WALKS ON ORDINALS

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<u>The structure on ω_1</u>:

$C_\alpha \quad (\alpha < \omega_1)$

(1) $C_\alpha \subseteq \alpha \quad , \quad \sup C_\alpha = \alpha$

(2) $C_{\alpha+1} = \{\alpha\}$

(3) $\alpha > 0$ limit $\longrightarrow$ $\mathrm{otp}\, C_\alpha = \omega$ and

C_α contains no limit ordinals.

A <u>step</u> from β towards smaller α is

$$\beta \longrightarrow \min(C_\beta \setminus \alpha)$$

$|C_\beta \cap \alpha|$ is the <u>weight</u> of this step

The <u>Full Code</u> of the walk is the function

$$\rho_0 : [\omega_1]^2 \to \omega^{<\omega}$$

defined by

$$\rho_0(\alpha,\beta) = \langle |C_\beta \cap \alpha| \rangle^\frown \rho_0(\alpha, \min(C_\beta \setminus \alpha)),$$

with the boundary value $\rho_0(\alpha,\alpha) = \emptyset$

The <u>Maximal Weight</u> is the function

$$\rho_1 : [\omega_1]^2 \to \omega$$

defined by

$$\rho_1(\alpha,\beta) = \max\{|C_\beta \cap \alpha|, \rho_1(\alpha, \min(C_\beta \setminus \alpha))\}$$

with the boundary value $\rho_1(\alpha,\alpha) = 0$.

<u>Lemma 1</u>. For all $\alpha < \beta < \omega_1$ and $n < \omega$:

(a) $\{\xi \leq \alpha : \rho_1(\xi, \alpha) \leq n\}$ is finite

(b) $\{\xi \leq \alpha : \rho_1(\xi, \alpha) \neq \rho_1(\xi, \beta)\}$ is finite.

<u>Proof:</u> Induction. $\square$

The <u>Number of Steps</u> function is

$$\rho_2 : [\omega_1]^2 \longrightarrow \omega$$

defined by

$$\rho_2(\alpha, \beta) = \rho_2(\alpha, \min(C_\beta \setminus \alpha)) + 1$$

with the boundary value $\rho_2(\alpha, \alpha) = 0$.

<u>Lemma 2</u>.

$$\forall \alpha < \beta < \omega_1 \quad \sup_{\xi < \alpha} |\rho_2(\xi, \alpha) - \rho_2(\xi, \beta)| < \infty$$

<u>Proof:</u> Use Full Lower Trace. $\square$

The Last Step Function

$$\rho_3 : [\omega_1]^2 \longrightarrow 2$$

is defined recursively by

$$\rho_3(\alpha,\beta)=1 \quad \text{iff} \quad \rho_0(\alpha,\beta)(\rho_2(\alpha,\beta)-1)=\rho_1(\alpha$$

In other words, $\rho_3(\alpha,\beta)=1$ just in case the last step of the walk $\beta \longrightarrow \alpha$ comes with the <u>maximal weig</u>

<u>Lemma</u>. 3

$\forall \alpha<\beta<\omega_1$ $\{\xi<\alpha : \rho_3(\xi,\alpha)\neq\rho_3(\xi,\beta)\}$ is finite

<u>Proof</u>: Use again the Full Lower Trace function. $\square$

Corollary 4. The tree $T(\varrho_0) = \{\varrho_0(\cdot,\beta)\restriction\alpha : \alpha \leq \beta <$
has countable levels and has no uncountable branches, $\therefore$ $T(\varrho_0)$ is Aronszajn (an A-tre

Question. Are the other trees $T(\varrho_0)$, $T(\varrho_2)$ and $T(\varrho_3)$ also A-trees?

Case of $T(\varrho_3)$ needs the following assumption on the $\bar{C}$-sequence.

(4) (a) If $\alpha = \lambda + \omega$ for λ limit,
$C_\alpha = \{\lambda + n : m < n < \omega\}$ for some

(b) If α is limit of limits and
$C_\alpha = (C_\alpha(n) : n < \omega)$ is the increasing enumeration, then for each n,
$C_\alpha(n) = \lambda_n + m$ for some $m > n$

Notation

Λ = the set of countable limit ordinals

$\Lambda + n = \{\lambda + n : \lambda \in \Lambda\}$ for $n < \omega$

Lemma 5. $\forall \beta\ \forall \lambda \in \Lambda\ \exists k\ \forall n$:

$\lambda + n < \beta \longrightarrow \varrho_3(\lambda+n, \beta) = 1$ i.e.

$\varrho_{3\beta} := \varrho_3(\cdot, \beta)$ restricted to $[\lambda, \lambda+\omega)$

is equal to 1 modulo finitely many ex

Proof: We may assume $\alpha = \lambda + \omega \leq \beta$.

Then there is m_0 such that

$\forall n \geq m_0$ the walk $\beta \to \lambda + n$ passes through

Let m_1 be such that (see (4))

$C_\alpha \supseteq \{\lambda + n : m_1 < n < \omega\}$

So for $n > \max(m_0, m_1)$ the last of the walk $\beta \to \lambda + n$ is the step $\alpha \to \lambda + n$.

Its weight $|\{k : \overset{\leq C_\alpha \cap (\lambda+n)}{m_0} < k < n\}| = n - m_0 - 1$ is eventually bigger than $\rho_1(\alpha, \beta)$, so eventually $\rho_3(\lambda + n, \beta) = 1$ $\square$

Lemma 6.

$\forall_\beta \forall n \; |\{\lambda \in \Lambda : \lambda + n < \beta \;\&\; \rho_3(\lambda+n, \beta) = 1\}| <$

Proof: Given an infinite $\Gamma \subseteq (\Lambda + n)$, we need to find $\lambda + n \in \Gamma$ such that

$$\rho_3(\lambda + n, \beta) = 0.$$

Shrink Γ so that we can assume

$$\forall \lambda + n \in \Gamma \quad \rho_1(\lambda + n, \beta) > n + 2.$$

So, if $\varrho_3(\lambda+n,\beta)=1$ for some $\lambda+n\in$ the last step $\alpha\to\lambda+n$ of the walk $\beta\to\lambda+n$ must have weight $>n$.
Note however that by the assumption (2) $\alpha\neq\lambda+n+1$, by the assumption 4(a) $\alpha\notin\lambda+\omega$, and by the assumption 4(b) α cannot be limit of limits. ✱. $\square$

<u>Lemma 7.</u> Let $B_\alpha=\{\xi<\alpha:\varrho_3(\xi,\alpha)=1\}$ for $\alpha<\omega_1$. Then

(1) $B_\alpha=^* B_\beta\cap\alpha$ for $\alpha<\beta$

(2) $(\Delta+n)\cap B_\beta$ is finite for all $n<\omega$ and $\beta<$

(3) $\{\lambda+n : n<\omega\}\subseteq^* B_\beta$ whenever $\lambda+\omega\leq\beta$

So in particular there is no

$$g : \omega_1 \longrightarrow 2$$

such that $g \upharpoonright \beta =^* \varrho_3(\cdot, \beta)$ for all β, so we have the following.

Lemma 8. $T(\varrho_3)$ is a coherent A-tree. $\square$

Case of $T(\varrho_2)$ needs the following fact.

Lemma 9. For every pair A and B of uncountable subsets of ω_1 and every $k < \omega$ there exist $\alpha \in A$, $\beta \in B$ such that $\varrho_2(\alpha, \beta) > k$.

In fact we prove a stronger statement

<u>Lemma 9*</u> For every uncountable $A \subseteq [\omega_1]^{<\omega}$ consisting of pairwise disjoint sets all of same fixed size n and every $k < \omega$ there exist uncountable $B \subseteq A$ such that

$$(\forall a < b \text{ in } B)(\forall i,j < n) \quad \rho_2(a(i), b(j)) \geq k$$

<u>Proof</u>: Induction on k. So suppose $(\forall a < b \text{ in } A)(\forall i,j < n) \ \rho_2(a(i), b(j)) \geq k$, and work for uncountable $B \subseteq A$ such that $(\forall a < b \text{ in } B)(\forall i,j < n) \ \rho_2(a(i), b(j)) \geq k+1$.

Fix limit $\delta < \omega$ and $b_\delta \in A$, $b_\delta > \delta$. The for some $\eta_\delta < \delta$,

$$\forall \xi \in [\eta_\delta, \delta) \ \forall \beta \in b_\delta \quad \rho_0(\xi, \beta) = \rho_0(\beta, \delta)^\frown \rho_0(\delta, \xi).$$

Find stationary $\Gamma \subseteq \omega_1$ and $\eta < \omega_1$ such that $\eta_\delta = \eta$ for all $\delta \in \Gamma$.

Choose stationary $\Sigma \subseteq \Gamma$ such that

$$\forall \gamma < \delta \text{ in } \Sigma \quad \gamma < b_\gamma < \delta < b_\delta$$

Consider the family

$$A^* = \{\{\gamma\} \cup b_\gamma : \gamma \in \Sigma\}$$

By induction hypothesis there is uncountable $B^* \subseteq A^*$ such that $\rho_2(a(i), b(j)) \geq k$ for all $i, j < n+1$ and $a < b$ in B. Let $B = \{b \setminus \{\min(b)\} : b \in B^*\}$. Then $B \subseteq A$ is uncountable and $\rho_2(a(i), b(j)) \geq k+1$ for all $i, j < n$ and $a < b$ in B. $\square$

<u>Corollary</u> 10. For every $g : \omega_1 \to \omega$ there is $\alpha <$ such that $\sup_{\xi < \alpha} |\rho_2(\xi, \alpha) - g(\xi)| = \infty$.

<u>Corollary 11.</u> $T(\varrho_2)$ is an A-tree.

<u>Question.</u> Is $T(\rho_0)$ an A-tree?

§2. <u>Traces</u>

The <u>upper trace</u> $Tr : [\omega_1]^2 \to [\omega_1]^{<\omega}$ is defined by

$$Tr(\alpha,\beta) = \{\beta\} \cup Tr(\alpha, \min(C_\beta \setminus \alpha))$$

with the boundary value is $Tr(\alpha,\alpha) = \{\alpha\}$.

The <u>lower trace</u> $L : [\omega_1]^2 \to [\omega_1]^{<\omega}$,

$$L(\alpha,\beta) = \{\lambda(\xi,\beta) : \xi \in Tr(\alpha,\beta),\ \xi \neq \beta\}$$

$$\lambda(\xi,\beta) = \max\{\max(C_\eta \cap \xi) : \eta \in Tr(\xi,\beta), \eta \neq$$

<u>Recursive definition:</u>

$$L(\alpha,\beta) = (L(\alpha, \min(C_\beta \setminus \alpha))) \setminus \max(C_\beta \cap \alpha)) \cup \{\max(C_\beta \cap \alpha$$

with the boundary value $L(\alpha,\alpha) = \emptyset$.

If $\beta=\beta_0>\beta_1>\dots>\beta_n=\alpha$ is the walk from β to α i.e.

$\beta_0=\beta$, $\beta_n=\alpha$ and $\beta_{i+1}=\min(C_{\beta_i}\setminus\alpha)$ $(i<n)$

then

$$Tr(\alpha,\beta)=\{\beta_i : i\le n\}$$

$$L(\alpha,\beta)=\{\lambda_i : i\le n\},$$

where $\lambda_i=\max\left(\bigcup_{j\le i} C_{\beta_j}\cap\alpha\right)$.

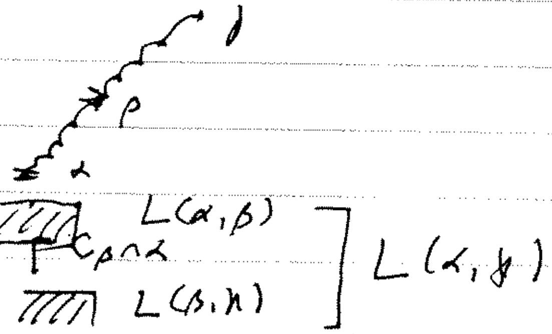
<u>Lemma 12</u>. For $\alpha\le\beta\le\gamma$,

(a) $\alpha>L(\beta,\gamma)$ implies $\rho_0(\alpha,\gamma)=\rho_0(\beta,\gamma)^\frown\rho_0(\alpha,\beta)$ and therefore $Tr(\alpha,\gamma)=Tr(\beta,\gamma)\cup Tr(\alpha,\beta)$.

