Some topics related to bounding by canonical functions

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Outline

1. The partial order $({}^{\kappa}ORD, \leq_{\mathcal{I}})$ and canonical functions
2. Self-generic structures ("antichain catching")
3. How antichain catching is related to bounding by canonical functions
4. Forcing Axioms vs. nice ideals on ω_2

The partial order $\leq_{\mathcal{I}}$ on ${}^{\kappa}ORD$

Let κ be regular, uncountable and $\mathcal{I} \subset \wp(\kappa)$ a **normal** ideal.

e.g.

- $\mathcal{I} := NS_{\kappa}$; or
- $\mathcal{I} := NS \restriction S$ for some stationary $S \subset \kappa$.

Define $\leq_{\mathcal{I}}$ on ${}^{\kappa}ORD$ by:

$$f \leq_{\mathcal{I}} g \iff \{\alpha < \kappa \mid f(\alpha) \leq g(\alpha)\} \in Dual(\mathcal{I})$$

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$\leq_{\mathcal{I}}$ is wellfounded

Canonical functions on κ

Definition (Canonical functions on κ)

By recursion: $h_\nu :\simeq$ the $\leq_{NS_\kappa}$-least upper bound of $\langle h_\mu \mid \mu < \nu \rangle$ (**if** such a l.u.b. exists)

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The "first few" (i.e. for $\nu < \kappa^+$); these all map into κ:

- $h_0 : \ \alpha \mapsto 0$
- $h_{\nu+1} : \ \alpha \mapsto h_\nu(\alpha) + 1$
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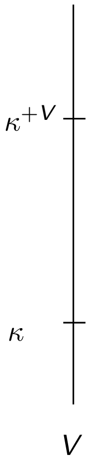
Theorem (Jech-Shelah; Hajnal)

Existence of h_{κ^+} is independent of ZFC.

Canonical functions and ultrapowers



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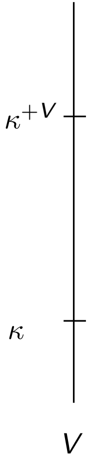
Let $\mathcal{U} \subset P(\kappa)$ be normal w.r.t. V



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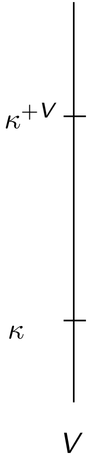

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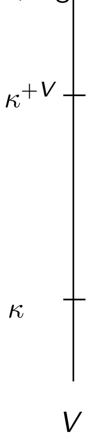

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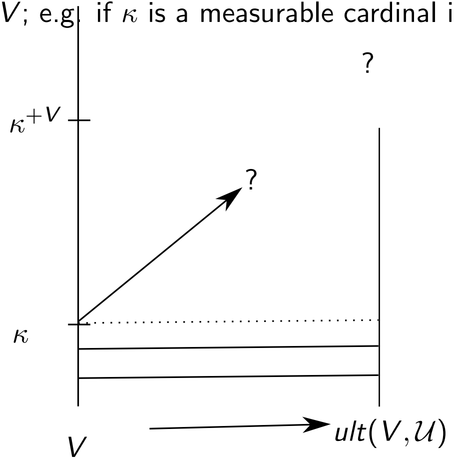

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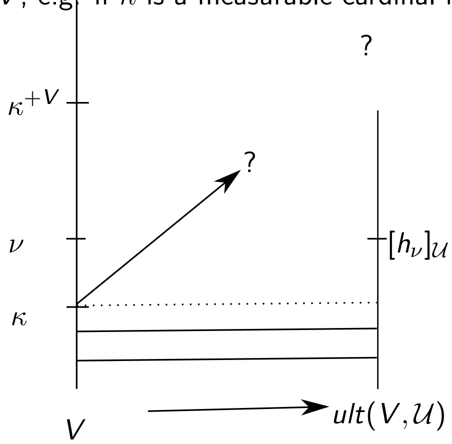

Other characterizations of the first κ^+ canonical functions

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Non-recursive characterizations of h_ν (for $\nu < \kappa^+$):

- "the" function which represents ν in any generic ultrapower by a normal ideal on κ
- Fix any surjection $g_\nu : \kappa \to \nu$ and set $h_\nu(\alpha) := otp(g_\nu'' \alpha)$
- Fix any wellorder Δ of H_{κ^+} and set

$$h_\nu(\alpha) :\simeq otp(M \cap \nu)$$

for **any** $M \prec (H_{\kappa^+}, \in, \Delta, \{\nu\})$ such that $\alpha = M \cap \kappa$

Bounding by canonical functions

Definition

For a normal ideal $\mathcal{I} \subset \wp(\kappa)$, *Bound*$(\mathcal{I})$ means that $\{h_\nu \mid \nu < \kappa^+\}$ is cofinal in $({}^\kappa\kappa, \leq_\mathcal{I})$.

