WALKS ON ORDINALS

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The structure on ω_1:

$C_\alpha \quad (\alpha < \omega_1)$

(1) $C_\alpha \subseteq \alpha$, $\sup C_\alpha = \alpha$

(2) $C_{\alpha+1} = \{\alpha\}$

(3) $\alpha > 0$ limit $\rightarrow$ $\mathrm{otp}\, C_\alpha = \omega$ and C_α contains no limit ordinals.

A <u>step</u> from β towards smaller α is

$$\beta \rightarrow \min(C_\beta \setminus \alpha)$$

$|C_\beta \cap \alpha|$ is the <u>weight</u> of this step

The <u>Full Code</u> of the walk is the function

$$\rho_0 : [\omega_1]^2 \to \omega^{<\omega}$$

defined by

$$\rho_0(\alpha,\beta) = \langle |C_\beta \cap \alpha| \rangle^\frown \rho_0(\alpha, \min(C_\beta \setminus \alpha)),$$

with the boundary value $\rho_0(\alpha,\alpha) = \emptyset$

The <u>Maximal Weight</u> is the function

$$\rho_1 : [\omega_1]^2 \to \omega$$

defined by

$$\rho_1(\alpha,\beta) = \max\{|C_\beta \cap \alpha|, \rho_1(\alpha, \min(C_\beta \setminus \alpha))\}$$

with the boundary value $\rho_1(\alpha,\alpha) = 0$.

<u>Lemma 1.</u> For all $\alpha < \beta < \omega_1$ and $n < \omega$:

(a) $\{\xi \leq \alpha : \rho_1(\xi, \alpha) \leq n\}$ is finite

(b) $\{\xi \leq \alpha : \rho_1(\xi, \alpha) \neq \rho_1(\xi, \beta)\}$ is finite.

<u>Proof:</u> Induction. □

The <u>Number of Steps</u> function is

$$\rho_2 : [\omega_1]^2 \to \omega$$

defined by

$$\rho_2(\alpha, \beta) = \rho_2(\alpha, \min(C_\beta \setminus \alpha)) + 1$$

with the boundary value $\rho_2(\alpha, \alpha) = 0$.

<u>Lemma 2.</u>

$$\forall \alpha < \beta < \omega_1 \quad \sup_{\xi < \alpha} |\rho_2(\xi, \alpha) - \rho_2(\xi, \beta)| < \omega$$

<u>Proof:</u> Use Full Lower Trace. □

The Last Step Function

$$\rho_3 : [\omega_1]^2 \to 2$$

is defined recursively by

$$\rho_3(\alpha,\beta) = 1 \quad \text{iff} \quad \rho_0(\alpha,\beta)(\rho_2(\alpha,\beta)-1) = \rho_1(\alpha$$

In other words, $\rho_3(\alpha,\beta)=1$ just in case the last step of the walk $\beta \to \alpha$ comes with the maximal weig

Lemma. 3

$\forall \alpha<\beta<\omega_1 \quad \{\xi<\alpha : \rho_3(\xi,\alpha) \neq \rho_3(\xi,\beta)\}$ is finite

Proof: Use again the Full Lower Trace function. ∎

Corollary 4. The tree $T(\rho_0) = \{\rho_0(\cdot,\beta)\restriction\alpha : \alpha \leq \beta <$ has countable levels and has no uncountable branches, $\therefore$, $T(\rho_0)$ is Aronszajn (an A-tre

Question. Are the other trees $T(\rho_0)$, $T(\rho_2)$ and $T(\rho_3)$ also A-trees?

Case of $T(\rho_3)$ needs the following assumption on the $\bar{C}$-sequence.

(4) (a) If $\alpha = \lambda + \omega$ for λ limit,
$C_\alpha = \{\lambda + n : m < n < \omega\}$ for some

(b) If α is limit of limits and
$C_\alpha = (C_\alpha(n) : n < \omega)$ is the increasing enumeration, then for each n,
$C_\alpha(n) = \lambda_n + m$ for some $m > n$

Notation

Λ = the set of countable limit ordinals

$\Lambda + n = \{\lambda + n : \lambda \in \Lambda\}$ for $n < \omega$

Lemma 5. $\forall \beta \ \forall \lambda \in \Lambda \ \exists k \ \forall n:$

$\lambda + n < \beta \longrightarrow \varrho_3(\lambda + n, \beta) = 1$ ...

