Comparison and Measures in Inner Models

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Outline

Inner model theory involves construction/analysis of canonical inner models. Simplest is L, Gödel's constructible universe.

- L is well understood, particularly through *fine structure*
- L satisfies GCH
- $\mathbb{R} \cap L$ can be wellordered, in fact there's a Δ^1_2 wellorder
- L is canonical: every proper class model of ZF computes the same L
- But L has no measurable cardinals (Scott)

Motivation: construct/analyze models like L, but containing large cardinals.

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$L[\mathcal{U}]$ is the inner model for one measurable cardinal.

Let $\mathcal{U}$ be a normal measure on κ.
Let $\mathcal{U}' = \mathcal{U} \cap L[\mathcal{U}]$. Then $L[\mathcal{U}] = L[\mathcal{U}']$ and

$$L[\mathcal{U}'] \models \text{“}\mathcal{U}' \text{ is a normal measure on } \kappa \text{ and } V = L[\mathcal{U}'].$$

Definition

Say $(M, \mathcal{V}, \kappa)$ is a κ-*model* iff

- M is transitive proper class, $M \models$ ZFC, and $\mathcal{V}, \kappa \in M$,
- $M \models$“$V = L[\mathcal{V}]$ and $\mathcal{V}$ is a normal measure on κ”.

(Implies that in V, $\mathcal{V}$ is a filter on κ; but if $M \neq V$ it need not be an ultrafilter.)

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Solovay proved that in a κ-model, κ is the unique measurable cardinal. This was improved by:

Theorem (Kunen)

Let $(M, \mathcal{V}, \kappa)$ be a κ-model.
In fact, $M \models$ "$\mathcal{V}$ is the unique normal measure, and all measures are equivalent to finite products of $\mathcal{V}$".

This follows from:

Theorem (Kunen)

Let $(M, \mathcal{V}, \kappa_{\mathcal{V}})$ and $(N, \mathcal{W}, \kappa_{\mathcal{W}})$ be $\kappa_{\mathcal{V}}, \kappa_{\mathcal{W}}$-models.

- *If $\kappa_{\mathcal{V}} = \kappa_{\mathcal{W}}$ then $\mathcal{V} = \mathcal{W}$.*
- *If $\kappa_{\mathcal{V}} < \kappa_{\mathcal{W}}$ then there's an elementary $j : M \to N$ such that $j(\kappa_{\mathcal{V}}) = \kappa_{\mathcal{W}}$ and $j(\mathcal{V}) = \mathcal{W}$. Moreover, $\mathcal{W} \in M$ and j is a class of M.*

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Outline

Proof Sketch (Kunen's second theorem).

We *compare* $(M, \mathcal{V})$ with $(N, \mathcal{W})$ (suppress "$\kappa_{\mathcal{V}}$" and "$\kappa_{\mathcal{W}}$").

Comparison Sketch.

- Form ultrapowers using $\mathcal{V}$, $\mathcal{W}$, and images of them, producing new models, until...
- Until we reach same model $(R, \mathcal{X})$ on either side.

$$M \longrightarrow \mathrm{Ult}(M, \mathcal{V}) \longrightarrow \ldots \longrightarrow R$$
$$\mathcal{V} \qquad\qquad\qquad\qquad \mathcal{X}$$
$$N \longrightarrow \mathrm{Ult}(N, \mathcal{W}) \longrightarrow \ldots \longrightarrow R$$
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Goal.

- Produce some γ-model $(R, \mathcal{X})$ and elementary embeddings

$$i : (M, \mathcal{V}) \to (R, \mathcal{X})$$

such that $\mathrm{crit}(i) = \kappa_{\mathcal{V}}$ (or i is the identity), and likewise

$$j : (N, \mathcal{W}) \to (R, \mathcal{X}).$$

- Therefore $(M, \mathcal{V}) \equiv (N, \mathcal{W})$ and $\mathbb{R}^M = \mathbb{R}^N$.
- Using how the embeddings i, j are defined, one can then prove that one side didn't "move" during comparison: either $(M, \mathcal{V}) = (R, \mathcal{X})$ or $(N, \mathcal{W}) = (R, \mathcal{X})$.