(b) $L(\alpha,\beta)>L(\beta,\gamma)$ implies

$$L(\alpha,\gamma)=L(\beta,\gamma)\cup L(\alpha,\beta)$$

<u>Proof.</u>



The Full lower trace

$$F : [\omega_1]^2 \to [\omega_1]^{<\omega}$$

$$F(\alpha,\beta) = F(\alpha, \min(C_\beta \setminus \alpha)) \cup \bigcup_{\xi \in C_\beta \cap \alpha} F(\xi, \alpha)$$

with the boundary value $F(\alpha,\alpha) = \{\alpha\}$

<u>Lemma 13</u>. For $\alpha \leq \beta \leq \gamma$

(a) $F(\alpha,\gamma) \subseteq F(\alpha,\beta) \cup F(\beta,\gamma)$

(b) $F(\alpha,\beta) \subseteq F(\alpha,\gamma) \cup F(\beta,\gamma)$.

<u>Proof</u>: Induction on γ. ~~To~~ prove (a), let

$$\gamma_1 = \min(C_\gamma \setminus \alpha)$$

We first show that

$$F(\alpha,\gamma_1) \subseteq F(\alpha,\beta) \cup F(\beta,\gamma).$$

<u>Case 1</u>. $\gamma_1 < \beta$.

By ind. hyp.

$$F(\alpha, \gamma_1) \subseteq F(\alpha, \beta) \cup F(\gamma_1, \beta)$$

$$\subseteq F(\alpha, \beta) \cup F(\beta, \gamma)$$

since $\gamma_1 \in C_\gamma \cap \beta$ and therefore $F(\gamma_1, \beta) \subseteq F(\beta, \gamma)$

<u>Case $\gamma_1 \geq \beta$:</u> Then

$$\gamma_1 = \min(C_\gamma \setminus \beta)$$

and therefore $F(\gamma_1, \beta) \subseteq F(\beta, \gamma)$. $\square$

<u>Consider a factor $F(\xi, \alpha)$</u> of $F(\alpha, \beta)$,

where $\xi \in C_\gamma \cap \alpha$.

By ind. hyp.

$$F(\xi, \alpha) \subseteq F(\alpha, \beta) \cup F(\xi, \beta)$$

$$\subseteq F(\alpha, \beta) \cup F(\beta, \gamma)$$

since $\xi \in C_\gamma \cap \beta$ and so $F(\xi, \beta) \subseteq F(\beta, \gamma)$

To prove (b) consider cases:

<u>Case</u> $\gamma_1 < \beta$. By ind hyp

$$F(\alpha,\beta) \subseteq F(\alpha,\gamma_1) \cup F(\gamma_1,\beta)$$

Since $\gamma_1 \in C_\gamma \cap \beta$, ~~the set~~ we get

$$F(\gamma_1,\beta) \subseteq F(\beta,\gamma)$$

By definition of $F(\alpha,\gamma)$,

$$F(\alpha,\gamma_1) \subseteq F(\alpha,\gamma).$$

<u>Case</u> $\gamma_1 \geq \beta$. Then

$$\gamma_1 = \min(C_\gamma \setminus \beta) \quad \text{so}$$

$$F(\beta,\gamma_1) \subseteq F(\beta,\gamma).$$

Using the ind hyp for $\alpha \leq \beta \leq \gamma_1$, we

$$F(\alpha,\beta) \subseteq F(\alpha,\gamma_1) \cup F(\beta,\gamma_1) \subseteq F(\alpha,\gamma) \cup F(\beta,\gamma)$$

$\square$

<u>Remark</u> Note that since $\alpha \in F(\alpha,\beta)$ for all $\alpha \leq \beta$, by the definition of the full lower trace, we have that

$$F(\alpha,\beta) \supseteq L(\alpha,\beta)$$

for all $\alpha \leq \beta < \omega_1$.

<u>Lemma 14.</u> If $\alpha \in F(\beta,\gamma)$ then

$$F(\alpha,\gamma) \cup F(\alpha,\beta) \subseteq F(\beta,\gamma)$$

<u>Proof</u>: Induction on γ. Let $\gamma_1 = \min(C_\beta \setminus$

<u>Case 1.</u> $\alpha \in F(\beta,\gamma_1)$. Then by ind. hyp.

$$F(\alpha,\beta) \subseteq F(\beta,\gamma_1) \subseteq F(\beta,\gamma)$$

On the other hand

$$F(\alpha,\gamma) \subseteq F(\alpha,\beta) \cup F(\beta,\gamma) = F(\beta,\gamma)$$

<u>Case 2</u> $\alpha \in F(\xi, \beta)$ for some $\xi \in C_\gamma \cap \beta$

Then by ind hyp for $\alpha \leq \xi \leq \beta$:

$$F(\alpha, \beta) \subseteq F(\xi, \beta) \subseteq F(\beta, \gamma)$$

On the other hand

$$F(\alpha, \gamma) \subseteq F(\alpha, \beta) \cup F(\beta, \gamma) = F(\beta, \gamma). \quad \square$$

<u>Lemma 15.</u> Let $\alpha \leq \beta \leq \gamma < \omega_1$ and let

$$\bar{\alpha} = \min(F(\beta, \gamma) \setminus \alpha).$$

Then

(a) $\rho_0(\alpha, \beta) = \rho_0(\bar{\alpha}, \beta) {}^\frown \rho_0(\alpha, \bar{\alpha})$

(b) $\rho_0(\alpha, \gamma) = \rho_0(\bar{\alpha}, \gamma) {}^\frown \rho_0(\alpha, \bar{\alpha})$

<u>Proof</u>: This is trivial if $\alpha = \bar{\alpha}$, so we assume $\alpha < \bar{\alpha}$. By Lemma 14,

$$F(\bar{\alpha}, \beta) \cup F(\bar{\alpha}, \gamma) \subseteq F(\beta, \gamma),$$

so

We have

$$\alpha > F(\beta,\gamma)\cap\bar{\alpha} \geq F(\bar{\alpha},\beta)\cap\bar{\alpha},\ F(\bar{\alpha},\gamma)\cap\bar{\alpha}$$
$$\geq L(\bar{\alpha},\beta),\ L(\bar{\alpha},\gamma),$$

and so we are done by Lemma 12 (a). ∎

<u>Corollary 16.</u> For all $\alpha<\omega_1$

$|\{\rho_0(\cdot,\beta)\restriction\alpha : \alpha\leq\beta<\omega_1\}|\leq\aleph_0$, and

$|\{\rho_2(\cdot,\beta)\restriction\alpha : \alpha\leq\beta<\omega_1\}|\leq\aleph_0$.

It follows that the tree

$T(\rho_2)=\{\rho_2(\cdot,\beta)\restriction\alpha : \alpha\leq\beta<\omega_1\}$

is Aronszajn. How about $T(\rho_0)$?.

<u>Definition 17.</u> The right lexicographical ordering $<^r_{lex}$ on $\omega^{<\omega}$ is defined by letting $s<^r_{lex}t$ if $s\sqsupset t$, ...

$s(i) < t(i)$ for $i = \min\{j : s(j) \neq t(j)\}$.

<u>Lemma 18</u>. For all $\beta < \gamma < \omega_1$ the set

$$\{\xi < \beta : \rho_0(\xi, \beta) = \rho_0(\xi, \gamma)\} =: X$$

is a closed subset of β.

<u>Proof</u>: Let $\alpha < \beta$ be an accumulation point of X. Let $\bar{\alpha} = \min(F(\beta,\gamma) \setminus \alpha)$. Pick a point $\xi \in X \cap \alpha$, such that

$$\xi > F(\beta,\gamma) \cap \alpha.$$

Then $\bar{\alpha} = \min(F(\beta,\gamma) \setminus \xi)$, so by Lemma 15,

$$\rho_0(\xi, \beta) = \rho_0(\bar{\alpha}, \beta) {}^\frown \rho_0(\xi, \bar{\alpha})$$
$$\shortparallel$$
$$\rho_0(\xi, \gamma) = \rho_0(\bar{\alpha}, \gamma) {}^\frown \rho_0(\xi, \bar{\alpha})$$

It follows that $\rho_0(\bar{\alpha}, \beta) = \rho_0(\bar{\alpha}, \gamma)$;

Applying Lemma 15 again, we have

$$\begin{array}{l} \rho_0(\alpha,\beta) = \rho_0(\bar{\alpha},\beta) ^\frown \rho_0(\alpha,\bar{\alpha}) \\ \qquad\qquad\quad \| \qquad\qquad\quad \| \\ \rho_0(\alpha,\gamma) = \rho_0(\bar{\alpha},\gamma) ^\frown \rho_0(\alpha,\bar{\alpha}), \end{array}$$

so $\rho_0(\alpha,\beta) = \rho_0(\alpha,\gamma)$. □

<u>Lemma 19</u> $\alpha < \beta < \gamma$ implies

$$\rho_0(\alpha,\gamma) <^{+}_{lex} \rho_0(\beta,\gamma).$$

<u>Proof</u>: Let $\gamma = \gamma_0(\alpha) > \cdots > \gamma_m(\alpha) = \alpha$ and $\gamma = \gamma_0(\beta) > \cdots > \gamma_n(\beta) = \beta$ be the walks from γ to α and γ to β, respectively. Let $j \leq \min\{m,n\}$ be the largest such that $\gamma_j(\alpha) = \gamma_j(\beta) = \gamma_j$. Then

$$(\alpha,\beta) \cap C_{\gamma_j} \neq \emptyset,$$ so

$$\neq |C_{\gamma_j} \cap \alpha| < |C_{\gamma_j} \cap \beta|$$

$$\begin{array}{ccc} \| & & \| \\ \rho_0(\alpha,\gamma)(j) & \cancel{\quad} & \rho_0(\beta,\gamma)(j) \end{array}$$

and $\rho_0(\alpha,\gamma)(i) = \rho_0(\beta,\gamma)(i)$ for $i<j$

It follows that $\rho_0(\alpha,\gamma) <^r_{lex} \rho_0(\beta,\gamma)$. $\square$

<u>Definition</u> 20 (Kurepa 1935) Let $\sigma\mathbb{Q}$ be the tree of all increasing transfinite sequences of rationals.

<u>Corollary</u> 21. $T(\rho_0)$ is a downward closed subtree of $\sigma\mathbb{Q}$. $\square$

<u>Theorem</u> 22 (Kurepa ~~1~~1935, 1953)
There is a strictly increasing $f: \sigma\mathbb{Q} \to \mathbb{R}$ but there is no str. increasing $f: \sigma\mathbb{Q} \to \mathbb{Q}$. $\square$

<u>Lemma 23</u> $T(\varrho_0)$ is a special A-tree.

<u>Proof:</u> In $T(\varrho_0)$ nodes of limit height have at most one immediate successor. This follows from Lemma 18. $\square$

<u>Lemma 24</u> (Peng) $T(\varrho_2)$ is also special.

§3 <u>ϱ-models</u>

<u>Proof:</u> For each $t = \varrho_2(\cdot, \beta) \restriction \alpha$ we assume β is minimal ordinal witnessing this and set

$$f(t) = \text{the isomorphism type of the } \varrho_2\text{-}\underline{\text{model}} \ (F(\alpha,\beta) \cup \{\beta\}, \leq, \varrho_2).$$

It suffices to show that if α and α' are limit ordinals and $t = \rho_2(\cdot,\beta)\restriction\alpha$ and $t' = \rho_2(\cdot,\beta')\restriction\alpha'$ are nodes of $T(\rho_2)$ such that

$$(F(\alpha,\beta)\cup\{\beta\},\leq,\rho_2) \cong (F(\alpha',\beta')\cup\{\beta'\},\leq,\rho_2)$$

then t and t' can't be comparable (and different). Assume $\alpha < \alpha'$.

<u>Case 1</u>. $F(\alpha,\beta)\cap\alpha \neq F(\alpha',\beta')\cap\alpha$.

Let $\xi = \min(F(\alpha,\beta)\,\Delta\,F(\alpha',\beta'))$. Then $\xi < \alpha$. May assume $\alpha \in F(\alpha,\beta)\setminus F(\alpha',\beta')$; the other ~~sub~~case is considered similarly.

Let $\xi' \in F(\alpha',\beta')$ be the ordinal that corresponds to ξ.

$\xi \bullet \longrightarrow \xi' \in F(\alpha',\beta')$

$F(\alpha,\beta)\cap\xi = F(\alpha',\beta')\cap\xi$

Then from Lemma 15 we get

$$\rho_2(\xi,\beta') = \rho_2(\xi',\beta') + \rho_2(\xi,\xi') > \rho_2(\xi',\beta')$$

From $(F(\alpha,\beta)\cup\{\beta\},\rho_2) \cong (F(\alpha',\beta')\cup\{\beta'\},\rho_2)$

we infer that

$$\rho_2(\xi,\beta) = \rho_2(\xi',\beta') \neq \rho_2(\xi,\beta')$$

so $t(\xi) \neq t'(\xi)$ so t and t' are incomparable.

<u>Case 2</u>: $F(\alpha,\beta)\cap\alpha = F(\alpha',\beta')\cap\alpha$.