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Lemma

Suppose κ is a **successor** *cardinal.*

Bound$(\mathcal{I})$ implies that if $\mathcal{U}$ is an ultrafilter on $V \cap \wp(\kappa)$ such that:

- *$\mathcal{U}$ is normal w.r.t. sequences from V*
- *$\mathcal{U}$ extends the dual of $\mathcal{I}$*

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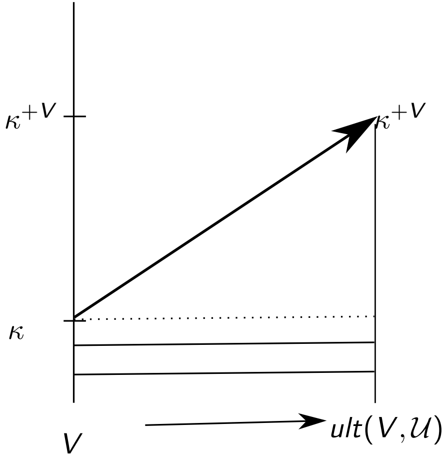
One can **always** obtain such a $\mathcal{U}$ (even if κ is a successor cardinal) by forcing with $\mathbb{P}_\mathcal{I} := (P(\kappa)/\mathcal{I}, \subseteq_\mathcal{I})$.

Assuming κ is successor, *Bound*$(\mathcal{I})$, and $\mathcal{U} \supset$ *Dual*$(\mathcal{I})$:



Saturation implies bounding

Definition

Let $\mathcal{I}$ be a normal ideal on κ. $\mathcal{I}$ is saturated iff $\mathbb{P}_{\mathcal{I}} := (\wp(\kappa)/\mathcal{I}, \subseteq_{\mathcal{I}})$ has the κ^+-cc.

Lemma (folklore)

If $\mathcal{I}$ is saturated then Bound$(\mathcal{I})$ holds.

Saturation implies bounding

- κ^+-cc of $\mathbb{P}_{\mathcal{I}}$ (and that κ is a successor cardinal) implies

$$\Vdash_{\mathbb{P}_{\mathcal{I}}} j_{\dot{G}}(\kappa) = \kappa^{+V}$$

- Then for every $f : \kappa \to \kappa$:

$$D_f := \{S \in \mathcal{I}^+ \mid \exists \nu < \kappa^+ \ f < h_\nu \text{ on } S\}$$

  is dense in $\mathbb{P}_{\mathcal{I}}$
- For each $S \in D_f$ pick a $\nu_S < \kappa^+$ such that $f < h_{\nu_S}$ on S
- Let $\mathcal{A}_f \subset D_f$ be a maximal antichain.
- Set $\mu := sup\{\nu_S \mid S \in \mathcal{A}_f\}$; $\mu < \kappa^+$ by κ^+-cc of $\mathbb{P}_{\mathcal{I}}$.
- Maximality of $\mathcal{A}_f$ implies that $f \leq_{\mathcal{I}} h_\mu$.

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Lemma (folklore?)

$$\Diamond_\kappa \implies \neg Bound(NS_\kappa)$$

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Suppose $\langle A_\alpha \mid \alpha < \kappa \rangle$ is a $\Diamond_\kappa$ sequence, $p : \kappa \times \kappa \leftrightarrow^{bij} \kappa$, and

$$f(\alpha) := \begin{cases} otp(A_\alpha) & \text{if } A_\alpha \text{ codes a wellorder (via } p \restriction (\alpha \times \alpha)) \\ 0 & \text{otherwise} \end{cases}$$

Fix $\nu < \kappa^+$. Fix $b \subset \kappa$ coding ν.

- $b \cap \alpha = A_\alpha$ for stationarily many α
- $otp(b \cap \alpha) = h_\nu(\alpha)$ for club-many α

So $f(\alpha) = h_\nu(\alpha)$ for stationarily many α. So $f \not<_{NS} h_\nu$

Chang's Conjecture and bounding

Lemma

$(\kappa^+, \kappa) \twoheadrightarrow (\kappa, < \kappa)$ *implies a weak variation of Bound*(NS_κ).

