# Davies trees and their applications

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May 4, 2012
5th Young Set Theory Workshop
CIRM, Luminy

# Some pre-Cohen set theory

**Theorem**

*CH implies that $^2\mathbb{R}$ is a countable union of rotations of graphs of functions.*

**Proof.**

- Replace $^2\mathbb{R}$ with $^2\omega_1$.
- Let $\varphi_\alpha \colon \omega \to \alpha + 1$ be surjective for all $\alpha < \omega_1$.
- Let $f_n(\alpha) = \varphi_\alpha(n)$ for all $n < \omega$ and $\alpha < \omega_1$.
- $^2\omega_1 = \bigcup_{n<\omega}(f_n \cup f_n^{-1})$.

**Question (Sierpinski, 1951).** Is the converse true?

**Theorem (Davies, 1963)**

*ZFC already implies that $^2\mathbb{R}$ is a countable union of rotations of graphs of functions.*

# Davies' proof

Actually, Davies proved something stronger:

## Theorem

*Let $F$ be an infinite field and let $(L_n : n < \omega)$ be a sequence of pairwise non-parallel lines in ${}^2F$. Then ${}^2F$ can be partitioned into sets $(S_n : n < \omega)$ such that for all $n < \omega$ and all lines $L \parallel L_n$, $|S_n \cap L| \leq 1$.*

**Proof.**

1. Let $\mathfrak{A} = (H(\theta), \in, F, +, \cdot, \vec{L})$ and find $(M_\alpha : \alpha < \kappa)$ such that
   1.1 ${}^2F \subseteq \bigcup_{\alpha<\kappa} M_\alpha$,
   1.2 each $M_\alpha$ is a countable and $M_\alpha \prec \mathfrak{A}$, and
   1.3 each $\bigcup_{\beta<\alpha} M_\beta$ is a **finite** union $\bigcup_{i<m_\alpha} N_\alpha^i$ where $N_\alpha^i \prec \mathfrak{A}$.
2. Ignore $M_\alpha$ if ${}^2F \cap M_\alpha \subseteq \bigcup_{i<m_\alpha} N_\alpha^i$.
3. Otherwise, let ${}^2F \cap M_\alpha \setminus \bigcup_{i<m_\alpha} N_\alpha^i = \{p_k : k < \omega\}$.
4. For each $p_k$ and $N_\alpha^i$, there is at most one $n(k,i)$ such that the line through $p_k$ parallel to $L_{n(k,i)}$ intersects $N_\alpha^i$.
5. Put $p_k$ in $S_{r(k)}$ where $r(k) \neq n(k,i)$ for all $i < m_\alpha$ and $r(k) \neq r(j)$ for all $j < k$.

# The tree

To achieve the finite union of 1.3, Davies constructs $\vec{M}$ as the leaves of what I will call a **Davies tree**.

- A Davies tree is of the form $(\mathfrak{A}_t : t \in T)$ where $T$ is a well-founded tree of finite sequences of ordinals.
- The lexicographic ordering of the leaves of any such $T$ is necessarily a well-ordering.
- ${}^2F \subseteq \mathfrak{A}_\varnothing \prec \mathfrak{A}$.
- If $\mathfrak{A}_t$ is countable, then $t$ is a leaf of $T$.
- If $\mathfrak{A}_t$ is uncountable, then
  - $\mathfrak{A}_t = \bigcup\{\mathfrak{A}_{t^\frown\alpha} : t^\frown\alpha \in T\}$,
  - $\alpha < \beta \Rightarrow \mathfrak{A}_{t^\frown\alpha} \subseteq \mathfrak{A}_{t^\frown\beta}$,
  - $\mathfrak{A}_{t^\frown\alpha} \prec \mathfrak{A}_t$, and
  - $|\mathfrak{A}_{t^\frown\alpha}| < |\mathfrak{A}_t|$.
- $T$ is well-founded because $s \subset t$ implies $|\mathfrak{A}_s| > |\mathfrak{A}_t|$.

# The finite union

- Let $t = (\alpha_0, \ldots, \alpha_{m_\alpha - 1})$ be a leaf of $T$.
- Every leaf $s <_{\mathrm{lex}} t$ is of the form $(\alpha_0, \ldots, \alpha_{i-1}, \beta, \gamma_{i+1}, \ldots, \gamma_{n-1})$ where $i < m_\alpha$ and $\beta < \alpha_i$.
- Set $p(t, i, \beta) = (\alpha_0, \ldots, \alpha_{i-1}, \beta)$.
- Set $\mathfrak{B}_{t,i} = \bigcup_{\beta < \alpha_i} \mathfrak{A}_{p(t,i,\beta)}$.
- $\bigcup_{s <_{\mathrm{lex}} t} \mathfrak{A}_s = \bigcup_{i < m_\alpha} \mathfrak{B}_{t,i}$.
- $\mathfrak{B}_{t,i} \prec \mathfrak{A}$ or $\mathfrak{B}_{t,i} = \varnothing$.