Then α and α' correspond to each other in the isomorphism between $(F(\alpha,\beta)\cup\{\beta\},\rho_2)$ and $(F(\alpha',\beta')\cup\{\beta'\},\rho_2)$

So by Lemma 15

$$\rho_2(\alpha,\beta') = \rho_2(\alpha',\beta') + \rho_2(\alpha,\alpha') > \underset{\substack{\| \\ \rho_2(\alpha,\beta)}}{\rho_2(\alpha',\beta')}$$

It follows that the two functions $\rho_2(\cdot,\beta)$ and $\rho_2(\cdot,\beta')$ disagree at the limit ordinal α, so they must disagree bellow α, and therefore t and t' are incomparable. This follows from the following analogue of Lemma 18.

<u>Lemma 18'</u>. For all $\alpha<\beta<\omega_1$, the set $\{\xi<\beta : \rho_2(\xi,\beta)=\rho_2(\xi,\gamma)\}$ is a closed subset of β. $\square$

§4. Positional Graphs

<u>Definition 25.</u> Two finite sets $F, G \subseteq \mathrm{Ord}$ are in <u>Δ_n-position</u> for some $n<\omega$

If their intersection admits a disjoint decomposition

$$F \wedge G = I \cup J$$

such that $I \sqsubseteq F$, $I \sqsubseteq G$ and $|J| \leq n$. A <u>positional graph</u> is a graph of the form

$$(\mathcal{V}, \neg\Delta_n)$$

where $\mathcal{V}$ is a family of finite sets of ordinals.

<u>Theorem</u> 26. $\mathrm{Chr}([\omega_1]^{<\omega}, \neg\Delta_0) = \aleph_0$.

<u>Proof</u>. Defered. ∎

<u>Exercise</u> 27. Show that for all $n<\omega$,

$$K_{\aleph_1} \leq ([\omega_2]^{<\omega}, \neg\Delta_n),$$

and therefore $\mathrm{Chr}([\omega_2]^{<\omega}, \neg\Delta_n) > \aleph_0$.

§5. Canonical Linear Orderings

Definition 28. For $\alpha, \beta < \omega_1$ put

$\alpha <_{\varrho_2} \beta$ iff $\varrho_2(\xi, \alpha) < \varrho_2(\xi, \beta)$

for $\xi = \Delta_{\varrho_2}(\alpha, \beta) = \min\{\eta \leq \min\{\alpha, \beta\} : \varrho_2(\eta, \alpha) \neq \varrho_2(\eta,$

Lemma 29. The cartesian square of the total ordering $<_{\varrho_2}$ on ω_1 can be decomposed into countably many chains.

Proof: It suffices to decompose the set $\{(\alpha, \beta) \in \omega_1^2 : \alpha < \beta\}$ into countably many chains. Fix a coloring

$$c : [\omega_1]^{<\omega} \to \omega$$

of the graph $([\omega_1]^{<\omega}, \neg\Delta_0)$ i.e.

$c(F) = c(G)$ implies F and G are in the Δ_0-position.

For $\alpha < \beta < \omega_1$, let $t(\alpha,\beta)$ be the isomorphism type of the $\underline{\varrho_2\text{-model}}$

$$(F(\alpha,\beta) \cup \{\beta\}, <, \varrho_2).$$

It suffices to prove the following fact.

<u>Claim</u>. Suppose $\alpha < \beta$ and $\alpha' < \beta'$ are such that

$$c(F(\alpha,\beta) \cup \{\beta\}) = c(F(\alpha',\beta') \cup \{\beta'\})$$

and

$$t(\alpha,\beta) = t(\alpha',\beta').$$

Then

$$\alpha <_{\varrho_2} \alpha' \text{ implies } \beta <_{\varrho_2} \beta'.$$

~~Let $\xi = \Delta_{\varrho_2}(\alpha,\alpha')$ and $\eta = \Delta_{\varrho_2}(\beta,\beta')$.~~

<u>Proof</u>: Assume $\alpha <_{\mathcal{S}_2} \alpha'$. This in particular means that $\alpha \neq \alpha'$ and since $F(\alpha,\beta) \cup \{\beta\}$ and $F(\alpha',\beta') \cup \{\beta'\}$ are in the Δ_0-position, $\beta \neq \beta'$ must hold, and in fact $\beta \notin F(\alpha',\beta')$ and $\beta' \notin F(\alpha,\beta)$. Moreover, $F(\alpha,\beta) \,\Delta\, F(\alpha',\beta') \neq \emptyset$ as α and α' are members of this set. Let

$$\xi = \min(F(\alpha,\beta) \,\Delta\, F(\alpha',\beta'))$$

and let

$$I = F(\alpha,\beta) \cap F(\alpha',\beta').$$

Then $I \sqsubseteq F(\alpha,\beta), F(\alpha',\beta')$, so

$$I = F(\alpha,\beta) \cap \xi = F(\alpha',\beta') \cap \xi.$$

Let $\xi' \in F(\alpha',\beta')$ corresponds to ξ in the isomorphism between the two $\mathcal{S}_2$-models.

Then $\xi < \xi'$ and

$$\xi' = \min(F(\alpha', \beta') \setminus \xi).$$

Then by Lemma 15,

$$\rho_2(\xi, \alpha') = \rho_2(\xi', \alpha') + \rho_2(\xi, \xi')$$
$$= \rho_2(\xi, \alpha) + \rho_2(\xi, \xi'),$$

and so in particular $\rho_2(\xi, \alpha') \neq \rho_2(\xi, \alpha)$ and we conclude that

$$\Delta_{\rho_2}(\alpha, \alpha') \leq \xi.$$

Similarly,

$$\Delta_{\rho_2}(\beta, \beta') \leq \xi.$$

We claim that $\Delta_{\rho_2}(\alpha, \alpha') > \max(I)$ and $\Delta_{\rho_2}(\beta, \beta') > \max(I)$.

To see this consider $\eta \leq \max(I)$ and let

$$\bar{\eta} = \min(I \setminus \eta).$$

By Lemma 15,

$$\rho_2(\eta, \alpha) = \rho_2(\bar{\eta}, \alpha) + \rho_2(\eta, \bar{\eta}) \quad \text{and}$$

$$\rho_2(\eta, \alpha') = \rho_2(\bar{\eta}, \alpha') + \rho_2(\eta, \bar{\eta}).$$

From

$$(F(\alpha,\beta) \cup \{\beta\}, <, \rho_2) \cong (F(\alpha',\beta') \cup \{\beta'\}, <, \rho_2)$$

we conclude that $\rho_2(\bar{\eta}, \alpha) = \rho_2(\bar{\eta}, \alpha')$, so $\rho_2(\eta, \alpha) = \rho_2(\eta, \alpha')$. Similarly, we get that $\rho_2(\eta, \beta) = \rho_2(\eta, \beta')$.

Consider an ordinal η such that

$$\max(I) < \eta < \xi.$$

By Lemma 15,

$$\rho_2(\eta, \alpha) = \rho_2(\xi, \alpha) + \rho_2(\eta, \xi), \quad \text{and}$$

$$\rho_2(\eta, \alpha') = \rho_2(\xi', \alpha') + \rho_2(\eta, \xi)$$

By $\rho_2(\xi, \alpha) = \rho_2(\xi', \alpha')$ follows from the isomorphism of the two ρ_2-models

so we conclude that $\rho_2(\eta, \alpha) = \rho_2(\eta, \alpha')$

It follows that $\Delta_{\rho_2}(\alpha, \alpha') \geq \xi$, and

Similarly $\Delta_{\rho_2}(\beta, \beta') \geq \xi$. It

follows that

$$\Delta_{\rho_2}(\alpha, \alpha') = \xi = \Delta_{\rho_2}(\beta, \beta').$$

Recall that we are working under

the assumption $\alpha <_{\rho_2} \alpha'$ and

$\xi = \min(F(\alpha, \beta) \Delta F(\alpha', \beta')) \in F(\alpha, \beta)$

and therefore $\xi < \xi'$ which by

Lemma 15 transfers to

$$\rho_2(\xi, \alpha') = \rho_2(\xi, \alpha) + \rho_2(\xi, \alpha') >$$
$$> \rho_2(\xi, \alpha)$$

So the symmetric assumption $\xi \in F(\alpha', \beta')$

would not agree with $\alpha <_{\varrho_2} \alpha'$.

Applying Lemma 15 again we get

$$\varrho_2(\xi,\beta') = \varrho_2(\xi',\beta') + \varrho(\xi,\xi')$$
$$= \varrho_2(\xi,\beta) + \varrho(\xi,\xi')$$
$$> \varrho_2(\xi,\beta).$$

From this we conclude that $\beta <_{\varrho_2} \beta'$. $\square$

<u>Definition 30</u>. For $\alpha,\beta<\omega_1$, let

$$\alpha <_{\varrho_0} \beta \text{ iff } \varrho_0(\xi,\alpha) <^r_{lex} \varrho_0(\xi,\beta),$$

for $\xi = \Delta_{\varrho_0}(\alpha,\beta) = \min\{\eta \le \min\{\alpha,\beta\} : \varrho_0(\eta,\alpha) \ne \varrho_0(\eta,\beta)\}$.

Then working as above we get

<u>Theorem. 31</u>. The cartesian square of the total ordering $<_{\varrho_0}$ on ω_1 can be decomposed into countably many chains.

Definition 32. For $\alpha, \beta < \omega_1$, set

$\alpha <_{\varrho_1} \beta$ iff $\varrho_1(\xi, \alpha) < \varrho_1(\xi, \beta)$ for

$$\xi = \Delta_{\varrho_1}(\alpha, \beta) = \min\{\eta \leq \min\{\alpha, \beta\} : \varrho_1(\eta, \alpha) \neq \varrho_1(\eta, \beta)\}$$

Theorem 33. The cartesian square of the total ordering $<_{\varrho_1}$ on ω_1 can be decomposed into countably many chains. $\square$.

Question 34. Are the ordering $<_{\varrho_0}$, $<_{\varrho_1}$ and $<_{\varrho_2}$ any different?

Let

$$C(\varrho_i) = (\omega_1, <_{\varrho_i}) \quad (i = 0, 1, 2, 3).$$

<u>Theorem 35.</u> Assume $\mathfrak{m} > \omega_1$. Then

(a) For every $i \leq 3$, the ordering $C(\varrho_i)$ is a minimal uncountable ordering i.e. for every uncountable linear ordering L such that $L \leq C(\varrho_i)$ we have $C(\varrho_i) \leq L$.

(b) Fix $i \leq 3$. Then for every uncountable ordering L whose square $L \times L$ can be decomposed into countably many chains, we have not either $C(\varrho_i) \leq L$ or $C(\varrho_i)^* \leq L$. $\dashv$

<u>Corollary 36.</u> If $\mathfrak{m} > \omega_1$ then
$$C(\varrho_i) \equiv C(\varrho_j) \text{ for all } i,j \leq 3.$$

Theorem (Baumgartner, Moore). Assume $\mathrm{mm} >$

If B is any uncountable set of reals of cardinality $\aleph_1$ then

$$\omega_1, \omega_1^*, B, C(\rho_2), C(\rho_2)^*$$

forms a basis for the class of all uncountable linear orderings, i.e., for every uncountable linearly ordered set L there is a member K of the basis such that $K \leq L$.

Problem 38. Determine the consistency strength of this conclusion. Does it involve large cardinals at all?

§6. Lipschitz trees

Definition 39. A partial map

$$g : S \to T$$

from a tree S into a tree T is *Lipschitz* if g is *level-preserving* and if

$$\Delta(g(x), g(y)) \geq \Delta(x, y)$$

for all $x, y \in \mathrm{dom}(g)$. [Recall, that for a tree T, $\Delta : T^2 \to \mathrm{Ord}$ is defined by

$$\Delta(s,t) = \mathrm{otp}\{x \in T : x <_T s \ \& \ x <_T t\}$$
]

Definition 40. A *Lipschitz tree* is any A-tree T with the property that every level preserving map from an uncountable subset A of T is Lipschitz on an uncountable subset of A.

~~Examples $T(\rho_0)$, $T(\rho_1)$, $T(\rho_2)$ and $T(\rho_3)$ are all Lipschitz~~

Definition 41. A *coherent tree* is any A-tree T that is isomorphic to a downward closed subset S of $I^{<\omega_1}$ for some countable set I such that for all $\alpha < \omega_1$ and $s, t \in S \cap I^{\alpha}$,

$$\{\xi < \alpha : s(\xi) \neq t(\xi)\} \text{ is finite.}$$

<u>Lemma 4.2</u> Suppose T is a coherent tree with the property that every uncountable subset of T contains an uncountable antichain. Then T is Lipschitz.

<u>Proof</u>: Consider uncountable $A \subseteq T$ and $g : A \to T$ such that

$(\forall s \in A)\ ht(s) = ht(g(s))$.