$Bound(NS_{\omega_1})$ is well-understood

Theorem (Larson-Shelah; Deiser-Donder)

The following are equiconsistent:

- $ZFC + Bound(NS_{\omega_1})$
- $ZFC +$ *there is an inaccessible limit of measurable cardinals*

Moreover, saturation of NS_{ω_1} (which implies $Bound(NS_{\omega_1})$) is known to be consistent relative to a Woodin cardinal (Shelah).

What about $Bound(NS_{\omega_2})$?

NOTATION: $S^m_n := \omega_m \cap cof(\omega_n)$

Theorem (Shelah)

Suppose $\mathcal{I}$ is a normal ideal on ω_2 such that $S^2_0 \in \mathcal{I}^+$. Then $\mathcal{I}$ is **not** *saturated.*

In particular, NS_{ω_2} is **never** *saturated.*

Theorem (Woodin; building on work of Kunen and Magidor)

It is consistent relative to an almost huge cardinal that there is some stationary $S \subseteq S^2_1$ such that $NS_{\omega_2} \restriction S$ is saturated.
(Recall this implies $Bound(NS_{\omega_2} \restriction S)$)

Question (Well-known open problems)

1. *Can $NS_{\omega_2} \restriction S^2_1$ be saturated?*
2. *Can $Bound(NS_{\omega_2})$ hold? What about $Bound(NS_{\omega_2} \restriction S^2_1)$?*

Big gap in known consistency bounds

Question

What is the consistency strength of: "Bound($\mathcal{I}$) holds for some normal ideal $\mathcal{I} \subset \wp(\omega_2)$"?

- Best known upper bound: almost huge cardinal (Kunen, Magidor, Woodin)
- Best known lower bound (even assuming that $F = NS_{\omega_2}$): inaccessible limit of measurables ! (Deiser-Donder)

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Lower bound for $Bound(\omega_2)$ hasn't even escaped "easy" inner model theory.

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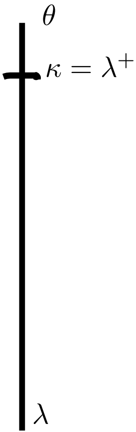
Derived ultrapowers



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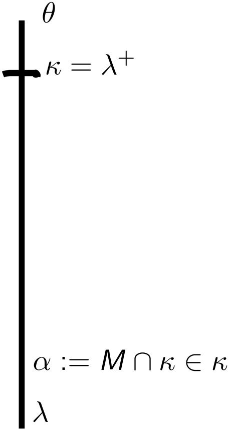

$M \prec (H_\theta, \in, \{\kappa\})$

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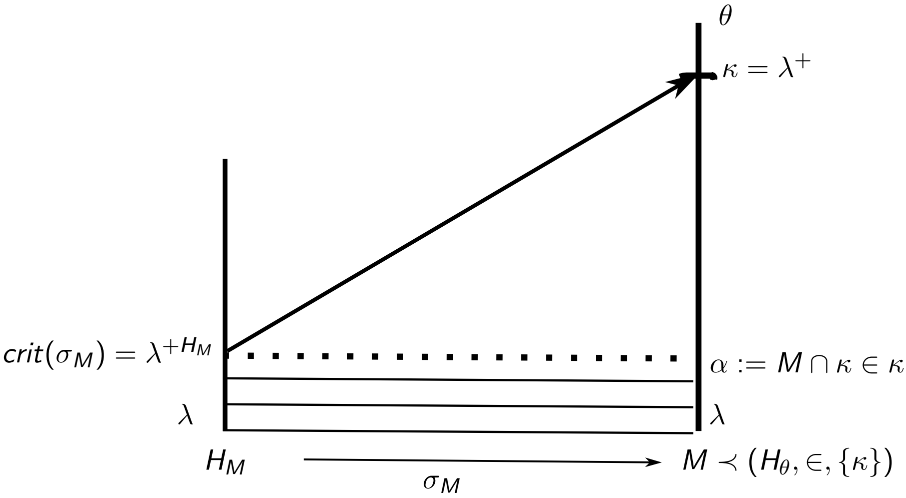
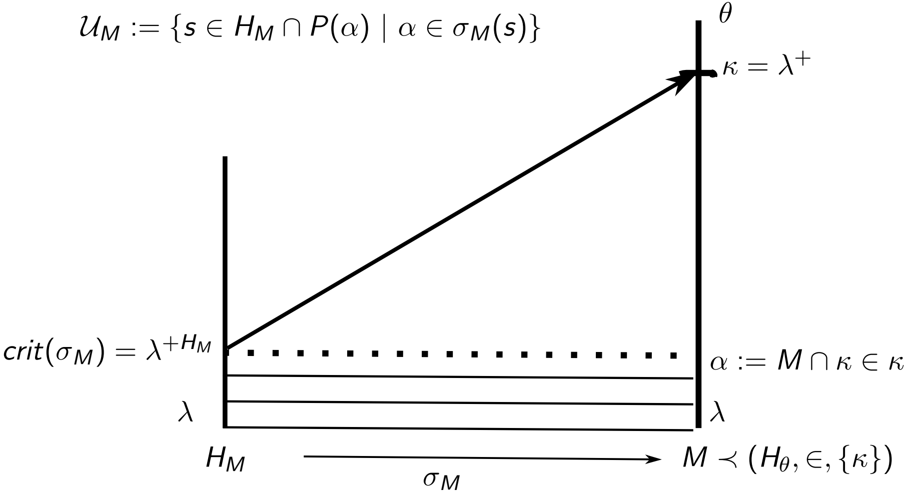


