Gale-Stewart games and Blackwell games

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Perfect & imperfect information for games

Perfect information: Players know about the previous moves of opponents.

E.g., Gale-Stewart games.

Imperfect information: Players do not know about what opponents did previously.

E.g., Blackwell games.

From now on...

Work in ZF+$\mathrm{DC}_{\mathbb{R}}$.

For a set X,

DC_X: For any relation $R \subseteq X \times X$ such that

$(\forall x \in X)\ (\exists y \in X)\ (x, y) \in R$, there is a function $f : \omega \to X$

such that $(f(n), f(n+1)) \in R$ for any natural number n.

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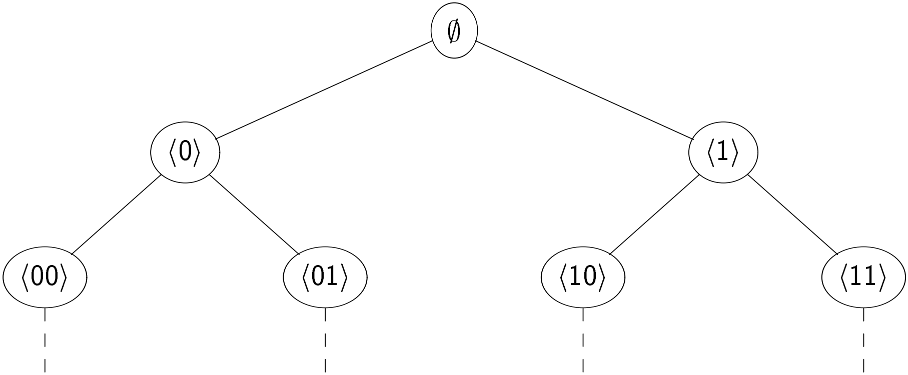
DC: DC_X holds for any set X.

Gale-Stewart games

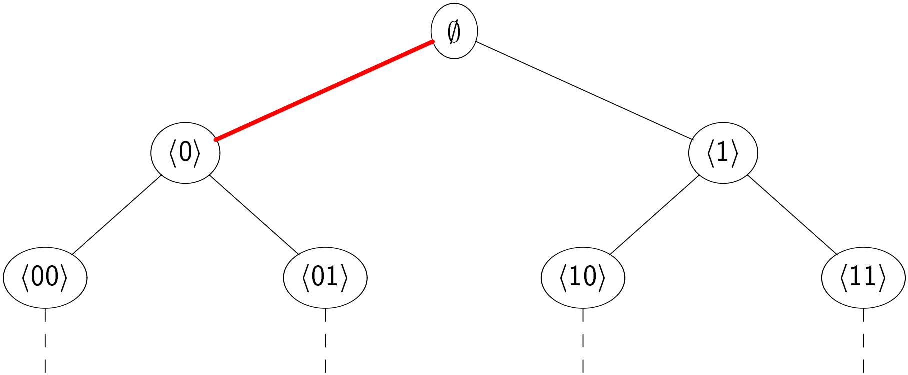
Fix a payoff set $A \subseteq 2^\omega$.

I's turn.



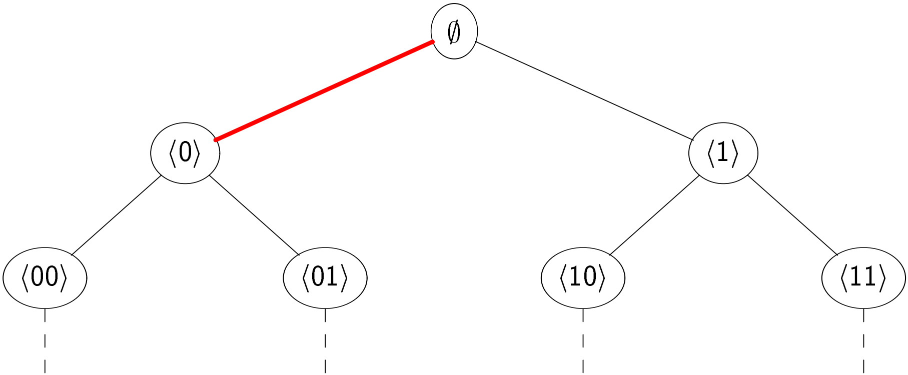
Gale-Stewart games ctd.