For each limit ordinal $\delta < \omega_1$ pick $t_\delta \in A$ of height $\geq \delta$ and a node $s_\delta \in T$ of height $= \delta$. Let

$$D_\delta = \{\xi < \delta : s_\delta(\xi) \neq t_\delta(\xi) \text{ or } s_\delta(\xi) \neq g(t_\delta)(\xi)\}$$

Then D_δ is a finite subset of δ for every countable limit ordinal δ, so applying the Pressing Down Lemma we get stationary $E \subseteq \Delta = \{\delta < \omega_1 : \delta \text{ limit}\}$ and $D \subseteq \omega_1$ such that

$$\forall \delta \in E \quad D_\delta = D.$$

Shrinking S, we may assume that for some $s, t \in T$ of height $\gamma_D = \max(D) + 1$, we have that

$$\forall \delta \in E \quad s_\delta \upharpoonright \gamma_D = s \;\&\; t_\delta \upharpoonright \gamma_D = t$$

Applying our assumption about T, we now find uncountable $F \subseteq E$ such that both sets

$$\{t_\delta \upharpoonright \delta : \delta \in F\} \text{ and } \{g(t_\delta) \upharpoonright \delta : \delta \in F\}$$

are antichains. It follows that for $\gamma \neq \delta$ in F

$$\Delta(t_\gamma, t_\delta) = \Delta(g(t_\gamma), g(t_\delta)),$$

so $g \upharpoonright \{t_\delta : \delta \in F\}$ is Lipschitz. $\square$

Lemma 43. If $\mathfrak{m} > \omega_1$ then every Lipschitz tree is coherent. $\square$

Question 44 Are $T(\rho_0)$ and $T(\rho_2)$ coherent without the assumption of $\mathfrak{m} > \omega_1$?

Theorem 45 (Martinez-Ranero) $T(\rho_0)$ is coherent.

Theorem 46 (Peng) $T(\rho_2)$ is coherent.

Proof: Define $a : [\omega_1]^2 \to \mathbb{Z}$ by

$a_2(\alpha, \beta) = \rho_2(\alpha, \beta) - \rho_2(\alpha \dot{-} 1, \beta) + \rho_2(\alpha \dot{-} 1, \alpha)$

Note that since $\alpha \dot{-} 1 = \alpha$ for α a limit ordinal we get that $a_2(\alpha, \beta) = 0$.

It follows that the tree

$T(a_2) = \{ a_2(\cdot, \beta) \upharpoonright \alpha : \alpha \leq \beta < \omega_1 \}$

does not split at limit levels. As we know this is also true for $T(\rho_2)$ (and $T(\rho_0)$). Note also that $\Delta_{a_2} = \Delta_{\rho_2}$ i.e. for all $\alpha < \beta < \omega_1$,

$$\min\{\xi \leq \alpha : \rho_2(\xi,\alpha) \neq \rho_2(\xi,\beta)\} = \min\{\xi \leq \alpha : a_2(\xi,\alpha) \neq a_2(\xi,\beta)\}$$

It follows that

$$\rho_2(\cdot,\beta) \restriction \alpha \longmapsto a_2(\cdot,\beta) \restriction \alpha$$

is an isomorphic embedding. So it remains to prove that $T(a_2)$ is coherent i.e. that for all $\beta < \gamma < \omega_1$

$$D = \{\xi < \beta : a_2(\xi,\beta) \neq a_2(\xi,\gamma)\}$$

is a finite set. Assume otherwise and let α be its first limit point. Choose $\xi \in D \cap \alpha$ such that

$\xi - 1 > \max(F(\alpha, \beta) \cap \alpha), \max(F(\alpha, \gamma) \cap \alpha)$

(Recall that each member of D must be a successor ordinal). Applying Lemma 15, we get

$$a_2(\xi, \beta) = \rho_2(\xi, \beta) - \rho_2(\xi - 1, \beta) + 1$$

$$= \rho_2\cancel{(\alpha, \beta)} + \rho_2(\xi, \alpha) - \rho_2\cancel{(\alpha, \beta)} - \rho_2(\xi-1, \alpha) + 1$$

$$= \rho_2(\xi, \alpha) - \rho_2(\xi - 1, \alpha) + 1$$

Similarly

$$a_2(\xi, \gamma) = \rho_2(\xi, \gamma) - \rho_2(\xi - 1, \gamma) + 1 =$$

$$= \rho_2(\alpha, \gamma) + \rho_2(\xi, \alpha) - \rho_2(\alpha, \gamma) - \rho_2(\xi - 1, \alpha) + 1$$

$$= \rho_2(\xi, \alpha) - \rho_2(\xi - 1, \alpha) + 1$$

So, $a_2(\xi, \beta) = a_2(\xi, \gamma)$, a contradiction!

It follows that $T(a_2)$ is coherent. $\square$

<u>Proof of Theorem 4.5</u>. The proof uses a similar idea. Let P be the sequence of primes $p_0, p_1, p_2, \ldots$ $(2, 3, 5, \ldots)$

For a sequence $t = (n_i : i < \ell)$ of integers, let

$$P^t = \prod_{i<\ell} p_i^{n_i}.$$

For $t = (n_i : i < \ell) \in \mathbb{Z}^{<\omega}$ let

$$-t = (-n_i : i < \ell) \in \mathbb{Z}^{<\omega}.$$

Finally, we are ready to define

$$a_0 : [\omega_1]^2 \to \omega$$

by

$$a_0(\alpha, \beta) = P^{\rho_0(\alpha, \beta)} \cdot P^{-\rho_0(\alpha - 1, \beta)} \cdot P^{\rho_0(\alpha - 1, \alpha)}$$

Working as above one checks that

$$\rho_0(\cdot, \beta) \restriction \alpha \longmapsto a_0(\cdot, \beta) \restriction \alpha$$

is an isomorphism between $T(\rho_0)$ and $T(a_0)$ and that $T(a_0)$ is coherent. $\Box$

<u>Definition 47</u>. For two trees S and T we let $S \leq T$ if there is a strictly increasing (equivalently, Lipshitz) map $f: S \to T$. Let $S < T$ whenever $S \leq T$ and $T \not\leq S$ and let $S \equiv T$ whenever $S \leq T$ and $T \leq S$; we call S and T <u>equivalent</u> whenever $S \equiv T$.

<u>Lemma 48</u> If $m > \omega_1$ then every coherent trees is equivalent to its homogeneous closure and two homogeneous coherent trees are <u>equivalent</u> iff they are <u>isomorphic</u>. $\square$

<u>Theorem 49.</u> Assume $\mathfrak{m} > \omega_1$.

(a) Every Lipschitz tree is comparable to every Aronszajn tree.

(b) If $\mathcal{L}$ denotes the class of Lipschitz trees then $(\mathcal{L}, \leq)$ is a discrete chain ~~&~~ every $T \in \mathcal{L}$ has an <u>immediate successor</u> $T^{(1)} \in \mathcal{L}$.

(c) If $\mathcal{A}$ denotes the class of Aronszajn trees then $\mathcal{L}$ is both cofinal and coinitial in $(\mathcal{A}, \leq)$.

d) (Martinez-Ranero – T.) $(\mathcal{L}/\mathbb{Z}, \leq)$ is isomorphic to the $\aleph_2$-saturated linear ordering of cardinality $\aleph_2$. ∎

Definition 50. Let g be a partial map from ω_1 into ω_1 and T a downward closed subset of some $I^{<\omega_1}$. Then the <u>g-shift</u>, $T^{(g)}$, is the downward closure of

$$\{t^{(g)} : t \in T \restriction \Omega\}, \text{ where}$$

$$\Omega = \{\delta < \omega_1 : g''\delta \subseteq \delta\},$$

and where $t^{(g)}$ is defined by

$$t^{(g)}(\xi) = t(g(\xi))$$

if $\xi \in \mathrm{dom}(g)$ and $t^{(g)}(\xi) = 0$, otherwise.

Theorem 51. Assuming $\mathrm{MM} > \omega_1$, for every pair S and T of Lipschitz trees there is strictly increasing partial map $g : \omega_1 \to \omega_1$ such that $S \equiv T^{(g)}$. $\square$

Corollary 52. Assuming $\mathfrak{m}\mathfrak{m} > \omega_1$,

$$\mathcal{L}/_{\equiv} = \{ T(\varrho_0)^{(g)} : g : \omega_1 \to \omega_1 \text{ partial increasing} \}.$$

Conclusion: So under $\mathfrak{m}\mathfrak{m} > \omega_1$ there is really only one $\mathcal{L}$-tree, the one obtained from a characteristic of walks on ω_1.

§7. Canonical filters on ω

Definition 53. Fix a Lipschitz tree T. For $X \subseteq T$, let

$$\Delta_T(X) = \{ \Delta(s,t) : s,t \in X,\ s,t \text{ incomparable} \}$$

Let

$$\mathcal{U}_{\omega_1}(T) = \{ \Gamma \subseteq \omega_1 : (\exists \text{ uncountable } X \subseteq T)\ \Delta_T(X) \subseteq \Gamma \}$$

<u>Lemma 54</u>. For every Lipschitz tree T, $\mathcal{U}_{\omega_1}(T)$ is a uniform filter on ω_1.

<u>Proof</u>: Given uncountable $X, Y \subseteq T$ we need to find uncountable $Z \subseteq T$ such that

$$\Delta_T(Z) \subseteq \Delta_T(X) \cap \Delta_T(Y).$$

For each $\alpha < \omega_1$ pick $x_\alpha \in X$, $y_\alpha \in Y$ of height $\geq \alpha$. Note that in any Lipschitz tree any uncountable set can be refined by an uncountable antichain. So we can find uncountable $\Gamma \subseteq \omega_1$ such that the sets

$$x_\delta \restriction \delta \ (\delta \in \Gamma) \quad \text{and} \quad y_\delta \restriction \delta \ (\delta \in \Gamma)$$

are antichains of T.

Applying the definition that T is Lipschitz successively first for the function $x_\delta \restriction \delta \mapsto y_\delta \restriction \delta$ and then to its inverse, we obtain an uncountable set $\Sigma \subseteq \Gamma$ such that

$$\Delta_T(x_\gamma \restriction \gamma, x_\delta \restriction \delta) = \Delta_T(y_\gamma \restriction \gamma, y_\delta \restriction \delta)$$

for all $\gamma, \delta \in \Sigma$, $\gamma \neq \delta$. Since $\Delta_T(x_\gamma \restriction \gamma, x_\delta \restriction \delta) = \Delta_T(x_\gamma, x_\delta)$ and $\Delta_T(y_\gamma \restriction \gamma, y_\delta \restriction \delta) = \Delta_T(y_\gamma, y_\delta)$, taking $Z = \{x_\gamma : \gamma \in \Sigma\}$ we get the desired conclusion $\Delta_T(Z) \subseteq \Delta_T(X) \cap \Delta_T(Y)$. $\square$

<u>Theorem 55</u>. If $m > \omega_1$ then $\mathcal{U}_{\omega_1}(T)$ is an ultrafilter for every Lipschitz tree T.

<u>Proof</u>: Let $\Gamma \subseteq \omega_1$ be a given set. We need to find uncountable $X \subseteq T$ such that either

$$\Delta_T(X) \subseteq \Gamma \text{ or } \Delta_T(X) \cap \Gamma = \emptyset.$$

Let $\mathcal{P}_\Gamma$ be the poset of all finite subsets p of T that take no more than one point from a given level of T such that $\Delta_T(p) \subseteq \Gamma$. If $\mathcal{P}_\Gamma$ satisfies the countable chain condition then an application of $m > \omega_1$ will give us an uncountable filter $\mathcal{G} \subseteq \mathcal{P}_\Gamma$, and therefore an uncountable set

$$X = \bigcup \mathcal{G}$$

such that $\Delta_T(X) \subseteq \Gamma$. So we analize

the alternative that $\mathcal{P}_\Gamma$ is not ccc.
Fix a sequence p_δ ($\delta<\omega_1$) of pairwise incompatible members of $\mathcal{P}_\Gamma$. Applying the Δ-system Lemma we may assume p_δ's are pairwise disjoint and all of some fix size n. So, re-enumerating, we may assume that for each δ, the nodes in p_δ have all height $\geq \delta$.
Applying the Lipschitz condition on T successively n^2 times we arrive at an uncountable set $\Gamma \subseteq \omega_1$ such that for all $i, j<n$,

$$p_\delta(i) \upharpoonright \delta \longmapsto p_\delta(j) \upharpoonright \delta \quad (\delta \in \Gamma)$$

is a Lipshitz map. (Here $p_\delta(i)$ is the ith element of p_δ in some fixed enumeration.)