# Costs and benefits

Benefits

- We can construct something arbitrarily large one countable piece at a time: $(S_n \cap M_\alpha : n < \omega)$ for $\alpha < \kappa$.
- Our work done prior to handling $M_\alpha$ is a finite union of very nice pieces: $(S_n \cap N^i_\alpha : n < \omega)$ for $i < m_\alpha$.

Unavoidable Costs

- The nice pieces might interact in nasty ways because generally $N^i_\alpha \not\subseteq N^j_\alpha$.
- Davies trees only work in contexts where these interactions are sufficiently benign.

Avoidable costs

- "$(N^i_\alpha : i < m_\alpha) \in M_\alpha$" is easy to arrange, but "$(S_n \cap N^i_\alpha : i < m_\alpha \wedge n < \omega) \in M_\alpha$" is not.
- This could be a problem in some contexts.
- The work-around is to build $\vec{S}$ and the Davies tree simultaneously...

# Long $\omega_1$-approximation sequences

(M., 2008) There is a simpler structure that induces a Davies tree.

- Let $\mathcal{L}$ be a countable language extending $\{\in\}$.
- Let $\mathfrak{A}$ be an $\mathcal{L}$-expansion of $(H(\theta), \in)$.
- Let $(M_\alpha : \alpha < \eta)$ be a **long $\omega_1$-approximation sequence**:
    - $M_\alpha$ is countable and $M_\alpha \prec \mathfrak{A}$.
    - $(M_\beta : \beta < \alpha) \in M_\alpha$.
- It is easy to build $\vec{M}$ and $\vec{S}$ simultaneously.
- Warning: If $\alpha \geq \omega_1$, then $\alpha \not\subseteq M_\alpha$ and $\exists \beta < \alpha \;\; M_\beta \notin M_\alpha \;\wedge\; M_\beta \not\subseteq M_\alpha$.
- There is $\varnothing$-definable well-founded class tree $\Upsilon$ of finite sequences of ordinals such that if the first $\eta$ leaves of $\Upsilon$, according to the lexicographic ordering, are $(u_\alpha : \alpha < \eta)$, then $(\mathfrak{A}_t : t \in T)$ is a Davies tree where:
    - $T = \{t : \exists \alpha < \eta \;\; t \subseteq u_\alpha\}$.
    - $\mathfrak{A}_{u_\alpha} = M_\alpha$.
    - $\mathfrak{A}_t = \bigcup \{\mathfrak{A}_{t^\frown \alpha} : t^\frown \alpha \in T\}$ if $t$ is not a leaf.

# Ordinal division

- $\forall \alpha, \beta > 0 \ \ \exists! \gamma, \delta \quad \alpha = \beta \cdot \gamma + \delta \ \wedge \ \delta < \beta$.
- $\forall \alpha > 0 \ \ \exists! Q\alpha, R\alpha \quad \alpha = |\alpha| \cdot Q\alpha + R\alpha \ \wedge \ R\alpha < |\alpha|$.
- Observe that $1 \leq Q\alpha < \alpha^+$.

Repeatedly divide the remainder by its cardinality and call the result the **cardinal normal form** of $\alpha$:

$$
\begin{aligned}
\alpha &= |\alpha| \cdot Q\alpha \quad + R\alpha \\
\alpha &= |\alpha| \cdot Q\alpha \quad + |R\alpha| \cdot QR\alpha \quad + R^2\alpha \\
&\ \ \vdots \\
\alpha &= |\alpha| \cdot Q\alpha \quad + |R\alpha| \cdot QR\alpha \quad + \left|R^2\alpha\right| \cdot QR^2\alpha + \cdots + R^{m_\alpha}\alpha
\end{aligned}
$$

Stop when $R^{m_\alpha}\alpha$ is countable.
Edge cases: $R^0 = \mathrm{id}$; $m_\alpha = 0$ for all $\alpha < \omega_1$.