I has played.



Gale-Stewart games ctd..

II's turn.



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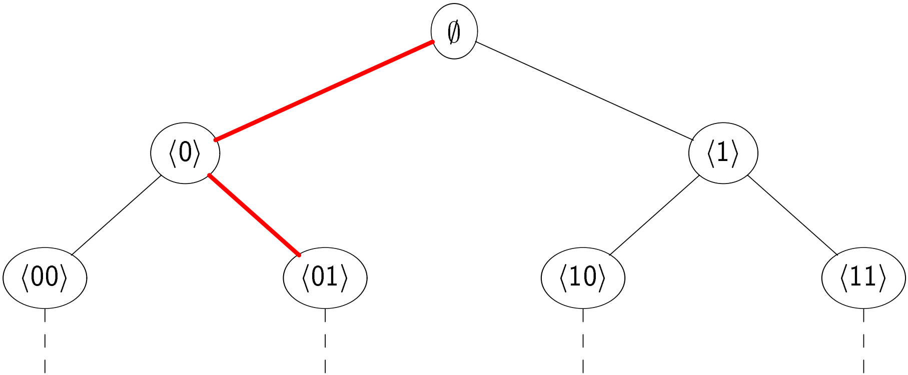

Gale-Stewart games ctd....

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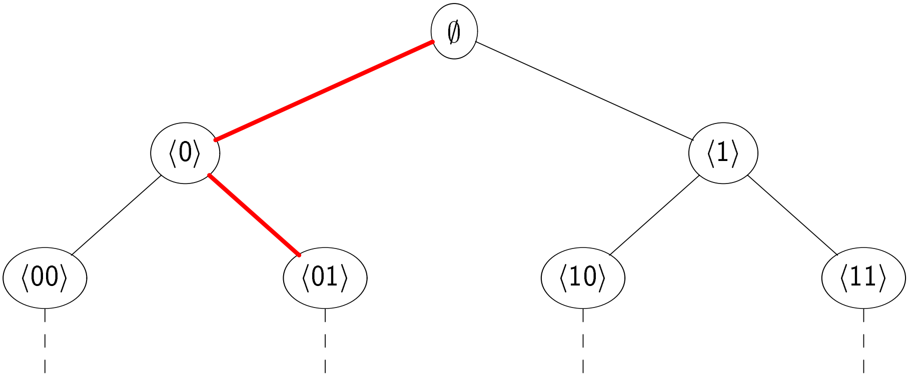
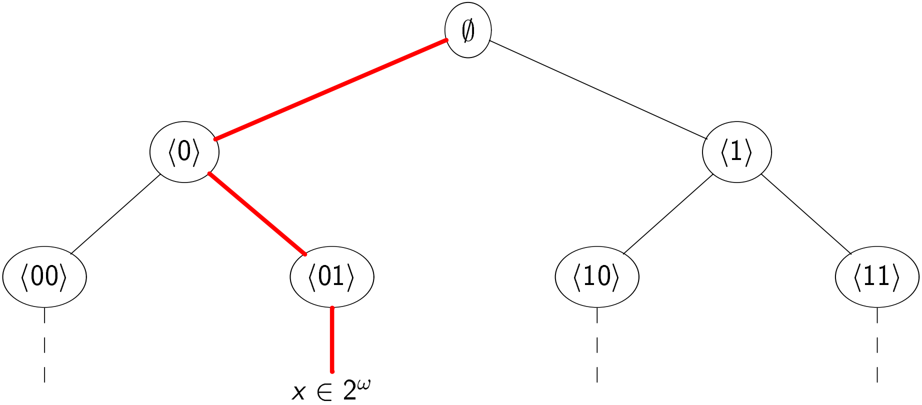


Gale-Stewart games ctd.....