Derived ultrapowers

$$\mathcal{U}_M := \{s \in H_M \cap P(\alpha) \mid \alpha \in \sigma_M(s)\}$$



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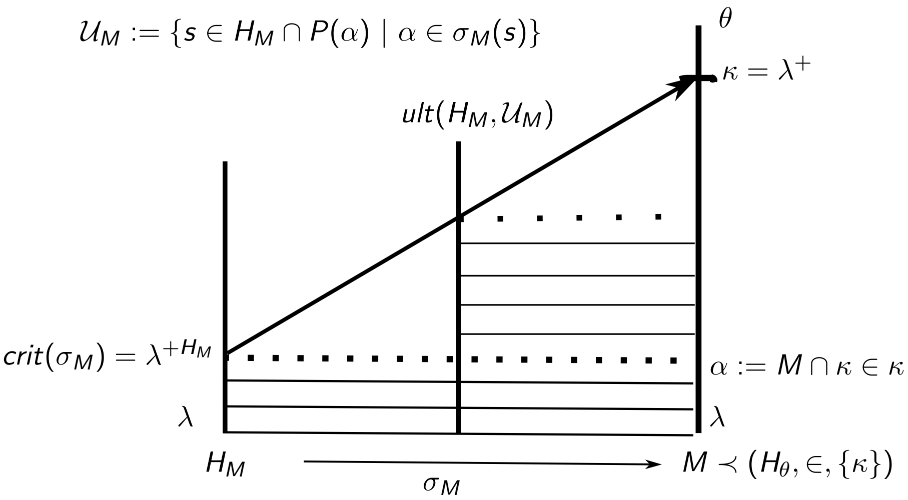

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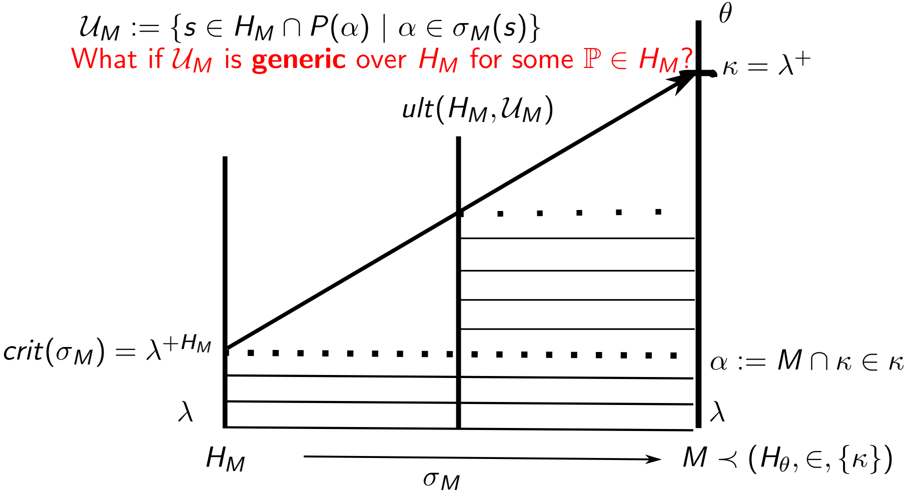