$\varrho_{3\beta} := \varrho_3(\cdot, \beta)$ restricted to $[\lambda, \lambda + \omega)$

is equal to 1 modulo finitely many ex

Proof: We may assume $\alpha = \lambda + \omega \leq \beta$.

Then there is m_0 such that

$\forall n \geq m_0$ the walk $\beta \to \lambda + n$ passes through

Let m_1 be such that (see (4))

$C_\alpha = \{\lambda + n : m_1 < n < \omega\}$

So for $n > \max(m_0, m_1)$ the last of the walk $\beta \to \lambda + n$ is the step $\alpha \to \lambda + n$.

Its weight $|\{k : m_0 < k < n\}| = n - m_0 - 1$ ($\leq C_\alpha \cap (\lambda+n)$) is eventually bigger than $\rho_1(\alpha, \beta)$, so eventually $\rho_3(\lambda + n, \beta) = 1$ □

Lemma 6.

$\forall_\beta \forall n \ |\{\lambda \in \Lambda : \lambda + n < \beta \ \& \ \rho_3(\lambda+n, \beta) = 1\}| <$

Proof: Given an infinite $\Gamma \subseteq (\Lambda + n) \cap$ we need to find $\lambda + n \in \Gamma$ such that

$$\rho_3(\lambda + n, \beta) = 0.$$

Shrink Γ so that we can assume

$$\forall \lambda + n \in \Gamma \quad \rho_1(\lambda + n, \beta) > n + 2.$$

So, if $\rho_3(\lambda+n,\beta)=1$ for some $\lambda+n\in$ the last step $\alpha\to\lambda+n$ of the walk $\beta\to\lambda+n$ must have weight $>n$.

Note however that by the assumption (2) $\alpha\neq\lambda+n+1$, by the assumption 4(a) $\alpha\notin\lambda+\omega$, and by the assumption 4(b) α cannot be limit of limits. ✗ . □

<u>Lemma 7.</u> Let $B_\alpha=\{\xi<\alpha : \rho_3(\xi,\alpha)=1\}$ for $\alpha<\omega_1$. Then

(1) $B_\alpha =^* B_\beta\cap\alpha$ for $\alpha<\beta$

(2) $(\Delta+n)\cap B_\beta$ is finite for all $n<\omega$ and $\beta<$

(3) $\{\lambda+n : n<\omega\}\subseteq^* B_\beta$ whenever $\lambda+\omega\leq\beta$

So in particular there is no

$$g : \omega_1 \longrightarrow 2$$

such that $g \upharpoonright \beta =^* \varrho_3(\cdot, \beta)$ for all β, so we have the following.

<u>Lemma 8.</u> $T(\varrho_3)$ is a coherent A-tree. □

<u>Case of $T(\varrho_2)$</u> needs the following fact.

<u>Lemma 9.</u> For every pair A and B of uncountable subsets of ω_1 and every $k < \omega$ there exist $\alpha \in A$, $\beta \in B$ such that $\varrho_2(\alpha, \beta) > k$.

In fact we prove a stronger statement

<u>Lemma 9*</u> For every uncountable $A \subseteq [\omega_1]^{<\omega}$ consisting of pairwise disjoint sets all of same fixed size n and every $k < \omega$ there exist uncountable $B \subseteq A$ such that

$$(\forall a < b \text{ in } B)(\forall i,j < n)\ \rho_2(a(i), b(j)) \geq k$$

<u>Proof</u>: Induction on k. So suppose $(\forall a < b \text{ in } A)(\forall i,j < n)\ \rho_2(a(i), b(j)) \geq k$, and work for uncountable $B \subseteq A$ such that $(\forall a < b \text{ in } B)(\forall i,j < n)\ \rho_2(a(i), b(j)) \geq k+1$.

Fix limit $\delta < \omega$ and $b_\delta \in A$, $b_\delta > \delta$. The for some $\eta_\delta < \delta$,

$\forall \xi \in [\eta_\delta, \delta)\ \forall \beta \in b_\delta \quad \rho_0(\xi, \beta) = \rho_0(\beta, \delta)^\frown \rho_0(\delta, \xi)$.

Find stationary $\Gamma \subseteq \omega_1$ and $\eta < \omega_1$ such that $\eta_\delta = \eta$ for all $\delta \in \Gamma$.