Refining Γ we may assume that for all $i < n$,

$$p_\gamma(i) \restriction \gamma \quad (\gamma \in \Gamma)$$

is a sequence of pairwise incompatible elements and that for some fixed ordinal $\bar{\gamma} < \omega_1$ and all $\gamma, \delta \in \Gamma$ and $i, j < n$:

(a) $p_\gamma(i) \restriction \gamma \neq p_\gamma(j) \restriction \gamma$ implies $p_\gamma(i) \restriction \bar{\gamma} \neq p_\gamma(j) \restriction \bar{\gamma}$,

(b) $p_\gamma(i) \restriction \bar{\gamma} = p_\delta(i) \restriction \bar{\gamma}$.

Combining all these properties we get that for $\gamma \neq \delta$ in Γ

$$\Delta_T(p_\gamma \cup p_\delta) = \Delta_T(p_\gamma) \cup \Delta_T(p_\delta) \cup \{\Delta_T(p_\gamma(0), p_\delta(0))\}$$

Since $\Delta_T(p_\gamma) \cup \Delta_T(p_\delta) \subseteq \Gamma$ and since p_γ and p_δ are incompatible we conclude that $\Delta_T(p_\gamma(0), p_\delta(0)) \notin \Gamma$ for all $\gamma \neq \delta$ in Γ.

Putting $X = \{p_\gamma^{-1}(0) : \gamma \in \Gamma\}$ we get an uncountable subset of T such that

$$\Delta_T(X) \cap \Gamma = \emptyset,$$

as required. ∎

<u>Theorem 56</u>. Assume $\mathfrak{m} > \omega_1$.

(a) For every pair S and T of Lipschitz trees,

$$S \equiv T \quad \text{iff} \quad \mathcal{U}_{\omega_1}(S) = \mathcal{U}_{\omega_1}(T)$$

(b) For every pair S and T of Lipschitz trees

$$\mathcal{U}_{\omega_1}(S) \equiv_{RK} \mathcal{U}_{\omega_1}(T). \;∎$$

<u>Conclusion</u>. Under $\mathfrak{m} > \omega_1$, there is a canonical ultrafilter $\mathcal{U}_{\omega_1}(\rho_3) = \mathcal{U}_{\omega_1}(T(\rho_3))$ that is Σ_1-definable in $(H(\omega_2), \in)$.

Theorem 57. Assume $m > \omega_1$.

(a) For every Lipschitz tree T and every $f : \omega_1 \to \omega$ the image

$$f[\mathcal{U}_{\omega_1}(T)] = \{M \subseteq \omega_1 : f^{-1}(M) \in \mathcal{U}_{\omega_1}(T)\}$$

is a selective ultrafilter on ω.

(b) For every pair S and T of Lipschitz trees and any pair of mappings $f : \omega_1 \to \omega$ and $g : \omega_1 \to \omega$, if the ultrafilters

$$f[\mathcal{U}_{\omega_1}(S)] \text{ and } g[\mathcal{U}_{\omega_1}(T)]$$

are nonprincipal then they are RK-equivalent.

Recall that an ultrafilter $\mathcal{U}$ on ω is <u>selective</u> if for every sequence $\{M_n : n<\omega\} \subseteq \mathcal{U}$ there is $N \in \mathcal{U}$ such that $N \setminus \{0,1,\dots,n\} \subseteq M_n$ for all $n \in N$; equivalently if for every $f:\omega \to \omega$ there is $M \in \mathcal{U}$ such that either $f \upharpoonright M$ is 1-1 or constant.

<u>Conclusion</u>. Under $m > \omega_1$ there is a canonical ultrafilter $\mathcal{U}_\omega(\varrho_3)$ on ω Σ_1-definable in $(H(\omega_2), \in)$ using the structure $(C_\alpha : \alpha < \omega_1)$ and characteristics of walks. It is rather remarkable

that this RK-unique ultrafilter is actually <u>selective</u> i.e. <u>generic</u>.

<u>Remark</u>: The ultrafilter $\mathcal{U}_{\omega_1}(\varrho_3)$ has a canonical map $d_\Delta : \omega_1 \to \omega$ such that

$$\mathcal{U}_\omega(\varrho_3) := d_\Delta[\mathcal{U}_{\omega_1}(\varrho_3)]$$

is nonprincipal. The map d_Δ is simply the distance map from the set Δ of all countable limit ordinals. More precisely

$$d_\Delta(\alpha) = \alpha - \lambda(\alpha),$$

where $\lambda(\alpha) = \max\{\lambda \in \Delta : \lambda \leq \alpha\}$.

<u>Summary</u>: The basic structure

$(\omega_1, C_\alpha\ (\alpha<\omega_1))$

allows walks to be defined and their characteristics such as

ρ_0, ρ_1, ρ_2 and ρ_3

to be analysed. They lead to structures such as

$C(\rho_i),\ T(\rho_i),\ \mathcal{U}_{\omega_1}(\rho_i),\ \mathcal{U}_\omega(\rho_i)$

that are at the same time <u>critical</u> for their own classes of structures and <u>unique</u>, and so, in particular, independent of the base structure $(\omega_1, C_\alpha(\alpha<\omega_1))$.

§8 The distance function ρ

Definition 58. Define $\rho : [\omega_1]^2 \to \omega$ recursively by

$$\rho(\alpha,\beta) = \max\{|C_\beta \cap \alpha|, \rho(\alpha, \min(C_\beta \setminus \alpha)), \rho(\xi,\alpha) : \xi \in C_\beta \cap \alpha\}$$

with the boundary condition $\rho(\alpha,\alpha) = 0$ for all α.

Lemma 59. For all $\alpha < \beta < \gamma < \omega_1$ and $n < \omega$,

(a) $\{\xi \le \alpha : \rho(\xi,\alpha) \le n\}$ is finite,

(b) $\rho(\alpha,\gamma) \le \max\{\rho(\alpha,\beta), \rho(\beta,\gamma)\}$,

(c) $\rho(\alpha,\beta) \le \max\{\rho(\alpha,\gamma), \rho(\beta,\gamma)\}$.

Proof: (a) This follows from the inequality

$$\rho(\alpha,\beta) \geq \rho_1(\alpha,\beta)$$

and the corresponding property of ρ_1 proved above (Lemma 1 (a)).

We prove (b) and (c) by induction. First we deal with (b). Let

$$n = \max\{\rho(\alpha,\beta), \rho(\beta,\gamma)\}$$

We need to show that $\rho(\alpha,\gamma) \leq n$. Let $\gamma_\alpha = \min(C_\gamma \setminus \alpha)$ and $\gamma_\beta = \min(C_\gamma \setminus \beta)$.

Case 1^b. $\gamma_\alpha = \gamma_\beta$. Then by the ind. hyp.,

$$\rho(\alpha, \gamma_\alpha) \leq \max\{\rho(\alpha,\beta), \rho(\beta, \gamma_\beta)\}$$

From the definition of $\rho(\beta,\gamma)$ we get

$$n \geq \rho(\beta,\gamma) \geq \rho(\beta, \gamma_\beta).$$

Plugging this into the above inequality, we get $n \geq \rho(\alpha, \gamma_\alpha)$.

Consider $\xi \in C_\mu \cap \alpha = C_\mu \cap \beta$.

By the ind. hyp

$$\rho(\xi, \alpha) \le \max\{\rho(\xi,\beta), \rho(\alpha,\beta)\}$$

From the definition of $\rho(\beta,\mu)$ we infer that $\rho(\beta,\mu) \geq \rho(\xi,\beta)$. Plugging this into the above inequality we get that $n \geq \rho(\xi,\alpha)$. Since

$$|C_\mu \cap \alpha| = |C_\mu \cap \beta| \le \rho(\beta,\mu) \le n$$

we get the last factor in the definition of $\rho(\alpha,\mu)$ to be dominated by n, and therefore $\rho(\alpha,\mu) \le n$.

<u>Case 2^b</u>. $\gamma_\alpha < \gamma_\beta$. Then

$$\gamma_\alpha \in C_\mu \cap \beta,$$

and so $\rho(\gamma_\alpha, \beta) \le \rho(\beta,\mu) \le n$.

similarly, for every

$$\xi \in C_\gamma \cap \alpha \subseteq C_\gamma \cap \beta,$$

$$\rho(\xi, \alpha) \leq \max\{\rho(\xi, \beta), \rho(\alpha, \beta)\} \leq n.$$

Finally, $|C_\gamma \cap \alpha| \leq |C_\gamma \cap \beta| \leq \rho(\beta, \gamma) \leq n$.

Combining all these inequalities we get the desired conclusion $\rho(\alpha, \gamma) \leq n$.

To prove (c), let

$$n = \max\{\rho(\alpha, \gamma), \rho(\beta, \gamma)\}.$$

We need to show that $\rho(\alpha, \beta) \leq n$.

Let γ_α and γ_β be as above.

<u>Case 1°</u>. $\gamma_\alpha = \gamma_\beta$. By the inductive hypothesis

$$\rho(\alpha, \beta) \leq \max\{\rho(\alpha, \gamma_\alpha), \rho(\beta, \gamma_\beta)\}$$

Since $\rho(\alpha, \gamma_\alpha) \leq \rho(\alpha, \gamma) \leq n$ and $\rho(\beta, \gamma_\beta) \leq \rho(\beta, \gamma) \leq n$ we get the desired inequality $\rho(\alpha, \beta) \leq n$.

<u>Case 2°</u>: $\gamma_\alpha < \gamma_\beta$. By ind. hyp.

$$\rho(\alpha, \beta) \leq \max\{\rho(\alpha, \gamma_\alpha), \rho(\gamma_\alpha, \beta)\}.$$

As before $\rho(\alpha, \gamma_\alpha) \leq \rho(\alpha, \gamma) \leq n$.
Since $\gamma_\alpha \in C_\gamma \cap \beta$ from the definition of $\rho(\beta, \gamma)$ we get that

$$\rho(\gamma_\alpha, \beta) \leq \rho(\beta, \gamma) \leq n.$$

Combining the two inequalities, we get that $\rho(\alpha, \beta) \leq n$, as required. $\square$.

Lemma 60

$\alpha < \beta < \gamma$ and $\varrho(\alpha,\beta) > \varrho(\beta,\gamma)$ implies

$$\varrho(\alpha,\gamma) = \varrho(\alpha,\beta).$$

Proof: Exercise. $\square$

Corollary 61. $T(\varrho) = \{\varrho(\cdot,\beta)\restriction\alpha : \alpha \leq \beta < \omega_1\}$ is also a coherent tree. $\square$

§9. The injective ϱ

Definition 62. Define $\bar{\varrho} : [\omega_1]^2 \to \omega$ by

$$\bar{\varrho}(\alpha,\beta) = 2^{\varrho(\alpha,\beta)} \cdot (2 \cdot |\{\xi \leq \alpha : \varrho(\xi,\alpha) \leq \varrho(\alpha,\beta)\}| + 1).$$

Lemma 63. For $\alpha < \beta < \gamma < \omega_1$,

(a) $\bar{\varrho}(\alpha,\gamma) \neq \bar{\varrho}(\beta,\gamma)$

(b) $\bar{\varrho}(\alpha,\gamma) \leq \max\{\bar{\varrho}(\alpha,\beta), \bar{\varrho}(\beta,\gamma)\}$

(c) $\bar{\varrho}(\alpha,\beta) \leq \max\{\bar{\varrho}(\alpha,\gamma), \bar{\varrho}(\beta,\gamma)\}$.

Proof: Exercise. $\square$

Lemma 64. $\bar{\varrho}(\alpha, \beta) \neq \bar{\varrho}(\beta, \gamma)$
for all $\alpha < \beta < \gamma < \omega_1$.

Proof: Suppose the conclusion fails for some triple $\alpha < \beta < \gamma$. Let

$$n = \bar{\varrho}(\alpha, \beta) = \bar{\varrho}(\beta, \gamma).$$

Write $n = 2^i(2j+1)$ for some integers i and j. Then

$$i = \varrho(\alpha, \beta) = \varrho(\beta, \gamma)$$

and

$$|\{\xi \leq \alpha : \varrho(\xi, \alpha) \leq i\}| = j = |\{\xi \leq \beta : \varrho(\xi, \beta) \leq i\}|$$

Since $\varrho(\alpha, \beta) = i$,

$$\alpha \in \{\xi \leq \beta : \varrho(\xi, \beta) \leq i\},$$

So the set

$$\{\xi \le \alpha : \rho(\xi, \alpha) \le i\}$$

is an initial segment of the set

$\{\xi \le \beta : \rho(\xi, \beta) \le i\}$ (EXERCISE).

