supremum, one can prove that m must actually be an integer, and that $m \in A$. Clearly $m+1 \notin A$, and this yields $\frac{m}{2^n} + \frac{1}{2^n} \geq x$, so that $\frac{m}{2^n} \geq x - \frac{1}{2^n}$. This proves that

$$x - \varepsilon \leq \frac{m}{2^n} \leq x \ .$$

We have just proved that the set $S = \{\frac{m}{2^n} \colon n \in \mathbb{N},\, m \in \mathbb{Z}\}$ is dense in $\mathbb{R}$. Our assumption on the function f is that $f(s) = 0$ for all $s \in S$, and that f is continuous, so reasoning as in the preceding exercise we obtain $f(x) = 0$ for all $x \in \mathbb{R}$.

5.2.14 : First, notice that if $g(x) = 0$ for some $x \in \mathbb{R}$ then one must actually have $g(y) = 0$ for all $y \in \mathbb{R}$: indeed,

$$g(y) = g(y - x + x) = g(y - x)g(x) = 0 \ .$$

Thus, we may as well assume that $g(x) \neq 0$ for all $x \in \mathbb{R}$; notice then that $g(0)g(x) = g(x)$ for all $x\ in\mathbb{R}$, so that $g(0) = 1$.
Assume now that g is continuous at 0, pick $x \in \mathbb{R}$ and a sequence (x_n) of reals that is convergent to x. Then we wish to prove that $\lim g(x_n) = g(x)$. For that, we have to use our assumption that g is continuous at 0 : it is then natural to look for a sequence that converges to 0 and that tells us something about our problem. Here this sequence is $(y_n) = (x_n - x)$: to use it, we write that

$$g(x_n) = g(x_n - x + x) = g(x_n - x)g(x) = g(y_n)g(x).$$

We know that $\lim g(y_n) = g(0) = 1$ because g is continuous at 0, so we obtain $\lim g(x_n) = g(x)$. Since the sequence (x_n) was arbitrary, we have proved that g is continous at x. This is true for all $x \in \mathbb{R}$, so g is continuous on $\mathbb{R}$.
Notice that then g doesn't have a zero, so it doesn't change sign (it is continuous!) ; since $g(0) = 1$, g takes only positive values. If one sets $h(x) = \ln(g(x))$, one has $h(x+y) = h(x) + h(y)$ for all $x \in \mathbb{R}$ and we can

use what we proved about such a function in the homework. This enables one to prove that there exists a real number λ such that $g(x) = e^{\lambda x}$ for all $x \in \mathbb{R}$.

5.3.8 : One has $f(1) = 2\ln(1) + \sqrt{1} - 2 = -1$, and $f(2) = 2\ln(2) + \sqrt{2} - 2 = \ln(4) + \sqrt{2} - 2 \geq 1 + \sqrt{2} - 2 > 0$. Thus the intermediate value theorem tells us that there exists $c \in [1,2]$ such that $f(c) = 0$. The bisection method proceeds as follows : $f(3/2)$ is greater than 0, hence there must be a solution of the equation in the interval $[1, 3/2]$; now let us look at what happens at the middle of this interval $(5/4)$: there we see that $f(5/4) \leq 0$. Hence there is a solution of the equation in the interval $[5/4, 3/2]$; the middle of this interval is $11/8$, and computation shows that $f(11/8) \leq 0$. Thus there is a solution in the interval $[11/8, 3/2]$. We need to keep going until we obtain an interval of length smaller than 10^{-2} in which we know that there is a solution to the equation : this process yields eventually that there is a solution in the interval $[189/128, 190/128]$ so an approximate (with at least 10^{-2} acccuracy) value of the solution is 1.48.
Remark. What we did above doesn't prove that the solution we obtained is unique ; to see that, one would need to study the variations of the function. The bisection method only gives one solution, not all of them...

5.4.11 : Assume that there exists a constant K such that $\sqrt{x} \leq Kx$ for all $x \in [0,1]$ (one can forget the absolute values here because all the numbers involved are nonnegative). First notice that necessarily $K \geq 1$ (that's what the inequality yields when $x = 1$). Apply then the inequality to $x = \dfrac{1}{K^4}$; this yields $\dfrac{1}{K^2} \leq \dfrac{K}{K^4}$, and this is equivalent to $K \leq 1$. Thus the only possible constant would be $K = 1$, and if it worked then one would have $\sqrt{\dfrac{1}{2}} = \dfrac{\sqrt{2}}{2} \leq \dfrac{1}{2}$, and this is not true. Thus there is no K as in the inequality above and this shows that g is not a Lipschitz function on $[0,1]$. Yet it is uniformly continuous on that interval because any continuous function on a closed bounded interval must be uniformly continuous on that interval.
Remark : To show that K could not exist, we used a statement of the form "for all x something happens" by saying "well, if it happens for all x, then it must also happen for *this* (well-chosen) x, and this can't be true".