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Comparison Details.

Start with

$$(M_0, \mathcal{V}_0) = (M, \mathcal{V}) \neq (N_0, \mathcal{W}_0) = (N, \mathcal{W}).$$

Let $\kappa_0 = \kappa_{\mathcal{V}}$ and $\mu_0 = \kappa_{\mathcal{W}}$.

We'll define models $(M_\alpha, \mathcal{V}_\alpha)$ and $(N_\alpha, \mathcal{W}_\alpha)$ for ordinals α.

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Case 1: $\kappa_0 < \mu_0$.
Form ultrapower on M side. Do nothing on N side.
Define:

- $M_1 = \mathrm{Ult}(M_0, \mathcal{V}_0)$.
- $i_{0,1} : M_0 \to M_1$ the ultrapower embedding.
- $\mathcal{V}_1 = i_{0,1}(\mathcal{V}_0)$ and $\kappa_1 = i_{0,1}(\kappa_0)$.
- $(N_1, \mathcal{W}_1, \mu_1) = (N_0, \mathcal{W}_0, \mu_0)$.
- $j_{0,1} : N_0 \to N_1$ the identity.

We have defined $(M_1, \mathcal{V}_1)$ and $(N_1, \mathcal{W}_1)$.

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Case 2: $\kappa_0 > \mu_0$. Symmetric to Case 1.

Case 3: $\kappa_0 = \mu_0$. Take an ultrapower on both sides, M_1 and N_1 are the resulting ultrapowers, $i_{0,1}, \mathcal{V}_1, j_{0,1}, \mathcal{W}_1$ defined as before.

Note M_1 is wellfounded since

$$M_0 \models \text{“}\mathcal{U}_0 \text{ is a normal measure and } M_1 = \mathrm{Ult}(V, \mathcal{U}_0).\text{”}.$$

Moreover,

$$M_0 \xrightarrow{i_{0,1}} M_1$$

is elementary and $i_{0,1}(\mathcal{V}_0) = \mathcal{V}_1$, so

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So $(M_1, \mathcal{V}_1)$ is a κ_1-model. Likewise $(N_1, \mathcal{W}_1)$ a μ_1-model.

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Now if $(M_1, \mathcal{V}_1) = (N_1, \mathcal{W}_1)$, we stop. Otherwise, proceed to:

2nd stage. Repeat 1st stage, working with $(M_1, \mathcal{V}_1)$ and $(N_1, \mathcal{W}_1)$. This produces $(M_2, \mathcal{V}_2)$ and $i_{1,2} : M_1 \to M_2$, and likewise $(N_2, \mathcal{W}_2)$ and $j_{1,2}$.

All successor stages are likewise; note we keep producing wellfounded models.

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Suppose we reach $n < \omega$ such that $(M_n, \mathcal{V}_n) = (N_n, \mathcal{W}_n) = (R, \mathcal{X})$.

Have elementary embeddings $i_{0,1}, i_{1,2}, \ldots, i_{n-1,n}$.
Let $i_{0,n}$ be their composition:

$$M_0 \xrightarrow{i_{0,1}} M_1 \xrightarrow{i_{1,2}} M_2 \longrightarrow \ldots \longrightarrow M_{n-1} \xrightarrow{i_{n-1,n}} M_n = R, \qquad i_{0,n} : M_0 \to M_n$$

Have $i_{0,n}(\mathcal{V}_0) = \mathcal{V}_n = \mathcal{X}$.

Likewise get embedding

$$j_{0,n} : N_0 \to N_n = R,$$

and $j_{0,n}(\mathcal{W}_0) = \mathcal{X}$. So $R, \mathcal{X}, i = i_{0,n}$ and $j = j_{0,n}$ are as required.

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Outline

What if $(M_n, \mathcal{V}_n) \neq (N_n, \mathcal{W}_n)$ for all $n < \omega$?