After infinitely many times. . .



Player I wins if x is in the payoff set A and otherwise Player II wins.

The Axiom of Determinacy

A subset A of 2^ω is *determined* if one of the players has a winning strategy in the Gale-Stewart game with the payoff set A.

Definition (Mycielski-Steinhaus)

The Axiom of Determinacy (AD) asserts the following:
Every subset A of 2^ω is determined.

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Definition (Mycielski-Steinhaus)

The Axiom of Determinacy (AD) asserts the following:
Every subset A of 2^ω is determined.

Remark

1. AD contradicts the Axiom of Choice (AC).
2. AD has many beautiful consequences, e.g., every set of reals is Lebesgue measurable.
3. Models of AD (or AD^+) are closely connected to models with Woodin cardinals.

Extensions of AD

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Remark

AD_X is inconsistent if there is an injection from ω_1 to X.

Out interest: AD and $\mathrm{AD}_{\mathbb{R}}$.

Blackwell games

Fix a payoff set $A \subseteq 2^\omega$.

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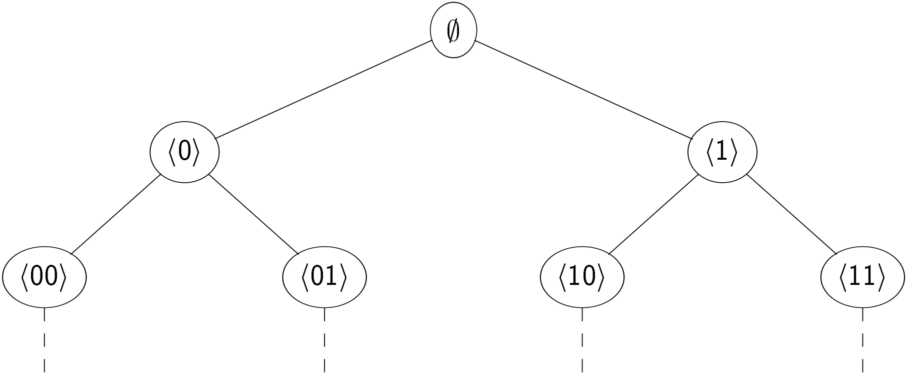
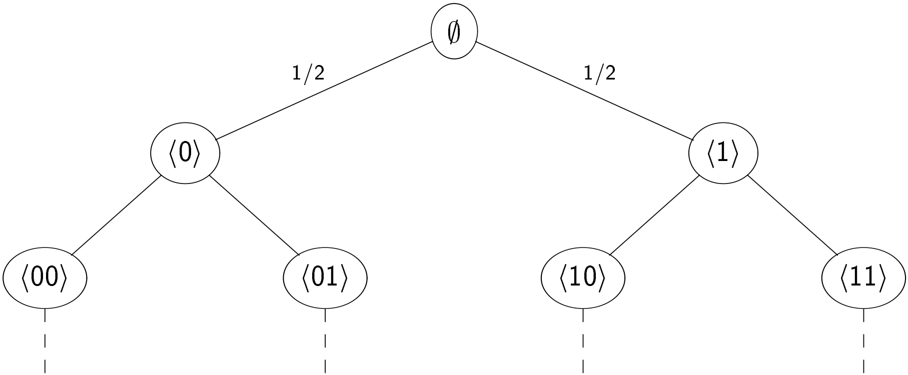


Blackwell games ctd.

I has played.