5.4.14 : Assume that f is a continuous function on $\mathbb{R}$ and that $f(x+p) = f(x)$ for some $p > 0$ and all $x \in \mathbb{R}$. Then notice first that $f(x) = f(x + Kp)$ for all $x \in \mathbb{R}$ and all $K \in \mathbb{Z}$ (prove this). Also, for all $x \in \mathbb{R}$ there exists $K \in \mathbb{Z}$ such that $x + Kp \in [0,p]$ (to prove it, use a method similar to that of exercise 5.2.9, or use the function $E(\dfrac{x}{p})$). Since f is continuous on $[0,p]$, it is bounded on that interval so there exists $m, M \in \mathbb{R}$ such that $m \leq f(y) \leq M$ for all $y \in [0,p]$. Given the choice of K, this implies that $m \leq f(x + Kp) \leq M$, and this is the same as saying that $m \leq f(x) \leq M$. We have thus proved that f is bounded on $\mathbb{R}$.
To show that f is uniformly continuous, the idea is again that f is essentially defined on a closed bounded interval (and then "repeats" its values), and continuous functions defined on closed bounded intervals are uniformly continuous. Thus this should be easy to write down ; there is, however, a slight problems due to the bounds of intervals of length p (try to write down a proof to convinve yourself). To avoid this problem, we use the interval $[0, 2p]$ instead of the interval $[0,p]$.
Pick $\varepsilon > 0$; we know that there exists δ such that for any two $x, y \in [0, 2p]$ one has $|x-y| \leq \delta \Rightarrow |f(x) - f(y)| \leq \varepsilon$. We would like this implication to hold for any two $x, y \in \mathbb{R}$; for this, notice that, if $\delta \leq p$, then for any $x, y \in \mathbb{R}$ such that $|x - y| \leq \delta$ there exists $K \in \mathbb{Z}$ such that both $x + Kp$ and $y + Kp$ belong to $[0, 2p]$ (are you able to prove this ?). Set now $\delta' = \min(\delta, p)$. Then pick any $x, y \in \mathbb{R}$ such that $|x-y| \leq \delta'$. One can find $K \in \mathbb{Z}$ such that $x + Kp$, $y + Kp$ both belong to $[0, 2p]$; since $|(x + Kp) - (y + Kp)| = |x - y| \leq \delta' \leq \delta$, we see that $|f(x + Kp) - f(y + Kp) \leq \varepsilon|$. Since f is p-periodic and $K \in \mathbb{Z}$, we have $f(x + Kp) = f(x)$, $f(y + Kp) = f(y)$; hence we finally obtained that for any $x, y \in \mathbb{R}$, $|x-y| \leq \delta' \Rightarrow |f(x) - f(y)| \leq \varepsilon$. This shows that f is uniformly continuous on $\mathbb{R}$.
Remark. There are a few assertions above that should be explained in more detail ; are you able to do so ? Do you see why the interval $[0, 2p]$ was used above instead of the interval $[0,p]$?

6.1.2 : One has $\dfrac{f(x) - f(0)}{x - 0} = x^{-2/3}$, and $x^{-2/3}$ doesn't have a limit at 0 (it is not locally bounded). This shows that $\dfrac{f(x) - f(0)}{x - 0}$ doesn't have a limit at $x = 0$, in other words f is not differentiable at 0.

6.1.9 : It is enough to apply the Chain Rule : given that the function $x \mapsto -x$ is differentiable on $\mathbb{R}$ and has a derivative equal to -1, we obtain (taking the derivative of both sides of the equation $f(x) = f(-x)$) that $f'(x) = (-1)f'(-x) = -f'(-x)$. This proves that the derivative of an even function is an odd function. The exact same proof (taking this time the derivative of both sides of $f(x) = -f(-x)$) yields that if f is an odd function then its derivative is an even function.