Must define $(M_\omega, \mathcal{V}_\omega)$ and $(N_\omega, \mathcal{W}_\omega)$ and carry on with comparison.

$$M_0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow \cdots \longrightarrow M_n \longrightarrow \cdots \quad M_\omega$$

Have $i_{0,n}$ for $n < \omega$. Likewise define e.g. $i_{1,3} = i_{2,3} \circ i_{1,2}$:

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Threads: For $m, m' < \omega$, $x \in M_m$ and $x' \in M_{m'}$, we say $(m, x) \approx (m', x')$ iff

$$m \leq m' \ \& \ i_{m,m'}(x) = x',$$

or

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Because the $i_{m,n}$'s commute and are 1-1, this is an equivalence relation. Let $[m, x]$ denote the thread (i.e. equivalence class) of (m, x).

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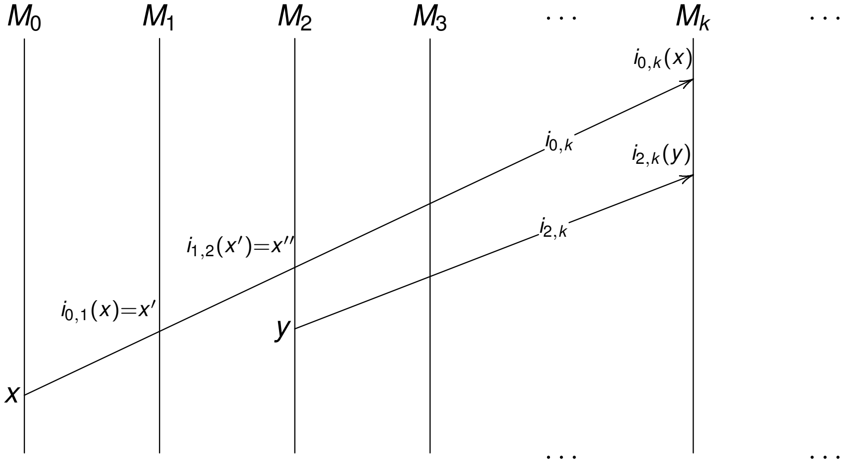

Here $[0, x] = [2, x'']$ but $[0, x] \neq [2, y]$.

Now M_ω consists of all threads:

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Define membership $\in^{M_\omega}$ of M_ω from membership of M_n's.
For $m \leq m'$,

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If so, and N_ω is also, can proceed with comparison. Why wellfoundedness important? Comparison algorithm depended on it to start with, and it's needed for the later parts of the proof (to be omitted).

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ω^{th} *stage:* define M_ω as the (wellfounded) direct limit, $i_{0,\omega}$ as direct limit embedding, $\mathcal{V}_\omega = i_{0,\omega}(\mathcal{V}_0)$.
Likewise for N_ω, $\mathcal{W}_\omega$.

$(\omega+1)^{\text{th}}$ stage: continue comparison with models $(M_\omega, \mathcal{V}_\omega)$ versus $(N_\omega, \mathcal{W}_\omega)$.

These methods produce M_α and $i_{\beta,\alpha}$ for all $\beta \leq \alpha \in \mathrm{OR}$, and likewise on N-side. All models produced are wellfounded.

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Outline

Related arguments can be used to show that in $L[\mathcal{U}]$, there is a Δ^1_3 wellorder of $\mathbb{R}$:

Say (for this slide) that $(M, \mathcal{U}, \kappa)$ is a *premouse* iff M is a transitive model of ZF $-$ $\{$Replacement$\}$ plus "$\mathcal{U}$ is a normal measure on κ, $V = L[\mathcal{U}]$, and Replacement for domains $\subseteq V_\kappa$". We can iterate M just like we did for the proper class models in comparison. Say M is a *mouse* iff it is a premouse all of whose iterates M_α are wellfounded.

In $L[\mathcal{U}]$, can wellorder $\mathbb{R}$ by: "$x < y$ iff there's a mouse $(M, \mathcal{U}, \kappa)$ such that $x, y \in \mathbb{R}^M$ and $M \models$"$x <_{L[\mathcal{U}]} y$". This is Σ^1_3, as it's Π^1_2 to assert that M is a mouse.