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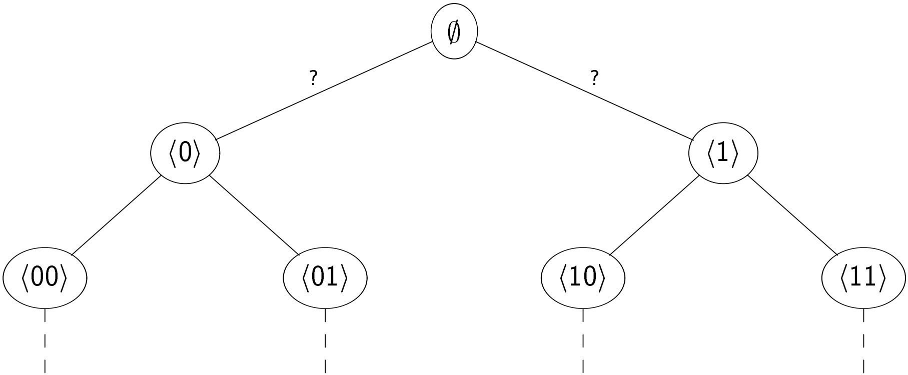

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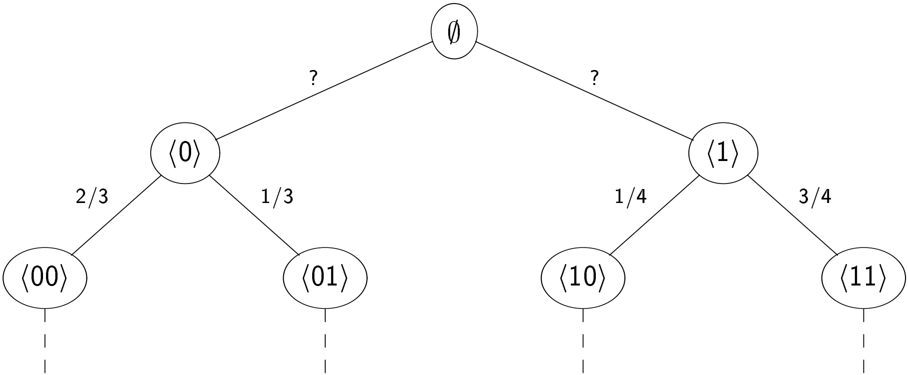

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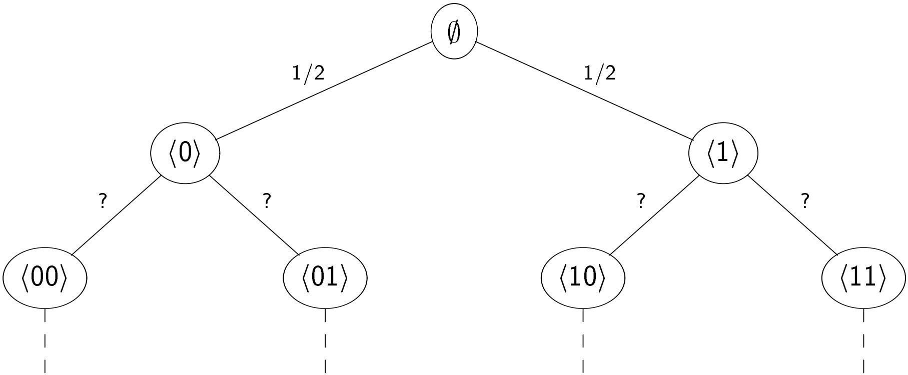