# A canonical Davies tree

- The non-leaves of $\Upsilon$ are the sequences $(\tau^j_\alpha : j < i)$, with $i \leq m_\alpha$, of **truncations** $\tau^j_\alpha = \sum_{k<j} \left|R^k\alpha\right| \cdot QR^k\alpha$.
- For convenience, set $\tau^{m_\alpha}\alpha = \alpha$.
- The leaves are the sequences $(\tau^j_\alpha : j \leq m_\alpha)$.

Actually, we don't strictly need the tree anymore:
Given a long $\omega_1$-approximation sequence $(M_\alpha : \alpha < \eta)$,

- the **eras** of $\alpha$ are the intervals $I^i_\alpha = [\tau^i_\alpha, \tau^{i+1}_\alpha)$ where $i < m_\alpha$;
- the **strata** of $M_\alpha$ are the unions $N^i_\alpha = \bigcup\{M_\beta : \beta \in I^i_\alpha\}$;
- $\bigcup_{\beta<\alpha} M_\beta = \bigcup_{i<m_\alpha} N^i_\alpha$;
- $N^i_\alpha \in M_\alpha$ and $\left|N^i_\alpha\right| \subseteq N^i_\alpha \prec \mathfrak{A}$;
- $i < j < m_\alpha \;\Rightarrow\; N^i_\alpha \in N^j_\alpha \;\wedge\; \left|N^i_\alpha\right| > \left|N^j_\alpha\right|$;
- $N^i_\alpha$ is uncountable, except possibly when $i = m_\alpha - 1$.

# Another application of Davies trees

**Theorem (Jackson and Mauldin, 2002)**

*There exists $S \subseteq {}^2\mathbb{R}$ such that $|S \cap L| = 1$ for every lattice $L$ isometric with ${}^2\mathbb{Z}$.*

- Jackson and Mauldin explicitly use a Davies tree in order to proceed one countable piece at a time, organizing all prior work into finitely many nice pieces.
- Proving that lattices $L_0 \in N_\alpha^0, \ldots, L_{m_\alpha - 1} \in N_\alpha^{m_\alpha - 1}$, $L_{m_\alpha} \in M_\alpha$ interact sufficiently benignly takes many pages of work.

# An implicit application to group theory

Theorem (Shelah, 1975)

*Let $G$ be a group and $\lambda$ a singular cardinal such that every subgroup of $G$ of size less than $\lambda$ is free. Every subgroup of $G$ of size $\lambda$ is then also free.*

- The above theorem is just a special case of Shelah's compactness theorem for singular cardinals.
- Shelah doesn't explicitly use a Davies tree, but he implicitly uses the first three non-root levels of a Davies tree, avoiding higher levels through an intricate inductive argument.

# The $\kappa$-Freese Nation property

Fuchino, Koppelberg, Shelah (1996):

- Let $\kappa$ be a regular cardinal.
- A boolean algebra $A$ has the $\kappa$-**FN** if there is a map $f \colon A \to [A]^{<\kappa}$ such that for all $a \leq b$ we have $f(a) \cap f(b) \cap [a, b] \neq \varnothing$.
- **Classic example:** free booleans algebras have the $\aleph_0$-FN.
- A subalgebra $B$ of $A$ is a $\kappa$-**subalgebra** of $A$, written $B \leq_\kappa A$, if every ideal of $B$ of the form $B \cap [0, a]$ where $a \in A$ is generated by fewer than $\kappa$-many elements of $B$.
- **Small substructure characterization:** $A$ has the $\kappa$-FN iff, for some club $\mathcal{E} \subseteq [A]^{<\kappa^+}$, every $B \in \mathcal{E}$ satisfies $B \leq_\kappa A$,

Using long $\kappa^+$-approximation sequences, a natural generalization of long $\omega_1$-approximation sequences, I proved:

- **Large substructure characterization:** $A$ has the $\kappa$-FN iff, for all $M \prec (H(\theta), \in, \leq_A)$ satisfying $\kappa^+ \cap M \in \kappa^+ + 1$, we have $A \cap M \leq_\kappa A$.

# Openly generated compacta

- Let $X$ be a compact (Hausdorff) space.
- Let $C(X)$ be the algebra of all continuous $f\colon X \to \mathbb{R}$.
- Given a class $A$, let $X/A$ denote the quotient space where points are identified iff no $f \in C(X) \cap A$ distinguishes them.
- **Small quotient characterization (Ščepin, 1981):** $X$ is **openly generated** iff, for some club $\mathcal{E} \subseteq [C(X)]^{<\aleph_1}$, the quotient map $q^X_M\colon X \to X/M$ is open for all $M \in \mathcal{E}$.
- **Large quotient characterization (M., 2008):** $X$ is openly generated iff, for all $M \prec (H(\theta), \in, C(X))$, the quotient map $q^X_M\colon X \to X/M$ is open.
- **Algebraic connection (classical):** If $X$ is zero-dimensional, then $X$ is openly generated iff $\mathrm{Clop}(X)$ has the $\aleph_0$-FN.
- **Example (easy):** Powers of 2 are openly generated.
- **Example (Šapiro, 1976):** The Vietoris hyperspace of ${}^{\aleph_2}2$ is openly generated but not a continuous image of a power of 2.