Self-generic structures

Suppose:

- $\mathcal{I}$ is normal ideal on a successor cardinal κ.
- $M \prec (H_\theta, \in, \{\mathcal{I}\}, ...)$ with $M \cap \kappa \in \kappa$
- $\sigma_M : H_M \to H_\theta$ and $\mathcal{U}_M$ are as on the previous slide
- $\mathbb{P} := (\wp(\kappa)/\mathcal{I}, \subseteq_{\mathcal{I}})$ and $\mathbb{P}_M := \sigma_M^{-1}(\mathbb{P})$.

Definition

M is called self-generic for $\mathcal{I}$ iff $\mathcal{U}_M$ is $\mathbb{P}_M$-generic over H_M.

Relation to saturation and precipitousness

$$S_{\mathcal{I}}^{SelfGen} := \{M \prec H_{(2^\kappa)^+} \mid M \text{ is self-generic for } \mathcal{I}\}$$

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$S_{\mathcal{I}}^{SelfGen}$ is stationary $\Longrightarrow$ $\mathcal{I}$ has a precipitous restriction

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$\mathcal{I}$-projective stationarity

A set $R \subset \wp_\kappa(H_\theta)$ is $\mathcal{I}$-**projective stationary** iff for every $S \in \mathcal{I}^+$:

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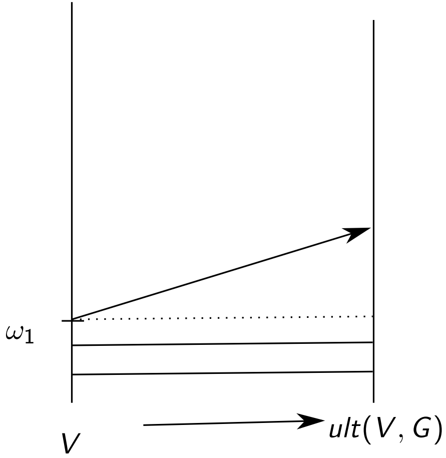
Special case of Ralf's observation:

Theorem (Schindler)

NS_{ω_1} is precipitous $\iff$ $S^{SelfGen}_{NS_{\omega_1}}$ is projective stationary.

(in original Feng-Jech sense of "projective stationary")

For $\mathcal{I}$ on ω_1, precipitousness implies $S_{\mathcal{I}}^{SelfGen}$ is large



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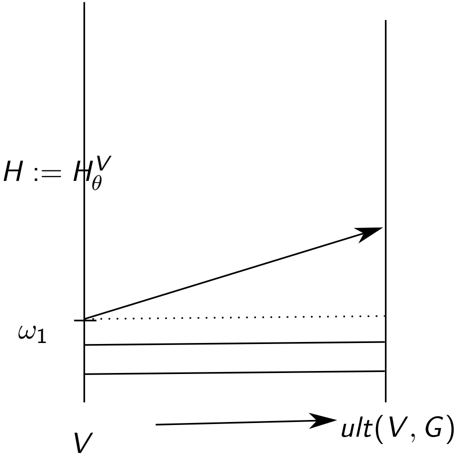

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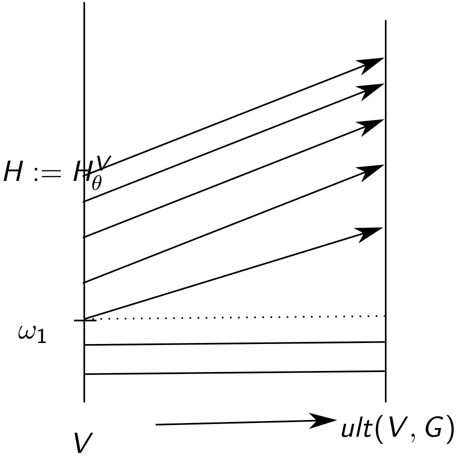