Since the two sets have the same cardinality, they must be equal, a contradiction since clearly β does not belong to $\{\xi \le \alpha : \rho(\xi, \alpha) \le i\}$. $\square$

<u>Lemma 65.</u>

$$\eta_\alpha \ne \eta_\beta < \min\{\alpha, \beta\} \text{ and}$$

$$\bar{\rho}(\eta_\alpha, \alpha) = \bar{\rho}(\eta_\beta, \beta) = n$$

imply not

$$\bar{\rho}(\eta_\alpha, \beta), \bar{\rho}(\eta_\beta, \alpha) > n. \quad \square$$

§10. A Souslin tree from ϱ

__Definition 66.__ For $p \in \omega^{<\omega}$ define $<_p \subseteq \omega_1 \times \omega_1$ by letting

$\alpha <_p \beta$ iff (a) $\alpha < \beta$,
(b) $\bar{\varrho}(\alpha, \beta) \in |p|$, and
(c) $(\forall \xi < \alpha)[\bar{\varrho}(\xi, \alpha) < p \Longrightarrow$
$\overleftarrow{p}(\bar{\varrho}(\xi, \alpha)) = p(\bar{\varrho}(\xi, \beta)).]$

__Lemma 67.__

(a) $<_p$ is a tree ordering on ω_1 of ~~height $\leq |p|+1$~~

(b) $p \subseteq q$ implies $<_p \subseteq <_q$.

__Definition 68.__ For $x \in \omega^\omega$, set

$$<_x = \bigcup_{n<\omega} <_{x \restriction n}$$

__Lemma 69.__ For every __infinite__ $\Gamma \subseteq \omega_1$, the set

$$U_\Gamma = \{x \in \omega^\omega : (\exists \alpha, \beta \in \Gamma)\ \alpha <_x \beta\}$$

is dense open subset of ω^ω. $\square$

<u>Lemma 70</u>. For every infinite $\Gamma \subseteq \omega_1$ that is an <u>antichain</u> in $T(\bar{\varrho})$ (i.e. $\varrho(\cdot, \alpha) \not\subseteq \varrho(\cdot, \beta)$ for all $\alpha \neq \beta$ in Γ), the set

$$V_\Gamma = \{x \in \omega^\omega : \exists \alpha, \beta \in \Gamma \ \ \alpha \not<_x \beta\}$$

is a dense open subset ~~~~ of ω^ω.

<u>Corollary 71</u>. If $c \in \omega^\omega$ is a Cohen real, $(\omega_1, <_c)$ is a Souslin tree. $\square$

§11 <u>A Hausdorff Gap From ϱ</u>

<u>Definition 72</u>. A function $a : [\omega_1]^2 \to \omega$ is <u>transitive</u> whenever

$$a(\alpha, \gamma) \leq \max\{a(\alpha, \beta), a(\beta, \gamma)\}$$

for all $\alpha < \beta < \gamma < \omega_1$.

<u>Example 73</u> Suppose

$$\{A_\alpha : \alpha < \omega_1\} \subseteq [\omega]^\omega$$

is such that $A_\alpha \subseteq^* B_\beta$ for $\alpha < \beta < \omega_1$.

Then $a : [\omega_1]^2 \to \omega$ defined by

$$a(\alpha,\beta) = \min\{n : A_\alpha \setminus n \subseteq A_\beta\}$$

is transitive.

<u>Definition 74.</u> Fix transitive

$$a : [\omega_1]^2 \to \omega.$$

Define $\rho_a : [\omega_1]^2 \to \omega$ recursively by

$$\rho_a(\alpha,\beta) = \max\{|C_\beta \cap \alpha|, a(\min(C_\beta \setminus \alpha),\beta), \rho_a(\alpha, \min(C_\beta \setminus \alpha)), \rho_a(\xi,\alpha) : \xi \in C_\beta \cap \alpha\}$$

where the boundary condition is that

$\rho_a(\alpha,\alpha) = 0$ for all α.

<u>Lemma 75</u>. For all $\alpha < \beta < \gamma < \omega_1$ and $n < \omega$,

(a) $\{\xi \le \alpha : \varrho_a(\xi, \alpha) \le n\}$ is finite,

(b) $\varrho_a(\alpha, \gamma) \le \max\{\varrho_a(\alpha, \beta), \varrho_a(\beta, \gamma)\}$,

(c) $\varrho_a(\alpha, \beta) \le \max\{\varrho_a(\alpha, \gamma), \varrho_a(\beta, \gamma)\}$

(d) $\varrho_a(\alpha, \beta) \geq a(\alpha, \beta)$. $\square$

<u>Lemma 76</u> $\varrho_a(\alpha, \beta) \geq \varrho_a(\alpha+1, \beta)$ for $0 < \alpha < \beta$, $\alpha \in \Lambda$. $\square$

<u>Fix</u> a strictly increasing $\subseteq^*$-sequence $\{A_\alpha : \alpha < \omega_1\} \subseteq [\omega]^\omega$ and let $a : [\omega_1]^2 \to \omega$ be the corresponding transitive map

$$a(\alpha, \beta) = \min\{n : A_\alpha \setminus n \subseteq A_\beta\}.$$

Let $\varrho_a : [\omega_1]^2 \to \omega$ be the corresponding ϱ-function and for $\alpha < \omega_1$, let

$$D_\alpha = A_{\alpha+1} \setminus A_\alpha .$$

Lemma 77.

$$(D_\alpha \setminus \varrho_a(\alpha,\gamma)) \cap (D_\beta \setminus \varrho_a(\beta,\gamma)) = \emptyset$$

for all $0 < \alpha < \beta < \gamma$ with $\alpha, \beta \in \Delta$. $\square$

Definition 78. Define ~~partial~~ $m : [\omega_1]^2 \to \omega$ by

$$m(\alpha,\beta) = \min(D_\alpha \setminus \varrho_a(\alpha,\beta)).$$

Lemma 77*. $m(\alpha,\gamma) \neq m(\beta,\gamma)$ for $\alpha \neq \beta$ in Δ and $\gamma > \alpha, \beta$. $\square$

Lemma 79. m is coherent on Δ, i.e.,

$$\{\xi < \min\{\alpha,\beta\} : m(\xi,\alpha) \neq m(\xi,\beta)\} \cap \Delta$$

is finite for all $\alpha, \beta < \omega_1$. $\square$.

Definition 80. For $\beta < \omega_1$, set
$$B_\beta = \{ m(\alpha,\beta) : \alpha \in \Lambda \cap \beta \}.$$

Lemma 81. $B_\beta =^* B_\gamma \cap A_\alpha$ for $\beta < \gamma$. ∎

Lemma 82. $m(\alpha,\beta) = \max(B_\beta \cap D_\alpha)$ for $\alpha \in \Lambda \cap \beta$. ∎

Lemma 83. There is <u>no</u> $B \subseteq \omega$ such that $B \cap A_\alpha =^* B_\beta$ for all β. ∎

Theorem 84. For every strictly $\subseteq^*$-increasing chain $\{A_\alpha : \alpha < \omega_1\} \subseteq [\omega]^\omega$ there is a sequence $\{B_\alpha : \alpha < \omega_1\} \subseteq [\omega]^\omega$ such that:

(a) $B_\alpha =^* B_\beta \cap A_\alpha$ for $\alpha < \beta$

(b) There is no B such that $B_\alpha =^* B \cap A_\alpha$ for all α. ∎

Problem 85. Investigate the structure of ω_1-gaps as in Theorem 84. What are the cannonical objects?

§12. Some high-dimensional theory

Fix a positive integer n. Then For every $0 < m \leq n$, we can find a C-sequence

$$\langle C^m_\alpha : \alpha < \omega_m \rangle$$

such that

(1) $C_{\alpha+1} = \{\alpha\}$

(2) C_α is a club in α

(3) $\operatorname{otp} C_\alpha = \operatorname{cf}(\alpha)$ for α limit

(4) $C_\alpha \setminus \lim(C_\alpha) \subseteq \{\xi+1 : \xi < \omega_m\}$.

and consider the corresponding walks and Their characteristics.

For example, we can consider

$$\rho^{(m)} : [\omega_m]^2 \longrightarrow \omega_{m-1}$$

defined by

$$\rho^{(m)}(\alpha,\beta) = \sup\{\mathrm{otp}(C_\beta \cap \alpha),\ \rho^{(m)}(\alpha, \min(C_\beta \setminus \alpha)),\ \rho^{(m)}(\xi, \alpha) : \xi \in C_\beta \cap \alpha\}$$

where $\rho^{(m)}(\alpha,\alpha) = 0$ is the boundary condition. We can also consider the injective version

$$\bar{\rho}^{(m)} : [\omega_m]^2 \longrightarrow \omega_{m-1}$$

and prove the following:

<u>Lemma 86</u>. For all $\alpha < \beta < \gamma < \omega_m$,

(a) $\bar{\rho}^{(m)}(\alpha,\gamma) \leq \max\{\bar{\rho}^{(m)}(\alpha,\beta), \bar{\rho}^{(m)}(\beta,\gamma)\}$,

(b) $\bar{\rho}^{(m)}(\alpha,\beta) \leq \max\{\bar{\rho}^{(m)}(\alpha,\gamma), \bar{\rho}^{(m)}(\beta,\gamma)\}$,

(c) $\bar{\rho}^{(m)}(\alpha,\gamma) \neq \bar{\rho}^{(m)}(\beta,\gamma)$;

(d) $\bar{\rho}^{(m)}(\alpha,\beta) \neq \bar{\rho}^{(m)}(\beta,\gamma)$.

Definition 87. For each $0 \le i \le n$ define recursively

$$f_i^{(n)} : [\omega_n]^{i+1} \to \omega_{n-i}$$

as follows:

(1) $f_0^{(n)} = Id_{\omega_n}$

(2) for $0 < i \le n$ and $\alpha_0 < \dots < \alpha_i$, set

$$f_i^{(n)}(\alpha_0, \alpha_1, \dots, \alpha_i) = \bar{\rho}^{(n-i+1)}\big(f_{i-1}^{(n)}(\alpha_0, \dots, \alpha_{i-1}),\ f_{i-1}^{(n)}(\alpha_1, \dots, \alpha_i)\big).$$

Finally, let $f_n = f_n^{(n)} : [\omega_n]^{n+1} \to \omega$.

We study some properties of these higher-dimensional ρ-functions.

Lemma 88. Suppose that

$$\alpha, \bar{\alpha} < \alpha_0 < \dots < \alpha_{i-1}$$

are such that

(a)
$$f_j^{(n)}(\alpha, \alpha_0, \dots, \alpha_{j-1}), f_j^{(n)}(\bar{\alpha}, \alpha_0, \dots, \alpha_j) <$$
$$< f_j^{(n)}(\alpha_0, \dots, \alpha_j)$$

for all $j < i$, and

(b) $$f_i^{(n)}(\alpha, \alpha_0, \dots, \alpha_{i-1}) = f_i^{(n)}(\bar{\alpha}, \alpha_0, \dots, \alpha_{i-1}).$$

Then $\alpha = \bar{\alpha}$.

<u>Proof</u>: Induction on i. The case $i = 0$ is trivial so we assume $i > 0$. Then

$$\bar{\rho}^{(n-i+1)}(f_{i-1}^{(n)}(\alpha, \alpha_0, \dots, \alpha_{i-2}), f_{i-1}^{(n)}(\alpha_0, \dots, \alpha_i)) =$$
$$f_i^{(n)}(\alpha, \alpha_0, \dots, \alpha_{i-1}) = f_i^{(n)}(\bar{\alpha}, \alpha_0, \dots, \alpha_{i-1}) =$$
$$\bar{\rho}^{(n-i+1)}(f_{i-1}^{(n)}(\bar{\alpha}, \alpha_0, \dots, \alpha_{i-2}), f_{i-1}^{(n)}(\alpha_0, \dots, \alpha_{i-1}))$$

From this and the injectivity of $\bar{\rho}^{(n-i+1)}$

(Lemma 86(c)) we conclude that

$$f^{(n)}_{i-1}(\alpha, \alpha_0, \dots, \alpha_{i-2}) = f^{(n)}_{i-1}(\bar{\alpha}, \alpha_0, \dots, \alpha_{i-2}),$$

~~so we are done by the ind. hyp~~ (i.e., we get the desired conclusion $\alpha = \bar{\alpha}$). □

Definition 89. (a) For $s, t \in [\omega_n]^{<\omega}$ we let

$s \lhd t$ iff $s \setminus \{\min(s)\} \sqsubseteq t$.

(b) We call a function*) $g : [\omega_n]^{<\omega} \to \mathrm{Ord}$ shift-increasing whenever

$s \lhd t$ implies $g(s) < g(t)$.

(c) We call a (possibly partial) function $g : [\omega_n]^{<\omega} \to X$ min-dependent whenever

$g(s) = g(t)$ implies $\min(s) = \min(t)$.

* possibly partial

Lemma 90.

$\forall A \in [\omega_n]^\omega \ \exists B \in [A]^\omega \ \forall i \leq n$

$f_i^{(n)} \restriction [B]^{i+1}$ is shift-increasing.

