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Research Statement, 5th Young Set Theory Workshop in Luminy

I am a PhD student mainly interested in (iterated) forcing and its applications to set theory of the reals; in particular, I am studying questions about small subsets of the real line and (variants of) the Borel Conjecture.

The *Borel Conjecture* (BC) is the statement that there are no uncountable strong measure zero sets (a set X is *strong measure zero* if for any sequence of ε_n's, X can be covered by intervals I_n of length ε_n, or, equivalently, if it can be translated away from each meager set). The *dual Borel Conjecture* (dBC) is the analogous statement about *strongly meager* sets (the sets which can be translated away from each measure zero set). Both BC and dBC fail under CH. In 1976, Laver [4] showed that BC is consistent (by a countable support iteration of Laver forcing of length ω_2). Carlson [2] showed that dBC is consistent (by a finite support iteration of Cohen forcing of length ω_2). What about $\mathrm{BC} + \mathrm{dBC}$?

Together with my advisor Martin Goldstern, Jakob Kellner and Saharon Shelah, I worked on the following theorem (see [3]):

> There is a model of ZFC in which both the Borel Conjecture and the dual Borel Conjecture hold, i.e., $\mathrm{Con}(\mathrm{BC} + \mathrm{dBC})$.

One of the difficulties in the proof is the fact that one is forced to obtain dBC *without adding Cohen reals* since Cohen reals inevitably destroy BC. This was first done by Bartoszyński and Shelah [1] using Shelah's non-Cohen oracle-c.c. framework from [5]. For this reason I made myself acquainted with this framework even though it turned out to be not directly applicable to the problem $\mathrm{Con}(\mathrm{BC} + \mathrm{dBC})$. A more digestible version of the non-Cohen oracle-c.c. framework will nevertheless be part of my thesis.

I also investigated another variant of the Borel Conjecture, which I call the *Marczewski Borel Conjecture* (MBC). It is the assertion that there are no uncountable sets in s_0^*, where s_0^* is the collection of those sets which can be translated away from each set in the *Marczewski ideal* s_0 (the Marczewski ideal s_0 is related to Sacks forcing: a set X is in s_0 if each perfect set contains a perfect subset disjoint from X). So MBC is the analogue to BC (dBC) with meager (measure zero) replaced by s_0 in its definition. The question arises whether MBC is consistent (the negation of MBC is consistent).

I do not know, but while exploring the family s_0^* under CH, I obtained the following result. Let's call $\mathcal{I} \subseteq \mathcal{P}(2^\omega)$ a *Sacks dense ideal* if

- $\mathcal{I}$ is a (non-trivial) translation-invariant σ-ideal
- $\mathcal{I}$ is "dense in Sacks forcing": each perfect set P contains a perfect subset $Q \subseteq P$ which belongs to $\mathcal{I}$.

Then the following holds:

> Assume CH. Then s_0^* is contained in every Sacks dense ideal $\mathcal{I}$.

So the question is whether we can (at least consistently) find "many Sacks dense ideals" (under CH). The meager sets as well as the measure zero sets form Sacks dense ideals, whereas the strong measure zero sets do not. Nevertheless the strong measure zero sets can be "approximated from above", meaning that each set in the intersection of all Sacks dense ideals (and hence each set in s_0^*) is strong measure zero (and strongly meager; in fact even null-additive).

So only "very small" sets can belong to all Sacks dense ideals. On the other hand, the intersection of $\aleph_1$ many Sacks dense ideals still contains an uncountable set of reals. It is unclear whether the same is true for the intersection of *all* Sacks dense ideals.

Very recently, I started to look at the analogous situation when the Marczewski ideal (connected to Sacks forcing) is replaced by other ideals connected to Axiom A forcing notions (e.g. v_0, which comes from Silver forcing). The above result seems to hold in general, i.e., each set in, e.g., v_0^* belongs to every "Silver dense ideal".

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References

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