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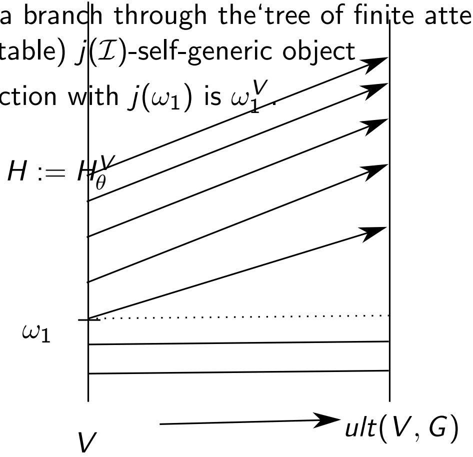

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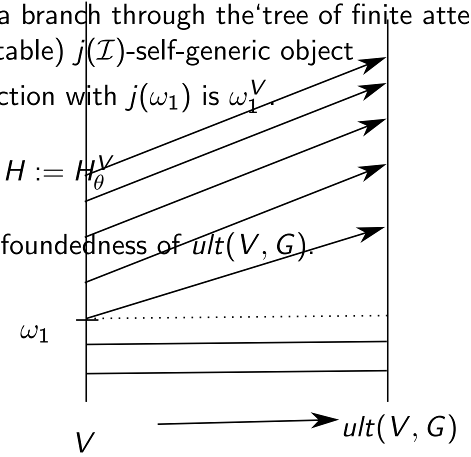
Then use wellfoundedness of $ult(V, G)$.



StatCatch, *ProjectiveCatch*, and *ClubCatch*

Definition

- *StatCatch*$(\mathcal{I})$ holds iff $S_{\mathcal{I}}^{\mathit{SelfGen}}$ is stationary
- *ProjectiveCatch*$(\mathcal{I})$ holds iff $S_{\mathcal{I}}^{\mathit{SelfGen}}$ is $\mathcal{I}$-projective stationary
- *ClubCatch*$(\mathcal{I})$ holds iff $S_{\mathcal{I}}^{\mathit{SelfGen}}$ contains a club (relative to "conditional club filter of $\mathcal{I}$")

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Unlike for ideals on ω_1, $StatCatch(\mathcal{I})$ is MUCH higher in consistency strength that precipitousness.

Note: $ProjectiveCatch(\mathcal{I})$ does **NOT** imply that generic ultrapowers by $\mathcal{I}$ have strong closure properties.

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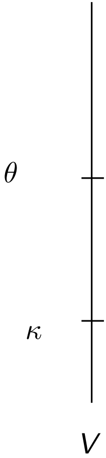

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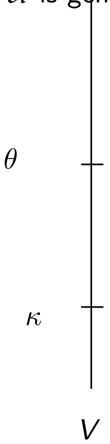
e.g. $\kappa = \omega_1$ and $\mathcal{U}$ is generic for $\wp(P_{\omega_1}(H_\theta))/NS$



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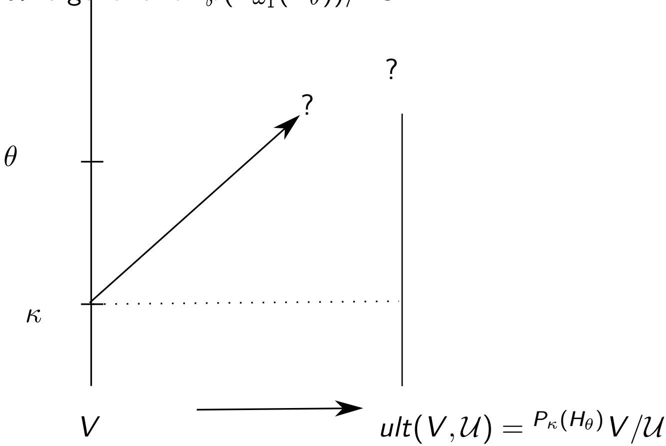

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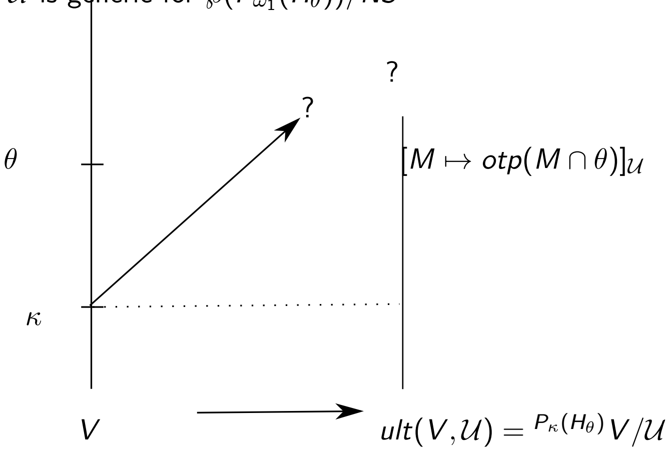

ProjectiveCatch$(\mathcal{I})$ implies weak version of *Bound*$(\mathcal{I})$

Observation (C.)