Proof: Define $\chi : [A]^{n+2} \to 3^{n+1}$ by

$$\chi(\alpha_0, \dots, \alpha_{n+1})(i) = \begin{cases} 0 & \text{if } f_i^{(n)}(\alpha_0, \dots, \alpha_i) < f_i^{(n)}(\alpha_1, \dots, \alpha_{i+1}) \\ 1 & \text{if } f_i^{(n)}(\alpha_0, \dots, \alpha_i) = f_i^{(n)}(\alpha_1, \dots, \alpha_{i+1}) \\ 2 & \text{if } f_i^{(n)}(\alpha_0, \dots, \alpha_i) > f_i^{(n)}(\alpha_1, \dots, \alpha_{i+1}). \end{cases}$$

By Ramsey's theorem there is $B \in [A]^\omega$ such that $\chi \restriction [B]^{n+1}$ is constant. Let $(\varepsilon_i : i \leq n) \in 3^{n+1}$ be the constant value.

Claim. $\forall i \leq n \quad \varepsilon_i = 0$

Proof: First of all note that $\varepsilon_i \neq 2$ for all $i \leq n$. Otherwise, fixing such $i \leq n$, choose $\triangleleft$-increasing sequence $\{s_k : k < \omega\} \subseteq [B]^{n+2}$

and conclude that

$$f_i^{(n)}(s_k \restriction i+1) > f_i^{(n)}(s_{k+1} \restriction i+1),$$

a contradiction.

Now we prove that $\varepsilon_i \neq 2$ for all $i \leq n$. We do this by induction on i. If $i = 0$ we know that $f_0^{(n)}(\alpha) = \alpha$, and this is clearly $\triangleleft$-increasing. Suppose $i > 0$. Let $\alpha_0 < \dots < \alpha_{i+1}$ are in B. By the ind. hyp.,

$$f_{i-1}^{(n)}(\alpha_0, \dots \alpha_{i-1}) < f_{i-1}^{(n)}(\alpha_1, \dots, \alpha_i) < f_{i-1}^{(n)}(\alpha_2, \dots, \alpha_{i+1})$$

Applying the property (d) of Lemma 86 of $\bar{g}^{(n-i+1)}$ we get that

$$f_i^{(n)}(\alpha_0,\dots,\alpha_i) = \bar{\rho}^{(n-i+1)}\left(f_{i-1}^{(n)}(\alpha_0,\dots,\alpha_{i-1}), f_{i-1}^{(n)}(\alpha_1,\dots,\alpha_i)\right) \neq$$

$$\neq \bar{\rho}^{(n-i+1)}\left(f_{i-1}^{(n)}(\alpha_1,\dots,\alpha_i), f_{i-1}^{(n)}(\alpha_2,\dots,\alpha_{i+1})\right) =$$

$$= f_i^{(n)}(\alpha_1,\dots,\alpha_{i+1}).$$

So $\varepsilon_i \neq 1$.

<u>Lemma 91.</u>

$\forall A \in [\omega_n]^\omega \; \exists B \in [A]^\omega \; \forall i \leq n$

$f_i^{(n)} \restriction [B]^{i+1}$ is min-dependent.

<u>Proof</u>: By Lemma 90 we find $C \in [A]^\omega$ such that $f_i^{(n)} \restriction [C]^{i+1}$ is shift-increasing for all $i \leq n$.

Using the Erdös-Rado canonical Ramsey theorem, we find $B \in [C]^\omega$ and for each $i \leq n$ a set $J_i \subseteq i+1$ such that for all $s, t \in [B]^{i+1}$

$$f_i^{(n)}(s) = f_i^{(n)}(t) \text{ iff } \{s(l) : l \in J_i\} = \{t(l) : l \in J_i\}$$

The following Claim finishes the proof.

Claim. $0 \in J_i$ for all $i \leq n$.

Proof: Fix $i \leq n$. It suffices to prove that

$$f_i^{(n)}(\alpha_0, \alpha_1, \ldots, \alpha_i) = f_i^{(n)}(\bar{\alpha}_0, \alpha_1, \ldots, \alpha_i)$$

implies $\alpha_0 = \bar{\alpha}_0$. Since $f_j^{(n)} \upharpoonright [B]^{j+1}$ are shift-increasing for all $j < i$, this conclusion follows from Lemma 88.

$\square$

§13. Positional graphs again

Recall the definitions from §4 above, two finite sets F and G of ordinals are in the Δ_k-position for some integer k whenever there is a partition

$$F \cap G = I \cup J$$

such that $I \sqsubseteq F$, $I \sqsubseteq G$ and $|J| \leq k$.

For a family $\mathcal{V}$ of finite sets and an integer k, the positional graph spaned by $\mathcal{V}$ is the graph

$$\mathcal{G}_k(\mathcal{V}) = (\mathcal{V}, \neg\Delta_k).$$

Fix a positive integer n and recall the higher-dimensional ρ-functions $f_i^{(n)}$ from the previous section.

<u>Definition 92</u>. Let $\mathcal{V}_n$ be the set of all finite subsets F of ω_n such that

(a) $f_n = f_n^{(n)}$ is min-dependent on $[F]^{n+1}$,

(2) $f_i^{(n)} \restriction [F]^{i+1}$ is shift-increasing for all $i < n$.

<u>Theorem 93</u>.

(a) $\forall A \in [\omega_n]^\omega \; \exists B \in [A]^\omega \; [B]^{<\omega} \subseteq \mathcal{V}_n$

(b) The positional graph

$$\mathcal{G}_{2n-1}(\mathcal{V}_n) = (\mathcal{V}_n, \neg \Delta_{2n-1})$$

is countably chromatic.

<u>Proof</u>: (a) follows from lemmas 90 and 91.
To prove (b) define

$$\mathcal{X} : \mathcal{V}_n \longrightarrow HF$$

by letting

$\chi(F)$ = the isomorphism type of the structure $(F, <, f_n \upharpoonright [F]^{n+1})$.

The proof is finished once we establish the following.

<u>Claim</u>. $\chi(s) = \chi(t)$ implies that s and t are in the Δ_{2n-1}-position.

<u>Proof</u>: If $|s \cap t| \leq 2n-1$ the conclusion is trivial, so let us consider the case $|s \cap t| \geq 2n$. Let

$\{\gamma_0, \gamma_1, \dots, \gamma_{n-1}, \gamma_n, \dots \gamma_{2n-1}\}_<$

be the last $2n$-elements of $s \cap t$. Let

$\Phi : s \to t$

be the increasing bijection.

<u>Subclaim 1</u>. $\forall i<n\ \Phi(\gamma_i)=\gamma_i$.

<u>Proof</u>: Fix $i<n$. Then

$$f_n(\gamma_i,\gamma_n,\dots,\gamma_{2n-1})=f_n(\Phi(\gamma_i),\Phi(\gamma_n),\dots,\Phi(\gamma_{2n-1}))$$

Since $\Phi''s\cap t\subseteq t$ and since $f_n\restriction[t]^{n+1}$ is min-dependent we conclude that $\Phi(\gamma_i)=\gamma_i$. $\square$

<u>Subclaim 2</u>. $s\cap\gamma_0=t\cap\gamma_0=(s\cap t)\cap\gamma_0$.

<u>Proof</u>: Consider $\gamma\in s\cap\gamma_0$. Then

$$f_n(\gamma,\gamma_0,\dots,\gamma_{n-1})=f_n(\Phi(\gamma),\Phi(\gamma_0),\dots,\phi(\gamma_{n-1}))=$$
$$=f_n(\Phi(\gamma),\gamma_0,\dots,\gamma_{n-1})$$

Since $f_i^{(n)}\restriction[s]^{i+1}$ and $f_i^{(n)}\restriction[t]^{i+1}$ are shift-increasing we conclude that

$$f_i^{(n)}(\gamma,\gamma_0,\dots,\gamma_i),\ f_i^{(n)}(\phi(\gamma),\gamma_0,\dots,\gamma_i)<$$
$$<f_i^{(n)}(\gamma_0,\dots,\gamma_{i+1})$$

holds for all $i \leq n$. This means that the hypotheses of Lemma 88 are satisfied. It follows that $\Phi(\gamma) = \gamma$, as required.

Let
$$I = s \cap \gamma_0 + 1 = t \cap \gamma_0 + 1 = (s \cap t) \cap \gamma_0 + 1$$
and let
$$J = \{\gamma_1, \dots \gamma_n, \gamma_{n+1}, \dots \gamma_{2n-1}\}.$$

Then $s \cap t = I \cup J$ is the decomposition witnessing that s and t are in the Δ_{2n-1}-position. $\square$

§14. Conditional weakly null sequences

Recall that a (semi) normalized sequence $(x_\alpha)_{\alpha<\mu}$ in some normed space $(X, \|\cdot\|)$, indexed by some ordinal Γ, is called a (Schauder) basic sequence if there is a constant $K \geq 1$ such that

$$\left\| \sum_{\alpha<\beta} a_\alpha x_\alpha \right\| \leq K \cdot \left\| \sum_{\alpha<\Gamma} a_\alpha x_\alpha \right\|$$

for all $\beta < \Gamma$ and all sequences $(a_\alpha)_{\alpha<\Gamma}$ of scalars. Such a basic sequence is called unconditional whenever there is a constant $K \geq 1$ such that

$$\left\| \sum_{\alpha\in A} a_\alpha x_\alpha \right\| \leq K \left\| \sum_{\alpha<\Gamma} a_\alpha x_\alpha \right\|$$

for all $A \subseteq \Gamma$ and all sequences $(a_\alpha)_{\alpha<\Gamma}$ of scalars. The unconditional basic sequence problem asking for conditions that guarantee the existence of infinite unconditional basic sequences has been one of the most fruitful problems in this area of mathematics.

<u>Lemma 94</u>. Suppose that Γ is an infinite ordinal for which we can find an integer $n \neq 0$ and a family $\mathcal{V} \subseteq [\Gamma]^{<\omega}$ such that

(a) $\forall A \in [\Gamma]^{\omega} \; \exists B \in [A]^{\omega} \; [B]^{<\omega} \subseteq \mathcal{V}$

(b) the postorder graph $\mathcal{G}_n(\mathcal{V}) = (\mathcal{V}, \neg\Delta_n)$ is countably chromatic

Then there is a norm $\|\cdot\|$ on $c_{00}(\Gamma)$ such that $(e_\gamma)_{\gamma<\Gamma}$ is <u>weakly null</u> normalized sequence in $(c_{00}(\Gamma), \|\cdot\|)$ but it contains no <u>infinite</u> unconditional <u>subsequence</u>. $\square$

<u>Notation</u>: $c_{00}(\Gamma)$ the family of all finitely supported mappings from Γ to $\mathbb{R}$. $e_\gamma : \Gamma \to \mathbb{R}$ is the mapping with support $\{\gamma\}$ such that $e_\gamma(\gamma) = 1$.

<u>Theorem 95</u> (Lopez Abad – T., 2011)
For every $n < \omega$ there is a weakly-null sequence $(x_\gamma)_{\gamma < \omega_n}$ of length ω_n with no infinite unconditional basis subsequence. □

<u>Theorem 96</u> (Dodos-Lopez Abad-T., 2009)
It is consistent that every weakly null sequence of length ω_ω contains an infinite unconditional basic subsequence. □

§ 15. The oscillation mapping and walks on ordinals

Fix an ordinal θ (typically regular and uncountable). Define

$$osc : \mathcal{P}(\theta) \to \omega$$

by

$$osc(x, y) = | \, x \setminus \max(\bar{x} \cap \bar{y}) + 1 \, / \sim \, |,^{*)}$$

where $\sim$ is the equivalence relation on the set

$$x \setminus \{\xi : \xi \leq \max(\bar{x} \cap \bar{y})\}$$

defined by

$\alpha \sim \beta$ iff $[\min\{\alpha, \beta\}, \max\{\alpha, \beta\}] \cap y = \emptyset$.

---

* $\bar{x}$ and $\bar{y}$ are the closures of x and y in θ

Remark 96. (a) If $\theta = \omega$ one has that $\bar{x} = x$ and $\bar{y} = y$ and one usually defines $osc(x,y)$ by letting it equal to the cardinality of the quotient $x \Delta y / \sim$ with $\sim$ defined by

$m \sim n$ iff $[\min\{m,n\}, \max\{m,n\}] \cap x = \emptyset$ or

$$[\min\{m,n\}, \max\{m,n\}] \cap y = \emptyset.$$

Thus, when $\theta > \omega$ we remove more from $x \cup y$ (as we are interested only in finite oscillations) and in fact we count only classes in $x \setminus y$.

(b) Recall the oscillation mapping

$$osc : (\omega^\omega)^2 \to \mathrm{Card}$$

defined by

$$osc(x,y) = |\{n < \omega : x(n) \leq y(n) \ \& \ x(n+1) > y(n+1)\}|.$$

The oscillation theory in essentially all known contexts rely on the fact that 'unbounded' families realize all possible oscillations. This is the case in this context as well.