# Flat bases

- A (local) base of a space is **flat** if every element of the (local) base has only finitely many supersets in the (local) base.
- (M., 2008) If $Y$ is a continuous image of an openly generated compact $X$, and $\mathcal{A}$ is a local base in $Y$, then $\mathcal{A}$ contains a flat local base $\mathcal{B}$ in $Y$.
  - This result didn't need a long $\omega_1$-approximation sequence for its proof, just a continuous elementary $\in$-chain.
- A space $Y$ is **homogeneous** if for all $p, q \in Y$, there is a homeomorphism $f\colon Y \to Y$ such that $f(p) = q$.
- (M., 2008) If $Y$ is a homogeneous continuous image of an openly generated compact $X$, and $\mathcal{A}$ is a base of $Y$, then $\mathcal{A}$ contains a flat base $\mathcal{B}$ of $Y$.
  - The result was proved with a long $\omega_1$-approximation sequence.
  - A stronger result for metrizable compacta was used to ensure that the inductive construction succeeded.
  - The crucial inductive argument is that if each $\mathcal{B} \cap N_\alpha^i$ is flat, then, for all nonempty open $U$, $q^Y_{N^i_\alpha}[U]$ has only finitely many supsersets in $\mathcal{B} \cap \bigcup_{\beta<\alpha} M_\alpha$.

# Why flatness?

(M., 2008) Every known homogeneous compact space (including all compact groups) has a flat local base.
(By homogeneity, this is equivalent to saying that every local base contains a flat local base.)

**Conjecture.** Every homogeneous compact space has a flat local base.

# Why homogeneity?

- A (local) base is $\kappa$-flat if every element of the (local) base has fewer than $\kappa$-many supersets in the (local) base.
- (M., 2008) Every known homogeneous compact space has a $\mathfrak{c}^+$-flat base.
- The **cellularity** $c(X)$ of a space $X$ is the supremum of the cardinalities of its pairwise disjoint families of open sets.
- **Van Douwen's Problem (c. 1970).** Is there a homogeneous compact space with cellularity greater than $\mathfrak{c}$?
- Van Douwen's Problem is still open in all models of ZFC.
- (M., 2008) If GCH holds, then, for all homogeneous compact $X$, every local base in $X$ contains a $c(X)$-flat local base.
- This is weak evidence for "no" to Van Douwen. Nevertheless...
- **Conjecture.** There are homogeneous compact $X$ with $c(X) > \mathfrak{c}$ because every compact $Y$ is a continuous image of a homogeneous compact $X$.

# A new application to homogeneous compacta

(M., 2012) If $Y$ is an openly generated compactum, then $Y$ is a continuous open image of a homogeneous openly generated compactum $X$.

- The result is proved with a long $\omega_1$-approximation sequence.
- The continuous surjection $f\colon X \to Y$ is built as an inverse limit of maps $f_{M_\alpha}\colon X_{M_\alpha} \to Y/M_\alpha$ with bonding maps $\pi^i_\alpha\colon X_{M_\alpha} \to X_{N^i_\alpha}/M_\alpha$ such that diagrams of the following form commute:

$$\begin{array}{ccc} X_{M_\alpha} & \xrightarrow{\quad f_{M_\alpha} \quad} & Y/M_\alpha \\ \Big\downarrow{\scriptstyle \pi^i_\alpha} & & \Big\downarrow{\scriptstyle q^{Y/M_\alpha}_{N^i_\alpha}} \\ X_{N^i_\alpha}/M_\alpha & \xrightarrow{f_{N^i_\alpha}/M_\alpha} & Y/(N^i_\alpha \cap M_\alpha) \end{array}$$

- Here $X_{N^i_\alpha}$ and $f_{N^i_\alpha}$ are themselves inverse limits of $(X_{M_\beta} : \beta \in I^i_\alpha)$ and $(f_{M_\beta} : \beta \in I^i_\alpha)$.
- That $N^i_\alpha = \bigcup\{M_\beta : \beta \in I^i_\alpha\}$ is a directed union is proved by induction on the Davies tree.