Contributions towards a fine structure theory of Aronszajn orderings

Carlos Martinez-Ranero

Centro de Ciencias Matematicas

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1. The sense of strength of such a classification result comes from the fact that whenever $(\mathcal{K}, \preceq)$ is well quasi-ordered then the complete invariants of the equivalence relation are only slightly more complicated than the ordinals.

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Theorem (Laver 1971)

The class of σ-scattered linear orders is well quasi-ordered by embeddability.

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(PFA) The uncountable linear orderings have a five element basis consisting of X, ω_1, ω_1^*, C, and C^* whenever X is a set of reals of cardinality $\aleph_1$ and C is a Countryman line.

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Theorem (Moore 2008)

(PFA) Exists a universal Aronszajn line, denoted by η_C. Moreover, η_C can be described as the subset of the lexicographical power $(\zeta_C)^\omega$ consisting of those elements which are eventually zero where ζ_C is the direct sum $C^* \oplus 1 \oplus C$.

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Let $\mathcal{A}_0$ denote the class of Countryman lines. For each $\alpha < \omega_2$, let $\mathcal{A}_\alpha$ denote the class of all elements of the form

$$\sum_{x \in I} A_x$$

such that $I \preceq C$ or $I \preceq C^*$ and $\forall x \in I \;\; A_x \in \mathcal{A}_\xi$ for some $\xi < \alpha$.

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Definition

Exists a natural rank associated to each fragmented A-line, given by $rank(A) = \min\{\alpha : A \in \mathcal{A}_\alpha\}$.

Sketch of the proof

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(i) C and C^* play the role of ω and ω^*,

(ii) η_C plays the role of the rationals

(iii) and being fragmented is analogous to being scattered in this context.

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Lemma (Main Lemma)

(MA_{ω_1}) For each $\alpha < \omega_2$, there exists two incomparable Aronszajn lines D_α^+, and D_α^- of rank α such that:

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2. $D^-_\alpha \preceq C^* \times D^+_\alpha$, $D^+_\alpha \preceq C \times D^-_\alpha$ and
3. For each $A \in \mathcal{A}_\alpha$ the following holds $A \equiv D^+_\alpha$ or $A \equiv D^-_\alpha$ or both $A \preceq D^+_\alpha$ and $A \preceq D^-_\alpha$.

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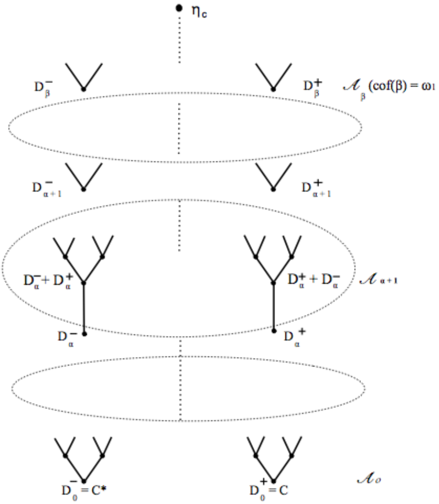

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The class $\mathcal{A}$ contains infinite strictly decreasing sequences as well as uncountable antichains. Thus, it fails to be well quasi-ordered.

This implies that the class $\mathcal{A}$ is too big to have a meaningful classification theorem.

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Definition

A tree T is *coherent* if it can be represented as a downward closed subtree of $\omega^{<\omega_1}$ with the property that for any two nodes $t, s \in T$ $\{\xi \in dom(t) \cap dom(s) : t(\xi) \neq s(\xi)\}$ is finite.

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Theorem (Todorcevic 2006)

(MA_{ω_1}) The class $\mathcal{C}$ of coherent Aronszajn trees is cofinal and coinitial in $(\mathcal{A}, \preceq)$ and $\mathcal{C}$ is totally ordered.

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(iii) Moreover, assuming PFA, any coherent Aronszajn tree T is comparable with any Aronszajn tree.

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We say that two Aronszajn trees T and S are equivalent $T \sim S$ if either T is the n-th successor of S or S is the n-th successor of T for some positive integer n.

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Theorem (M-R, Todorcevic)

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Corollary (M-R, Todorcevic)

(PFA) The class of Aronszajn trees is universal for linear orders of cardinality at most ω_2.

Sketch of the proof

Theorem

(MA_{ω_1}) Every coherent tree T is irreducible, i.e, $T \preceq U$ for every downward closed subtree $U \subseteq T$.