6.1.10 : This is more or less the same exercise as exercise 3 of HW11.

6.2.6 We know that $\sin'(x) = \cos(x)$ for all $x \in \mathbb{R}$; thus, if $x < y \in \mathbb{R}$, the mean value theorem yields $f(x) - f(y) = \cos(c)(x-y)$ for some $c \in (x, y)$. Since $|\cos(c)| \leq 1$ for all $x \in \mathbb{R}$, we obtain $|f(x) - f(y)| \leq |x-y|$ for any x, y such that $x < y$. This inequality is also true if $x = y$, and since x, y play symmetric roles it must also be true if $x > y$. Thus we have proved that $|\sin(x) - \sin(y)| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

6.2.8 : To show that $f'(a)$ exists, we need to look at $\dfrac{f(x) - f(a)}{x - a}$. Applying the mean value theorem to f (which satisfies its assumptions) we obtain that $\dfrac{f(x) - f(a)}{x - a} = f'(c)$ for some $c \in (a, x)$. Since $\lim\limits_{x \to a} f'(x) = A$, we know that for any $\varepsilon > 0$ there exists δ such that $a < x \leq a + \delta \Rightarrow |f'(x) - A| \leq \varepsilon$. Given what we've written before, and since when $x \leq a + \delta$ one also has $c \leq a + \delta$, we get $a < x \leq a + \delta \Rightarrow \left|\dfrac{f(x) - f(a)}{x - a} - A\right| \leq \varepsilon$. Thus, $\lim \left(\dfrac{f(x) - f(a)}{x - a} \right) = A$, and this shows that $f'(a)$ exists and equals A.

6.2.11 : Define $f(x) = \sqrt{x}$. Then f is uniformly continuous on $[0, 1]$ since it is continuous on that closed,bounded interval, it is differentiable on $(0, 1)$ and $f'(x) = \dfrac{1}{2\sqrt{x}}$ is not bounded on $(0, 1)$.
Remark. The point of this exercise is that when $|f'|$ is bounded on (a, b) then one gets that f is uniformly continuous on $[a, b]$ (notice that one interval is closed and one is open) ; actually, one gets that f is Lipschitz on $[a, b]$. Here we see that the converse is false, i.e a function can be uniformly continuous on $[a, b]$ even if its derivative is not bounded on (a, b).

6.2.13 : Pick $x < y \in I$. Then $f(y) - f(x) = f'(c)(x - y)$ for some $c \in (x, y)$ because of the Mean Value theorem applied to the function f on $[x, y]$ (notice that f satisfies all the assumptions of this theorem, since it is differentiable on $[x, y]$ and hence is certainly continuous on $[x, y]$ and differentiable on (x, y)). Since f' only takes positive values, we see that $f(y) - f(x) > 0$ as soon as $y > x$: f is strictly increasing.
Remark. This is one of the reasons why the Mean Value Theorem is so important : it justifies the results about variations of functions.

6.2.14 : We saw in class that derivatives satisfy the conclusion of the intermediate value theorem (that's Darboux's theorem). Thus if there were two points $x, x' \in I$ such that $f'(x) \leq 0$ and $f'(x) \geq 0$ then f' would necessarily have a zero somewhere between x and x'. Thus if f' does not take the value 0 on an interval I then it must keep a constant sign, in other words one must have either $f'(x) > 0$ for all $x \in I$ or $f'(x) < 0$ for all $x \in I$.
Remark. This is related to the preceding exercise : if one wants to study the variations of a function f, one needs to establish where $f(x) > 0$ and where $f(x) > 0$. This exercise tells you that to do so, you need to look first for points where $f'(x) = 0$: if f' changes sign then it's at one of those points (of course it doesn't have to actually change sign at a zero, look at what happens when $f(x) = x^3$).

6.2.17 : One has $g'(x) - f'(x) \geq 0$ for all x, thus (applying the mean value theorem to the differentiable function $g - f$, as in exercise 6.2.13), we get that $g - f$ is an increasing function. Since $(g - f)(0) = g(0) - f(0) = 0$, we get $(g - f)(x) = g(x) - f(x) \geq 0$ for all $x \geq 0$ (and $(g - f)(x) = g(x) - f(x) \leq 0$) for all $x \leq 0$.
In other words, $g(x) \geq f(x)$ for all $x \geq 0$ (and $g(x) \leq f(x)$ for all $x \leq 0$).