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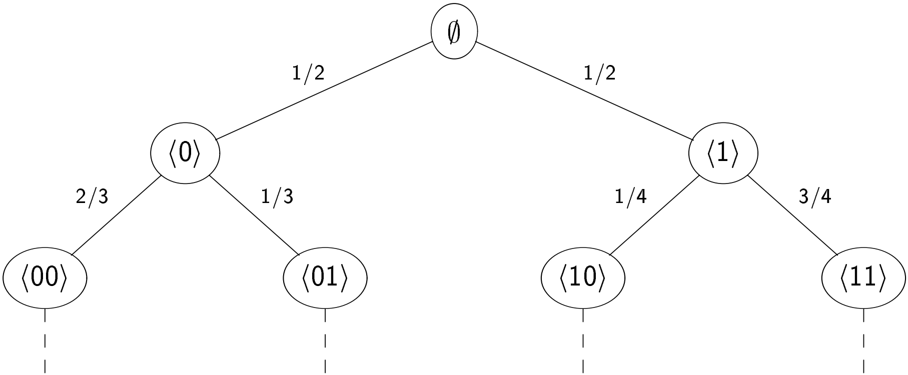
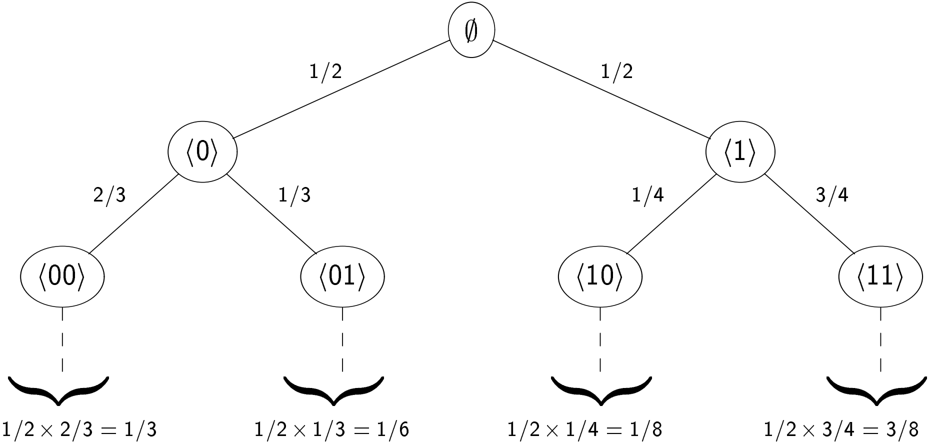


Blackwell games ctd......

Calculate the probability as below.



Blackwell games ctd.......

Player I wins if the probability of the payoff set is 1.
Player II wins if the probability of the payoff set is 0.

Formal definitions; Blackwell games

- σ is a *mixed strategy for I* if $\sigma : 2^{\mathsf{Even}} \to \mathrm{Prob}(2)$.
- τ is a *mixed strategy for II* if $\tau : 2^{\mathsf{Odd}} \to \mathrm{Prob}(2)$.

Formal definitions; Blackwell games

- σ is a *mixed strategy for I* if $\sigma\colon 2^{\text{Even}} \to \text{Prob}(2)$.
- τ is a *mixed strategy for II* if $\tau\colon 2^{\text{Odd}} \to \text{Prob}(2)$.
- For a mixed strategy σ for I and a mixed strategy τ for II, define $\sigma * \tau\colon 2^{<\omega} \to \text{Prob}(2)$ as follows:

$$\sigma * \tau(s) = \begin{cases} \sigma(s) & \text{if } \text{lh}(s) \text{ is even,} \\ \tau(s) & \text{if } \text{lh}(s) \text{ is odd.} \end{cases}$$

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Then define $\mu_{\sigma,\tau}\colon 2^{<\omega} \to [0,1]$ as follows:

$$\mu_{\sigma,\tau}(s) = \prod_{i<\mathrm{lh}(s)} \sigma * \tau(s \upharpoonright i)(s(i)).$$

With the help of $\mathrm{DC}_{\mathbb{R}}$, one can uniquely extend $\mu_{\sigma,\tau}$ to a Borel probability measure on the Cantor space.

Formal definitions; Blackwell games ctd.

Let $A \subseteq 2^\omega$.

- A mixed strategy σ for I is *optimal in A* if for any mixed strategy τ for II, $\mu_{\sigma,\tau}(A) = 1$.
- A mixed strategy τ for II is *optimal in A* if for any mixed strategy σ for I, $\mu_{\sigma,\tau}(A) = 0$.

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- A is *Blackwell determined* if either I or II has an optimal strategy in A.
- Bl-AD: Every $A \subseteq 2^\omega$ is Blackwell determined.

Note: There is another formulation of Blackwell games coming from game theory.

Formal definitions; Blackwell games ctd..

Let X be a non-empty set.

- σ is a *mixed strategy for I* if $\sigma\colon X^{\mathrm{Even}} \to \mathrm{Prob}_\omega(X)$.
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If we have $\mathrm{DC}_{(\mathbb{R}\times X^\omega)}$, we can uniquely extend $\mu_{\sigma,\tau}$ to a Borel probability measure on X^ω.