Choose stationary $\Sigma \subseteq \Gamma$ such that

$$\forall \gamma < \delta \text{ in } \Sigma \quad \gamma < b_\gamma < \delta < b_\delta$$

Consider the family

$$A^* = \{\{\gamma\} \cup b_\gamma : \gamma \in \Sigma\}$$

By induction hypothesis there is uncountable $B^* \subseteq A^*$ such that $\rho_2(a(i), b(j)) \geq k$ for all $i, j < n+1$ and $a < b$ in B. Let $B = \{b \setminus \{\min(b)\} : b \in B^*\}$. Then $B \subseteq A$ is uncountable and $\rho_2(a(i), b(j)) \geq k+1$ for all $i, j < n$ and $a < b$ in B. ∎

<u>Corollary</u> 10. For every $g : \omega_1 \to \omega$ there is $\alpha <$ such that $\sup_{\xi < \alpha} |\rho_2(\xi, \alpha) - g(\xi)| = \infty$.

Corollary 11. $T(\varrho_2)$ is an A-tree.

Question. Is $T(\rho_0)$ an A-tree?

§2. Traces

The *upper trace* $Tr : [\omega_1]^2 \to [\omega_1]^{<\omega}$ is defined by

$$Tr(\alpha,\beta) = \{\beta\} \cup Tr(\alpha, \min(C_\beta \setminus \alpha))$$

with the boundary value is $Tr(\alpha,\alpha) = \{\alpha\}$.

The *lower trace* $L : [\omega_1]^2 \to [\omega_1]^{<\omega}$,

$$L(\alpha,\beta) = \{\lambda(\xi,\beta) : \xi \in Tr(\alpha,\beta),\ \xi \neq \beta\}$$

$$\lambda(\xi,\beta) = \max\{\max(C_\eta \cap \xi) : \eta \in Tr(\xi,\beta),\ \eta \neq$$

Recursive definition:

$$L(\alpha,\beta) = (L(\alpha, \min(C_\beta \setminus \alpha)) \setminus \max(C_\beta \cap \alpha)) \cup \{\max(C_\beta \cap \alpha$$

with the boundary value $L(\alpha,\alpha) = \emptyset$.

If $\beta = \beta_0 > \beta_1 > \dots > \beta_n = \alpha$ is the walk from β to α i.e.

$\beta_0 = \beta$, $\beta_n = \alpha$ and $\beta_{i+1} = \min(C_{\beta_i} \setminus \alpha)$ $(i<n)$

then

$$Tr(\alpha,\beta) = \{\beta_i : i \le n\}$$

$$L(\alpha,\beta) = \{\lambda_i : i \le n\},$$

where $\lambda_i = \max\left(\bigcup_{j \le i} C_{\beta_j} \cap \alpha\right)$.

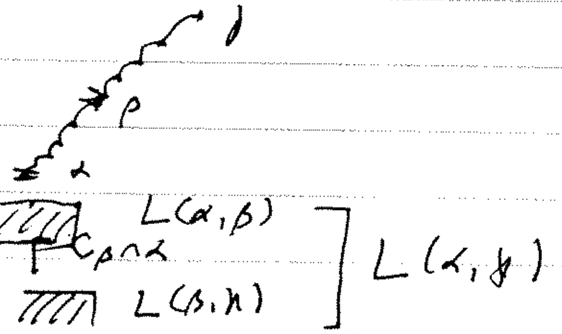
<u>Lemma 12</u>. For $\alpha \le \beta \le \gamma$,

(a) $\alpha > L(\beta,\gamma)$ implies $\rho_0(\alpha,\gamma) = \rho_0(\beta,\gamma)^\frown \rho_0(\alpha,\beta)$ and therefore $Tr(\alpha,\gamma) = Tr(\beta,\gamma) \cup Tr(\alpha,\beta)$.

(b) $L(\alpha,\beta) > L(\beta,\gamma)$ implies

$$L(\alpha,\gamma) = L(\beta,\gamma) \cup L(\alpha,\beta)$$

<u>Proof.</u>



The Full lower trace

$$F : [\omega_1]^2 \to [\omega_1]^{<\omega}$$

$$F(\alpha,\beta) = F(\alpha, \min(C_\beta \setminus \alpha)) \cup \bigcup_{\xi \in C_\beta \cap \alpha} F(\xi,\alpha)$$

with the boundary value $F(\alpha,\alpha) = \{\alpha\}$

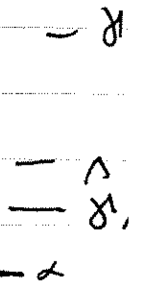
Lemma 13. For $\alpha \leq \beta \leq \gamma$

(a) $F(\alpha,\gamma) \subseteq F(\alpha,\beta) \cup F(\beta,\gamma)$

(b) $F(\alpha,\beta) \subseteq F(\alpha,\gamma) \cup F(\beta,\gamma)$.