<u>Definition 97</u>. A family $\mathcal{X} \subseteq \mathcal{P}(\Theta)$ is <u>unbounded</u> if for every closed and unbounded set $C \subseteq \Theta$ there exist an arbitrarily long finite increasing sequence $\{\delta_k : k \leq l\} \subseteq C$ such that for all $k < l$

$$\sup(x \cap \delta_{k+1}) < \delta_{k+1} \text{ and } [\delta_k, \delta_{k+1}) \cap x \neq \emptyset.$$

Here is a typical result of the oscillation theory in this context.

<u>Lemma 98.</u> If $\mathcal{X}$ is an unbounded family of bounded subsets of θ then for every positive integer n there exist x and y in $\mathcal{X}$ such that $\mathrm{osc}(x, y) = n$. $\square$

<u>Definition 99.</u> A C-sequence $(C_\alpha : \alpha < \theta)$ on θ is called <u>nontrivial</u> if there is no club $C \subseteq \theta$ such that for all $\alpha \in \lim(C)$ there is $\beta \geq \alpha$ such that $C \cap \alpha \subseteq C_\beta$.

<u>Exercises</u> (a) Show that every successor cardinal θ supports a nontrivial C-sequence.

(b) Show that the first inaccessible cardinal θ supports a nontrivial C-sequence.

(c) Show that first Mahlo cardinal θ has a nontrivial C-sequence.

<u>Definition 100</u>. Fix a C-sequence $(C_\alpha : \alpha < \theta)$ on θ. A <u>partial action</u> of $\theta^{<\omega}$ on θ $(\alpha, t) \mapsto \alpha_t$ is defined recursively as follows:

$$\alpha_{\langle\rangle} = \alpha$$

$\alpha_{\langle\xi\rangle}$ = the ξth element of C_α if $\xi < tp(C_\alpha)$

$\alpha\langle\xi\rangle$ is undefined if $\xi \geq tp(C_\alpha)$

$$\alpha_{t^\frown\langle\xi\rangle} = (\alpha_t)_{\langle\xi\rangle}.$$

<u>Exercise</u>. Let $\beta = \beta_0 > \dots > \beta_n = \alpha$ be the walk from β to α along the C-sequence. Let $t = \rho_0(\alpha, \beta)$. Then $\beta_t = \alpha$ and $\beta_{t\restriction i} = \beta_i$ for $i \leq n$. Show, however, that in general $\beta_t = \alpha$ does not imply $\rho_0(\alpha, \beta) = t$.

<u>Definition 101</u>. To every C-sequence $(C_\alpha : \alpha<\theta)$ on θ we attach the corresponding oscillation mapping

$$o = o(C_\alpha : \alpha<\theta) : [\theta]^2 \to \omega$$

as follows. Given $\alpha<\beta<\theta$, if there is $t \sqsubseteq \varrho_0(\alpha,\beta)$ such that

(i) $\mathrm{osc}(\alpha_t, \beta_t) > 1$, but

(ii) $\mathrm{osc}(\alpha_s, \beta_s) = 1$ for all $s \sqsubset t$,

let $o(\alpha,\beta) = \mathrm{osc}(C_{\alpha_t}, C_{\beta_t})$; otherwise let $o(\alpha,\beta) = 0$.

<u>Theorem 102</u>. If $(C_\alpha : \alpha<\theta)$ is a nontrivial C-sequence on θ and if $o : [\theta]^2 \to \omega$ is the corresponding oscillation mapping then for every unbounded $\Gamma \subseteq \theta$ and integer $n \geq 2$ there exist $\alpha<\beta$ in Γ such that $o(\alpha,\beta) = n$.

Proof: Fix $n \geq 2$ and unbounded $\Gamma \subseteq \theta$. Choose a continuous $\in$-chain $\mathcal{M}$ of elementary submodels M of H_{θ^+} such that $\delta_M = M \cap \theta \in \theta$ and such that $C(C_\alpha : \alpha < \theta), \Gamma \in M$. Let

$$C = \{\delta_M : M \in \mathcal{M}\}$$

Then C is a club in θ.

Choose now $N \prec H_{(2^\theta)^+}$ containing all these objects such that $\delta = N \cap \theta \in \theta$. Pick $\beta \in \Gamma$ such that $\beta \geq \delta$. Let

$$\beta = \beta_0 > \beta_1 > \cdots > \beta_k \geq \delta$$

be the part of the walk from β to δ such that

$$\sup(C_{\beta_k} \cap \delta) = \delta .$$

Thus either $\beta_k = \delta$ or $\beta_{k+1} = \delta$ is the last step of the walk from β to δ. Note that this in particular means that for all $i < k$

$$\xi_i = \max(C_{\beta_i} \cap \delta) < \delta.$$

Let $t = \rho_0(\beta_k, \beta)$. Then $\beta_i = \beta_{t+i}$ for $i \leq k$.

Since $(C_\alpha : \alpha < \theta)$ is nontrivial, applying this to the club $C = \{\delta_M : M \in \mathcal{M}\}$ and using the elementarity of N, we conclude that

$$\sup((C \cap \delta) \setminus C_{\beta_k}) = \delta.$$

So, we can pick an $\in$-chain M_i $(i \leq n)$ from $\mathcal{M} \cap N$ such that:

(1) $\delta_i = \delta_{M_i} \notin C_{\beta_k}$ for all $i \leq n$

(2) $[\delta_i, \delta_{i+1}) \cap C_{\beta K} \neq \emptyset$ for all $i < n$

(3) $\delta_0 > \xi_i$ for all $i < K$.

Let $J_0 = [0, \max(C_{\beta K} \cap \delta_1)]$,

$J_i = [\delta_i, \max(C_{\beta K} \cap \delta_{i+1}]$ $(0 < i < n)$

$J_n = [\delta_n, \beta]$.

Then $J_0 < J_1 < \ldots < J_n$ is a block sequence of closed intervals which covers $C_{\beta K}$ with the property that for every $0 < j \leq n$,

(*) $(J_i : i < j) \in M_j$ and $J_j \geq M_j \cap \theta = \delta_j$.

Let $\mathcal{F}$ be the family of all block-sequences $(I_i : i \leq n)$ of closed intervals of θ for which we can find $\alpha \in \Gamma$ such that:

(4) $C_{\alpha t} \subseteq \bigcup_{i \leq n} I_i$,

(5) $\xi_i = \max C_{\alpha_{t+r_i}} \cap \alpha_t$ for $i<k$

(6) $\max I_i \in C_{\alpha_t}$ for all $i<k$

(6) $\max I_n = \alpha$ and $C_{\alpha_t} \cap I_n \neq \emptyset$

(i.e., α_t is a limit ordinal from the interior of I_n)

(7) $I_0 = J_0$

Clearly $(J_i : i \leq n) \in \mathcal{F}$ and $\mathcal{F} \in M_i$ for all $0 < i \leq n$.

Let $\partial \mathcal{F}$ be the collection of all sequences $(I_i : i < n)$ of intervals such that

$$\forall \mu < \theta \ \exists \text{ interval } I \geq \mu \ \ (I_i : i<n)^\frown I \in \mathcal{F}$$

Then using $(*)$ for $j = n$ and the elementarity of M_n we conclude that

$$(J_i : i < n) \in \partial \mathcal{F}.$$

Let $\partial^2 \mathcal{F}$ be the collection of all block sequences $(I_i : i<n-1)$ of closed interval ~~for which~~ such that

$\forall \mu<\omega \; \exists$ interval $I \geq \mu$ $(I_i : i<n-1)^\frown I \in \partial \mathcal{F}$.

Then using $(*)$ for $j=n-1$ and the elementarily of M_{n-1} we conclude that $(J_i : i<n-1) \in \partial^2 \mathcal{F}$. Proceeding this way we arrive at the conclusion that

$$(J_0) \in \partial^n \mathcal{F}$$

Using the definition of $\partial^n \mathcal{F}$ and the elementarily of M_1 we can find $I_1 \in M_1$ such that $I_1 > J_0$ and $(J_0, I_1) \in \partial^{n-1} \mathcal{F}$. Now, using this and the elementarily of M_2

we can find interval $I_2 \in M_2$ such that $I_2 > J_1$ and $(J_0, I_1, I_2) \in \partial^{n-2} \mathcal{F}$, and so on. Proceeding this way we arrive at

$$(J_0 I_1, \ldots I_n) \in M_n \cap \mathcal{F}$$

such that $I_{i+1} > J_i$ for all $i < n$. Pick an $\alpha \in \Gamma$ witnessing the fact that $(J_0, I_1, \ldots, I_n)$ belongs to $\mathcal{F}$ i.e., satisfying the conditions (4) — (7). It follows that the intersections

$$C_{\alpha t} \cap I_i \qquad (0 < i \leq n)$$

are the convex pieces the set

$$C_{\alpha t} \setminus \max(C_{\alpha t} \cap C_{\beta t}) + 1$$

is split by the set $C_{\beta t}$ i.e., $\operatorname{osc}(C_{\alpha t}, C_{\beta t}) = n$. On the other hand $\operatorname{osc}(C_{\alpha t + i}, C_{\beta t + i}) = 1$ for all $i < k$. It follows that $o(\alpha, \beta) = n$, as required. $\square$

Definition 103. For $\alpha < \beta < \theta$ let

$$[\alpha\beta] = \beta_t$$

where $t \sqsubseteq \rho_0(\alpha, \beta)$ is minimal for which α_t is defined and if $\xi = tp(C_{\beta_t} \cap \alpha)$, then

ξth element of $C_{\alpha_t} \neq \xi^{th}$ element of C_{β_t}.

Definition 104. A C-sequence $(C_\alpha : \alpha < \theta)$ on θ avoids a subset S of θ whenever $C_\alpha \cap S = \emptyset$ for all limit ordinals $\alpha < \theta$.

Theorem 105. Suppose $(C_\alpha : \alpha < \theta)$ is a C-sequence on θ which avoids a set $S \subseteq \theta$. Then for every unbounded $\Gamma \subseteq \theta$,

$$S \setminus \{[\alpha, \beta] : \alpha, \beta \in \Gamma,\ \alpha < \beta\}$$

is not stationary in θ. $\square$.

§16. Initial Motivations

Definition 106. For a cardinal (structure) θ and positive integers r, k, t let

$$\theta \to (\theta)^r_{k,t}$$

denote the statement

$$\forall f : [\theta]^r \to \{0,1,\dots,k-1\}\ \exists \Gamma \in [\theta]^\theta\ |f[\Gamma]^r| \le t.$$

The minimal such t that works for all k (if it exists) is called (big) Ramsey degree of r, $t_r = t_r(\theta)$.

Let

$$\theta \to [\theta]^r_k \text{ iff } \theta \to (\theta)^r_{k,k-1}$$

Examples 107

(1) (Erdös-Hajnal-Rado, 1965)

$$2^\omega \not\to [2^\omega]^r_{2^{r-1}} \text{ but } \forall \ell \ \ 2^\omega \to (2^\omega)^r_{\ell, 2^{r-1}}$$

(2) (D. Devlin, 1979)

$$\mathbb{Q} \not\to [\mathbb{Q}]^r_{\tan^{2r-1}(0)} \text{ but } \forall \ell \ \ \mathbb{Q} \to (\mathbb{Q})^r_{\ell, \tan^{(2r-1)}(0)}$$

Proposition 108. Suppose

$$\theta \not\to [\theta]^r_t \text{ but } \forall \ell \ \ \theta \to (\theta)^r_{\ell, t}$$

Let E_t be the equivalence relation on $[\theta]^r$ witnessing $\theta \not\to [\theta]^r_t$. Then for every other equivalence relation E on $[\theta]^r$ ~~there is $\Gamma \in [\theta]^\theta$~~ such that $[\theta]^r/E$ is finite there is $\Gamma \in [\theta]^\theta$ such that

$$E \restriction [\Gamma]^r \text{ is coarser than } E_r \restriction [\Gamma]^r.$$

Example 109 (Galvin-Shelah 1973)

Fix orderings $<_S$ and $<_A$ on ω_1 such that $(\omega_1, <_S)$ is separable while $(\omega_1, <_A)$ contains no ω_1, ω_1^* nor an uncountable separable ordering.

Define E_{GS} on $[\omega_1]^2$ by letting

$$\{\alpha, \beta\}_< E \{\gamma, \delta\}_< \text{ iff } \forall R \in \{<, <_S, <_A\} \; [\alpha R \beta \leftrightarrow \gamma R \delta].$$

Question 110 (Galvin-Shelah 1973)

Does $\omega_1 \to (\omega_1)^2_{4,\ell}$ for all $\ell < \omega$?

In other words, does for every equivalence relation E on $[\omega_1]^2$ we can find $\Gamma \in [\omega_1]^{\omega}$ such that

$$E_{GS} \restriction [\Gamma]^2 \leq E \restriction [\Gamma].$$