Note: $\mathrm{DC}_{(\mathbb{R}\times\mathbb{R}^\omega)}$ follows from $\mathrm{DC}_{\mathbb{R}}$

Formal definitions; Blackwell games ctd...

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- $\mathsf{Bl\text{-}AD}_X$: every $A \subseteq X^\omega$ is Blackwell determined.

Formal definitions; Blackwell games ctd...

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- A is *Blackwell determined* if either I or II has an optimal strategy in A.
- Bl-AD$_X$: every $A \subseteq X^\omega$ is Blackwell determined.

Remark

Bl-AD$_X$ is inconsistent if there is an injection from ω_1 to X

Our interest: Bl-AD and Bl-AD$_\mathbb{R}$.

Observation 1

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Let X be a nonempty set and A be a subset of X^ω. If A is determined, then A is Blackwell determined.

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Corollary (Martin)

AD implies Bl-AD. $\mathrm{AD}_{\mathbb{R}}$ implies $\text{Bl-AD}_{\mathbb{R}}$.

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Conjecture (Martin)

Bl-AD implies AD.

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If a finite game is Blackwell determined, then it is determined when X is totally ordered.

Finite games = games ending at some fixed round $n < \omega$.

Sketch of proof.

In blackboards. □

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Corollary (Löwe)

Assume Bl-$AD_{\mathbb{R}}$. Then Uniformization holds, i.e., every relation on the reals can be uniformized by a function.

Observation 2 ctd.

By the same argument. . .

Proposition

If a clopen set is Blackwell determined, then it is determined.

Observation 2 ctd.

By the same argument...

Proposition

If a clopen set is Blackwell determined, then it is determined.

Theorem (Neeman)

Assume Bl-AD. Then every Suslin & co-Suslin subset of the Cantor space is determined.

Definition

1. A subset A of the Cantor space is *Suslin* if there is an ordinal γ and a tree T on $2 \times \gamma$ such that $A = \mathrm{p}[T]$.
2. A subset A of the Cantor space is *co-Suslin* if the complement of A is Suslin.

Observation 2 ctd..

Theorem (Kechris & Woodin)

If every Suslin & co-Suslin set is determined, then $AD^{L(\mathbb{R})}$ holds.

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If every Suslin & co-Suslin set is determined, then $\mathrm{AD}^{\mathrm{L}(\mathbb{R})}$ holds.

Corollary (Martin, Neeman & Vervoort)

$\mathrm{L}(\mathbb{R}) \vDash$ "AD $\iff$ Bl-AD". In particular, AD and Bl-AD are equiconsistent.

Observation 3

Observation

Assume BI-$AD_{\mathbb{R}}$. Let $A \subseteq \mathbb{R}^\omega$. If A is range-invariant, then A is determined.

Definition

A set $A \subseteq \mathbb{R}^\omega$ is *range-invariant* if for any $\vec{x}, \vec{y} \in \mathbb{R}^\omega$ with the same range, $\vec{x} \in A \iff \vec{y} \in A$.

Observation 3 ctd.

Theorem (de Kloet, Löwe, I.)

Assume BI-$AD_{\mathbb{R}}$. Then there is a fine, normal, σ-complete ultrafilter on $\mathcal{P}_{\omega_1}(\mathbb{R})$.

Definition

Let U be a filter on $\mathcal{P}_{\omega_1}(\mathbb{R})$.

1. U is *fine* if for any $x \in \mathbb{R}$, $\{a \in \mathcal{P}_{\omega_1}(\mathbb{R}) \mid x \in a\} \in U$.
2. U is *normal* if for any family $\{A_x \in U \mid x \in \mathbb{R}\}$, $\triangle_{x \in \mathbb{R}} A_x = \{a \in \mathcal{P}_{\omega_1}(\mathbb{R}) \mid (\forall x \in a)\ a \in A_x\} \in U$.

Observation 3 ctd..

