Research Statement: **Aspects of Borel ideals**

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My research focuses on combinatorics and cardinal invariants of nice ideals on countable sets, and on related methods of forcing and descriptive set theory.

0. Motivation. The following families are classical examples of ideals on ω: the ideal of finite sets: $\mathrm{Fin} = [\omega]^{<\omega}$; the *density zero ideal*: $\mathcal{Z} = \{A \subseteq \omega : \lim\big(|A \cap n|/n\big) = 0\}$; the *summable ideal*: $\mathcal{I}_{1/n} = \{A \subseteq \omega : \sum\{1/n : n \in A\} < \infty\}$; and the *van der Waerden ideal*: $\mathcal{W} = \{A \subseteq \omega : A$ does not contain arbitrary long arithmetic progressions$\}$.

In the past few years the study of these ideals and their generalizations has become a central topic of infinite combinatorics, forcing theory, and applications of set theory.

Ideals on ω can be seen as subsets of the Cantor set $\mathcal{P}(\omega) \simeq 2^\omega$ which is a

(i)	Polish space	⇝ Borel-complexity of ideals;
(ii)	Boolean algebra	⇝ factor algebras and their invariants;
(iii)	pre-ordered set (by $\subseteq^*$)	⇝ star-invariants of ideals.

For example, Fin, $\mathcal{I}_{1/n}$, and $\mathcal{W}$ are F_σ ideals but $\mathcal{Z}$ is $F_{\sigma\delta}$. Furthermore, Fin, $\mathcal{Z}$ and $\mathcal{I}_{1/n}$ are *P-ideals*, that is, for $\mathcal{I} = \mathrm{Fin}, \mathcal{Z}, \mathcal{I}_{1/n}$ the pre-ordered set $(\mathcal{I}, \subseteq^*)$ is σ-directed (where $A \subseteq^* B$ iff $A \setminus B$ is finite). This property can be seen as an analogue of σ-closedness of the ideal of subsets of the real line with measure zero (denoted by $\mathcal{N}$).

1. Tukey-reducibility and star-invariants. Tukey-reducibility between pre-orders, or in general, Galois-Tukey connections between relations can be applied both for proving inequalities between cardinal invariants and for proving implications between bounding and dominating properties of forcing notions.

I. Farah, D. Fremlin, M. Hrušák, and many others proved numerous deep results in this topic. In particular, it turned out that there are natural inequalities between the star-invariants ($\mathsf{add}^*, \mathsf{non}^*, \mathsf{cov}^*, \mathsf{cof}^*$) of tall Borel ideals and the classical additivity, uniformity, covering, and cofinality numbers of measure and category

I am interested in the following still open problems:

(P1) Are $(\mathcal{I}, \subseteq^*)$ and $(\mathcal{N}, \subseteq)$ Tukey-equivalent for each tall Borel P-ideal $\mathcal{I}$?

(P2) Is $\mathfrak{b} < \mathsf{cov}^*(\mathcal{Z})$ consistent with ZFC?

(P3) Is $\mathsf{non}^*(\mathcal{Z}) < \mathsf{cov}^*(\mathcal{Z})$ consistent with ZFC?

2. $\mathcal{I}$-almost-disjoint families. Assume $\mathcal{I}$ is an ideal on ω. An infinite family $\mathcal{A} \subseteq \mathcal{I}^+ = \mathcal{P}(\omega) \setminus \mathcal{I}$ is *$\mathcal{I}$-almost-disjoint* ($\mathcal{I}$-AD) if $A \cap B \in \mathcal{I}$ for all distinct $A, B \in \mathcal{A}$. An $\mathcal{I}$-AD family $\mathcal{A}$ is *maximal* ($\mathcal{I}$-MAD) if for all $X \in \mathcal{I}^+$, there is an $A \in \mathcal{A}$ such that $X \cap A \in \mathcal{I}^+$, in other words $\mathcal{A}$ is $\subseteq$-maximal among $\mathcal{I}$-AD families.

For example, it is not hard to see that there is a countable (infinite) $\mathcal{Z}$-MAD family but all Fin-MAD, $\mathcal{I}_{1/n}$-MAD, and $\mathcal{W}$-MAD families are uncountable. The following interesting problem is still open:

(P4) Is there any reasonable characterization of those Borel ideals for which there exist countable $\mathcal{I}$-MAD families?

I introduced the notion of *Borel-regular embeddings* between factor Boolean algebras $\mathcal{P}(\omega)/\mathcal{I}$ and $\mathcal{P}(\omega)/\mathcal{J}$, and I applied these embeddings to prove (a) inequalities between the almost-disjointness numbers of ideals, and (b) forcing indestructibility results.

3. Idealized Ramsey theory and monotonic ideal. Two of my favorite questions:

(P5) Does there exist a tall Borel ideal $\mathcal{I}$ with the following property (Mon): for all sequence $(a_n)_{n\in\omega}$ of reals, there is an $X \in \mathcal{I}^+$ such that $(a_n)_{n\in X}$ is monotonic?

(P6) Assume $\mathcal{I}$ has the property Mon. Does it imply that for any coloring $c : [\omega]^2 \to 2$, there is a c-homogeneous set $X \in \mathcal{I}^+$?