Proof: Induction on γ. To prove (a), let

$$\gamma_1 = \min(C_\gamma \setminus \alpha)$$



We first show that

$$F(\alpha,\gamma_1) \subseteq F(\alpha,\beta) \cup F(\beta,\gamma).$$

Case 1. $\gamma_1 < \beta$.

By ind. hyp.

$$F(\alpha,\gamma_1) \subseteq F(\alpha,\beta) \cup F(\gamma_1,\beta).$$

$$\subseteq F(\alpha,\beta) \cup F(\beta,\gamma)$$

since $\gamma_1 \in C_\gamma \cap \beta$ and therefore $F(\gamma_1,\beta) \subseteq F(\beta,\gamma)$

<u>Case $\gamma_1 \geq \beta$:</u> Then

$$\gamma_1 = \min(C_\gamma \setminus \beta)$$

and therefore $F(\gamma_1,\beta) \subseteq F(\beta,\gamma)$. $\square$

<u>Consider a factor $F(\xi,\alpha)$</u> of $F(\alpha,\beta)$,

where $\xi \in C_\gamma \cap \alpha$.

By ind. hyp.

$$F(\xi,\alpha) \subseteq F(\alpha,\beta) \cup F(\xi,\beta)$$

$$\subseteq F(\alpha,\beta) \cup F(\beta,\gamma)$$

since $\xi \in C_\gamma \cap \beta$ and so $F(\xi,\beta) \subseteq F(\beta,\gamma)$

To prove (b) consider cases:

<u>Case</u> $\gamma_1 < \beta$. By ind hyp

$$F(\alpha,\beta) \subseteq F(\alpha,\gamma_1) \cup F(\gamma_1,\beta)$$

Since $\gamma_1 \in C_\gamma \cap \beta$, ~~the set~~ we get

$$F(\gamma_1,\beta) \subseteq F(\beta,\gamma)$$

By definition of $F(\alpha,\gamma)$,

$$F(\alpha,\gamma_1) \subseteq F(\alpha,\gamma).$$

<u>Case</u> $\gamma_1 \geq \beta$. Then

$$\gamma_1 = \min(C_\gamma \setminus \beta) \quad \text{so}$$

$$F(\beta,\gamma_1) \subseteq F(\beta,\gamma).$$

Using the ind hyp for $\alpha \leq \beta \leq \gamma_1$, we

$$F(\alpha,\beta) \subseteq F(\alpha,\gamma_1) \cup F(\beta,\gamma_1) \subseteq F(\alpha,\gamma) \cup F(\beta,\gamma)$$

$\square$

Remark Note that since $\alpha \in F(\alpha,\beta)$ for all $\alpha \leq \beta$, by the definition of the full lower trace, we have that

$$F(\alpha,\beta) \supseteq L(\alpha,\beta)$$

for all $\alpha \leq \beta < \omega_1$.

Lemma 14. If $\alpha \in F(\beta,\gamma)$ then

$$F(\alpha,\gamma) \cup F(\alpha,\beta) \subseteq F(\beta,\gamma)$$

Proof: Induction on γ. Let $\gamma_1 = \min(C_\gamma \setminus \beta)$.

Case 1. $\alpha \in F(\beta,\gamma_1)$. Then by ind. hyp.

$$F(\alpha,\beta) \subseteq F(\beta,\gamma_1) \subseteq F(\beta,\gamma)$$

On the other hand

$$F(\alpha,\gamma) \subseteq F(\alpha,\beta) \cup F(\beta,\gamma) = F(\beta,\gamma)$$

Case 2 $\alpha \in F(\xi, \beta)$ for some $\xi \in C_\gamma \cap \beta$

Then by ind hyp for $\alpha \leq \xi \leq \beta$:

$F(\alpha, \beta) \subseteq F(\xi, \beta) \subseteq F(\beta, \gamma)$

On the other hand

$F(\alpha, \gamma) \subseteq F(\alpha, \beta) \cup F(\beta, \gamma) = F(\beta, \gamma)$. $\square$