Outline

Generalizations? To produce models with larger cardinals:

- Build models from *extenders* instead of measures
- Must deal with more complex iterations

An extender:

- Is a set-sized object coding an elementary embedding
- Consists of a collection of measures, which cohere appropriately

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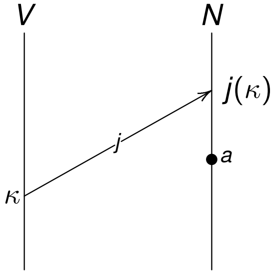
Given $j : V \to N$ elementary with $\kappa = \mathrm{crit}(j)$ and $a \in j(\kappa)^{<\omega}$, have measure E_a over $\kappa^{<\omega}$ defined by:

$$X \in E_a \iff a \in j(X).$$



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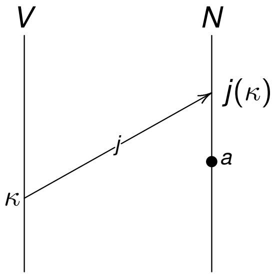

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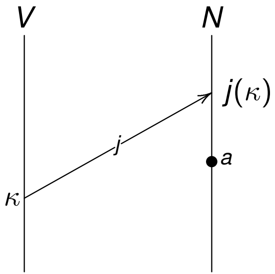

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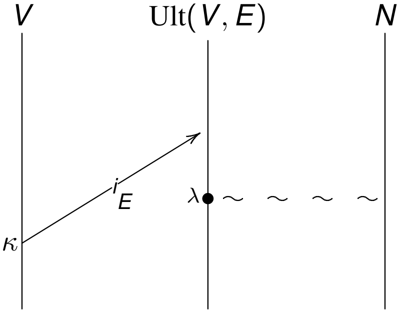

If κ is a strong or Woodin cardinal, then it is so via embeddings from extenders.

To obtain models with strong or Woodin cardinals, we can build from extenders.

Consider models of form $L[\mathbb{E}]$, where, ignoring some details, $\mathbb{E}$ is a sequence of extenders: $\mathbb{E} = \langle \mathbb{E}_\alpha \rangle_{\alpha \in I}$.

The extenders appear on the sequence $\mathbb{E}$ in a canonical order. In fact for the standard (fine-structural) models, some $\mathbb{E}_\gamma$'s are not literally extenders of $L[\mathbb{E}]$ in the sense defined earlier; their component measures can be partial.

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Outline

Iterations for $L[\mathcal{U}]$ were simpler than the general case:

(a) At stage α, we always used $\mathcal{U}_\alpha = i_{0,\alpha}(\mathcal{U})$ for next ultrapower

(b) $\mathcal{U}_\alpha$ was *applied to M_α* to form $M_{\alpha+1} = \mathrm{Ult}(M_\alpha, \mathcal{U}_\alpha)$

Comparing $L[\mathbb{E}]$ versus $L[\mathbb{F}]$, where $\mathbb{E} \neq \mathbb{F}$, we choose the extenders E, F involved in the *least difference* between $\mathbb{E}$ and $\mathbb{F}$, and form ultrapowers using E, F.
I.e., choose $E = \mathbb{E}_\gamma$ and $F = \mathbb{F}_\gamma$, where γ is least such that $\mathbb{E}_\gamma \neq \mathbb{F}_\gamma$.
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More general iterations allow (starting with a base model M_0):

(a) At stage α, may choose any $E_\alpha \in M_\alpha$ such that $M_\alpha \models$"E_α is an extender".

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Why (b)? The comparison proof breaks down if we require E_α to apply to M_α.

This obstacle overcome by (b) and *iteration trees*. This was a key innovation due to Mitchell, Martin and Steel.

Given models M, N such that $M \models$"E is an extender with $\mathrm{crit}(E) = \kappa$", and such that $V^M_{\kappa+1} = V^N_{\kappa+1}$, it makes sense to define $\mathrm{Ult}(N, E)$, even if $E \notin N$.

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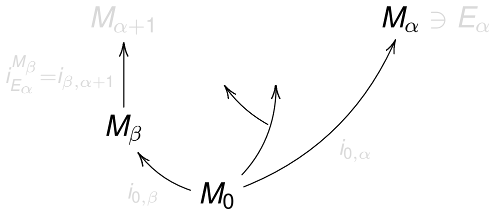
Roughly, an iteration tree $\mathcal{T}$ is a tree on some ordinal λ, with a model M_α attached to each node $\alpha < \lambda$. 0 is the root node.