Theorem (Solovay)

If there is a fine, normal σ-complete ultrafilter on $\mathcal{P}_{\omega_1}(\mathbb{R})$, then $\mathbb{R}^\#$ exists.

Points:

1. By assumption, ω_1 is measurable and hence $a^\#$ exists for all $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$.
2. Letting U be a fine normal measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$,

$$\phi \in \mathbb{R}^\# \iff \{a \in \mathcal{P}_{\omega_1}(\mathbb{R}) \mid \phi \in a^\#\} \in U.$$

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Corollary (de Kloet, Löwe & I.)

Assume Bl-AD$_{\mathbb{R}}$. Then $\mathbb{R}^\#$ exists. Hence Bl-AD$_{\mathbb{R}} \vdash \mathrm{Con}(\mathrm{AD})$.

$AD_{\mathbb{R}}$ vs BI-$AD_{\mathbb{R}}$

Question

Does BI-$AD_{\mathbb{R}}$ imply $AD_{\mathbb{R}}$?

$\mathsf{AD}_{\mathbb{R}}$ vs $\mathsf{Bl}\text{-}\mathsf{AD}_{\mathbb{R}}$

Question

Does $\mathsf{Bl}\text{-}\mathsf{AD}_{\mathbb{R}}$ imply $\mathsf{AD}_{\mathbb{R}}$?

Theorem (Woodin & I.)

Under ZF+DC, $\mathsf{AD}_{\mathbb{R}}$ and $\mathsf{Bl}\text{-}\mathsf{AD}_{\mathbb{R}}$ are equivalent.

$\mathsf{AD}_{\mathbb{R}}$ vs BI-$\mathsf{AD}_{\mathbb{R}}$

Question

Does BI-$\mathsf{AD}_{\mathbb{R}}$ imply $\mathsf{AD}_{\mathbb{R}}$?

Theorem (Woodin & I.)

Under ZF+DC, $\mathsf{AD}_{\mathbb{R}}$ and BI-$\mathsf{AD}_{\mathbb{R}}$ are equivalent.

But!

Remark (Solovay)

$\mathsf{AD}_{\mathbb{R}}$+DC implies the consistency of $\mathsf{AD}_{\mathbb{R}}$.

So assuming DC is not optimal for the above theorem.

$\mathsf{AD}_{\mathbb{R}}$ vs. $\mathsf{BI\text{-}AD}_{\mathbb{R}}$ ctd.

Question

Are $\mathsf{AD}_{\mathbb{R}}$ are $\mathsf{BI\text{-}AD}_{\mathbb{R}}$ equiconsistent?

$AD_{\mathbb{R}}$ vs. $BI\text{-}AD_{\mathbb{R}}$ ctd.

Question

Are $AD_{\mathbb{R}}$ are $BI\text{-}AD_{\mathbb{R}}$ equiconsistent?

Conjecture (Woodin)

$AD_{\mathbb{R}}$ and $BI\text{-}AD_{\mathbb{R}}$ are equiconsistent.

BI-$AD_{\mathbb{R}}$ and generic embeddings

Theorem (Sargsyan)

Assume CH and that there is a generic embedding $j\colon V \to M$ such that

1. M is transitive and $M^\omega \cap V[G] \subseteq M$,
2. G is a generic filter of a homogeneous forcing, and
3. $j \restriction \mathrm{Ord}$ is definable in V.

Then there is a model of ZF+$AD_{\mathbb{R}}$+"Θ is regular".

The method: Core Model Induction

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The method: Core Model Induction

Theorem

Assume BI-AD$_{\mathbb{R}}$. Then for any $\alpha < \Theta$ and $A \subseteq \mathbb{R}$, there is a generic embedding $j \colon \mathrm{L}(A, \mathbb{R}) \to M$ such that

1. M is transitive, $\mathbb{R}^{V[G]} \subseteq M$, and α is countable in M,
2. G is a generic filter of a homogeneous forcing, and
3. $j \restriction \mathrm{Ord}$ is definable in V.

Gratitude

Thank you very much for your attention!