Let $\theta = (2^\kappa)^+$. *StatCatch*$(\mathcal{I})$ *implies: for every* $f : \kappa \to \kappa$ *there are stationarily many* $M \in \wp_\kappa(H_\theta)$ *such that:*

- $otp(M \cap \theta) > f(M \cap \kappa)$

Ideals that bound their completeness

Definition (C.)

Suppose $\mathcal{J}$ is a normal ideal over $\wp_\kappa(H_\theta)$ with completeness κ. We say $\mathcal{J}$ **bounds its completeness** iff for every $f : \kappa \to \kappa$:

$$S_f := \{M \in \wp_\kappa(H_{(2^\kappa)^+}) \mid otp(M) > f(M \cap \kappa)\}$$

is in the dual of $\mathcal{J}$.

Lemma (C.)

- *It is consistent for κ to be supercompact, yet* **no** *normal measures on any $\wp_\kappa(H_\theta)$ bound their completeness*
- *If κ is almost huge, many normal measures that bound completeness.*
- *If $\mathcal{T}$ is a presaturated tower of ideals with critical point κ, then a tail end of the ideals in the tower bound their completeness.*

ProjectiveCatch and bounding

Recall from earlier:

Theorem (C.-Zeman)

ProjectiveCatch($\mathcal{I}$) (for $\mathcal{I}$ on ω_2) gives inner model with Woodin cardinal

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Suppose $\mathcal{I}$ is a normal ideal on κ and ProjectiveCatch($\mathcal{I}$) holds. Set $\mathcal{J} := NS \restriction S^{SelfGen}_{\mathcal{I}}$. Then $\mathcal{J}$ bounds its completeness (which is κ).

Conjecture

The consistency strength of "there is an ideal concentrating on IU_{ω_1} which bounds its completeness, where the completeness is ω_2" is strictly between a supercompact and almost huge cardinal.

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Note: *Bound*($\mathcal{I}$) implies existence of a $\mathcal{J}$ which bounds its completeness.

Outline

1. The partial order $({}^{\kappa}ORD, \leq_{\mathcal{I}})$ and canonical functions
2. Self-generic structures ("antichain catching")
3. How antichain catching is related to bounding by canonical functions
4. Forcing Axioms vs. nice ideals on ω_2

Conflict between forcing axioms and nice ideals on ω_2

MA: Martin's Axiom (MA_{ω_1})
PFA: Proper Forcing Axiom
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PFA $\implies$ *there is* **no** *presaturated ideal on* ω_2

PFA $\implies$ *failure of* $(\omega_3, \omega_2) \twoheadrightarrow (\omega_2, \omega_1)$

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Theorem (C.)

MM is consistent with weakened versions (e.g. $(\theta, \omega_2) \twoheadrightarrow (\omega_2, \omega_1)$*; instances of ProjectiveCatch for ideals with completeness* ω_2*)*

Some related results

Theorem (C.-Viale)

$WRP([\omega_2]^\omega) \implies$ there is **no** *ideal which bounds its completeness and concentrates on the class GIC_{ω_1} (ω_1-guessing, internally club sets).*

$sat(NS_{\omega_1})$ + $TP(\omega_2)$ yields stronger result (with GIS_{ω_1} in place of GIC_{ω_1}).

(WRP and SRP follow from PFA^+ and MM, respectively)

Corollary

PFA^+ (resp. MM) implies there is **no** *presaturated tower that concentrates on GIC_{ω_1} (resp. GIS_{ω_1}).*

Bounding completeness and trees

Define a partial order on $\wp_\kappa(H_\theta)$ by:

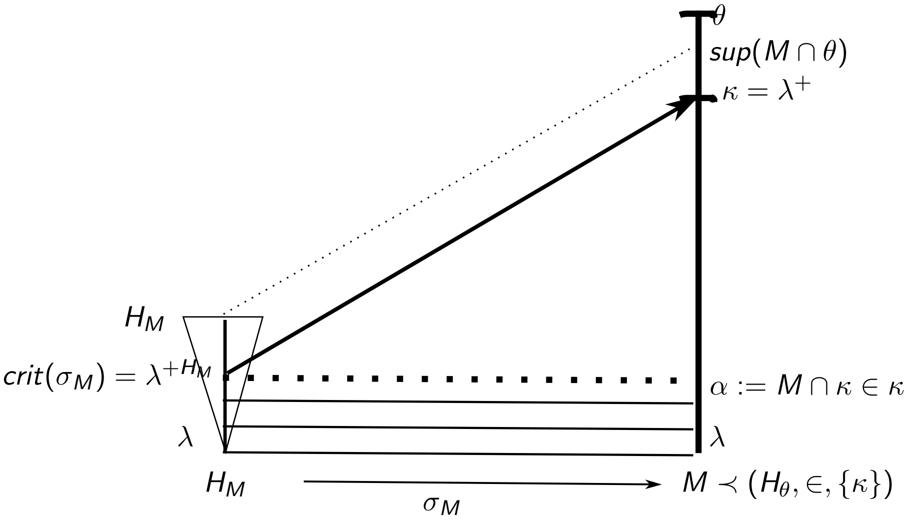
$$M \leq_r M' \iff \exists \beta < \theta \ \ M = M' \cap V_\beta$$

For each $\alpha < \kappa$ set:

$$T_\alpha^{\wp_\kappa(H_\theta)} := \{M \in \wp_\kappa(H_\theta) \mid M \cap \kappa = \alpha\}$$

$(T_\alpha^{\wp_\kappa(H_\theta)}, \leq_r)$ is a tree of height $\leq \kappa$.

Tree of models at α



Bounding completeness and trees

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height$(T_\alpha^{\wp_\kappa(H_\theta)}) \leq \kappa$

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Lemma

Suppose $\mathcal{J}$ is a normal ideal on $\wp_\kappa(H_\theta)$ with completeness κ. Let $\mathcal{I}$ be the projection of $\mathcal{J}$ to a normal ideal on κ.

If $\mathcal{J}$ bounds its completeness, then

$$\textit{height}(T_\alpha^{\wp_\kappa(H_\theta)}) = \kappa$$

for $\mathcal{I}$-measure one many $\alpha < \kappa$.

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Resembles "Strong Chang's Conjecture".

Useful result of Gitik

Theorem (Gitik)

For any club $D \subset [\omega_2]^\omega$ and any $x \in \mathbb{R}$, there are $a, b, c \in D$ such that $x \in L_{\omega_2}[a, b, c]$.

Corollary

If W is a transitive ZF^- model of height ω_2 and $\mathbb{R} - W \neq \emptyset$, then $[\omega_2]^\omega - W$ is stationary.

Velickovic strengthened Gitik's Theorem in a way that shows: $[\omega_2]^\omega - W$ is in fact **projective** *stationary.*

Some corollaries

Corollary

*$WRP([\omega_2]^\omega)$ (**resp.** $SRP([\omega_2]^\omega)$) $\implies$ if W is a transitive ZF^- model of height ω_2 and every proper initial segment of W is internally club (**resp. internally stationary**), then $\mathbb{R} \subset W$.*

Yet another corollary of Gitik's Theorem

Observation (C.)

Neeman's and Friedman's recent models of PFA are **not** *models of* $WRP([\omega_2]^\omega)$*; in particular, they're not models of* PFA^+*.*

Fundamentally different from Baumgartner's classic model of PFA:

If

- κ is supercompact
- $\mathbb{P}$ is any countable support iteration of proper posets which has the κ-cc

Then $V^{\mathbb{P}} \models WRP([\kappa]^\omega)$

Ongoing work and questions

Recall that $Bound(NS_{\omega_1})$ is well-understood.

1. Is $Bound(NS_{\omega_2})$ consistent?
2. Find better lower bounds for consistency strength
   - even need to escape "easy" inner model theory
   - I suspect that our proof that obtains a Woodin cardinal from $StatCatch(\mathcal{I})$ will help
3. Can $ProjectiveCatch(NS \restriction S^2_1)$ hold? Can $NS \restriction S^2_1$ be saturated?
4. Exactly how much can Forcing Axioms tolerate nice ideals/towers on ω_2?
   - Some partial results with Viale, Weiss (using ideas from Neeman's PFA forcing)

Note:
Bound(NS_{ω_2}) *together* with precipitousness of NS_{ω_2} has very high consistency strength.