Let $<_\mathcal{T}$ be the tree order. Whenever $\gamma \leq_\mathcal{T} \delta$ we have an iteration embedding $i_{\gamma,\delta} : M_\gamma \to M_\delta$.

Arrows in diagram represent tree order and embeddings.

Stage α: choose $E_\alpha \in M_\alpha$, form $M_{\alpha+1} = \mathrm{Ult}(M_\beta, E_\alpha)$, some $\beta \leq \alpha$. And $i_{\beta,\alpha+1}$ is the ultrapower embedding.

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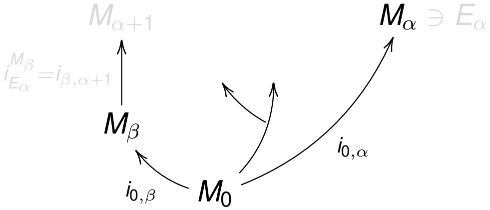

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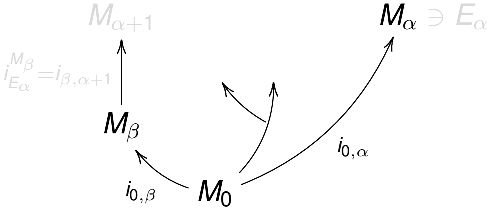

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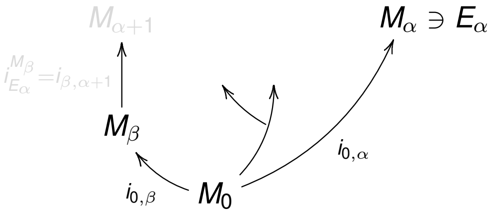

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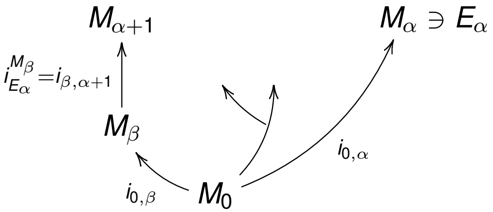

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With iteration trees, limit stages of iteration introduce a very significant new issue. Given an iteration tree $\mathcal{T}$ of length ω, how to define M_ω? Idea:

- Choose an ω-cofinal branch b through the tree order $<_{\mathcal{T}}$.
- Let M_ω be the direct limit of the models M_γ for $\gamma \in b$, under the iteration embeddings.
- (Note that for $\gamma \leq_{\mathcal{T}} \delta \in b$, we have $i_{\gamma,\delta}$ exists; we also maintain commutativity of these embeddings, so the direct limit works.)
- Ensure by choice of b that M_ω is wellfounded.

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Outline

A *fine iteration tree* is a refinement of *iteration tree*, due to Mitchell and Steel, to better suit *fine-structural* inner models $L[\mathbb{E}]$. For fine trees, some of the differences are:

- Each $M_\alpha = L_\gamma[\mathbb{E}^{M_\alpha}]$ for some γ
- E_α is anything indexed on $\mathbb{E}^{M_\alpha}$ (partial or total extender)
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The definition of *iterability* adapts. Standard canonical inner models are fine-structural and iterable.

What can we say about measures that appear in fine-structural inner models? All measures come from the sequence of extenders $\mathbb{E}$ in a canonical way.

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Theorem (S.)

Let M be an iterable, fine-structural inner model satisfying $\mathrm{ZFC} - $ Replacement. Suppose $M \models$"$\mathcal{U}$ is a countably complete ultrafilter".
Then there is a finite *fine iteration tree $\mathcal{T}$ on $M = M_0$, with last model $R = M_n$, with iteration embedding $i_{0,n} : M \to R$, such that:*

- $R = \mathrm{Ult}(M, \mathcal{U})$,
- *The ultrapower embedding $i_{\mathcal{U}}$ equals $i_{0,n}$.*

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