Lemma 15. Let $\alpha \leq \beta \leq \gamma < \omega_1$ and let

$$\bar{\alpha} = \min(F(\beta, \gamma) \setminus \alpha).$$

Then

(a) $\rho_0(\alpha, \beta) = \rho_0(\bar{\alpha}, \beta) ^\frown \rho_0(\alpha, \bar{\alpha})$

(b) $\rho_0(\alpha, \gamma) = \rho_0(\bar{\alpha}, \gamma) ^\frown \rho_0(\alpha, \bar{\alpha})$

Proof: This is trivial if $\alpha = \bar{\alpha}$, so we assume $\alpha < \bar{\alpha}$. By Lemma 14,

$$F(\bar{\alpha}, \beta) \cup F(\bar{\alpha}, \gamma) \subseteq F(\beta, \gamma),$$

so

We have

$$\alpha > F(\beta,\gamma)\cap\bar\alpha \geq F(\bar\alpha,\beta)\cap\bar\alpha,\ F(\bar\alpha,\gamma)\cap\bar\alpha$$
$$\geq L(\bar\alpha,\beta),\ L(\bar\alpha,\gamma),$$

and so we are done by Lemma 12 (a). ∎

<u>Corollary 16.</u> For all $\alpha < \omega_1$

$|\{\rho_0(\cdot,\beta)\restriction\alpha : \alpha\leq\beta<\omega_1\}| \leq \aleph_0$, and

$|\{\rho_2(\cdot,\beta)\restriction\alpha : \alpha\leq\beta<\omega_1\}| \leq \aleph_0$.

It follows that the tree

$T(\rho_2) = \{\rho_2(\cdot,\beta)\restriction\alpha : \alpha\leq\beta<\omega_1\}$

is Aronszajn. How about $T(\rho_0)$?.

<u>Definition 17.</u> The right lexicographical ordering $<^r_{lex}$ on $\omega^{<\omega}$ is defined by letting $s <^r_{lex} t$ if $s \sqsupset t$,

$s(i) < t(i)$ for $i = \min\{j : s(j) \neq t(j)\}$.

<u>Lemma 18</u>. For all $\beta < \gamma < \omega_1$ the set

$$\{\xi < \beta : \rho_0(\xi, \beta) = \rho_0(\xi, \gamma)\} =: X$$

is a closed subset of β.

<u>Proof</u>: Let $\alpha < \beta$ be an accumulation point of X. Let $\bar{\alpha} = \min(F(\beta,\gamma) \setminus \alpha)$

Pick a point $\xi \in X \cap \alpha$, such that

$$\xi > F(\beta,\gamma) \cap \alpha.$$

Then $\bar{\alpha} = \min(F(\beta,\gamma) \setminus \xi)$, so by Lemma 15,

$$\rho_0(\xi, \beta) = \rho_0(\bar{\alpha}, \beta) ^\frown \rho_0(\xi, \bar{\alpha})$$
$$\parallel$$
$$\rho_0(\xi, \gamma) = \rho_0(\bar{\alpha}, \gamma) ^\frown \rho_0(\xi, \bar{\alpha})$$

It follows that $\rho_0(\bar{\alpha}, \beta) = \rho_0(\bar{\alpha}, \gamma)$;

Applying Lemma 15 again, we have

$$\rho_0(\alpha,\beta) = \rho_0(\bar{\alpha},\beta) ^\frown \rho_0(\alpha,\bar{\alpha})$$
$$\rho_0(\alpha,\gamma) = \rho_0(\bar{\alpha},\gamma) ^\frown \rho_0(\alpha,\bar{\alpha}),$$

so $\rho_0(\alpha,\beta) = \rho_0(\alpha,\gamma)$. □

<u>Lemma 19</u> $\alpha < \beta < \gamma$ implies

$$\rho_0(\alpha,\gamma) <^{+}_{lex} \rho_0(\beta,\gamma).$$

<u>Proof</u>: Let $\gamma = \gamma_0(\alpha) > \dots > \gamma_m(\alpha) = \alpha$ and $\gamma = \gamma_0(\beta) > \dots > \gamma_n(\beta) = \beta$ be the walks from γ to α and γ to β, respectively. Let $j \leq \min\{m,n\}$ be the largest such that $\gamma_j(\alpha) = \gamma_j(\beta) = \gamma_j$. Then

$(\alpha,\beta) \cap C_{\gamma_j} \neq \emptyset$, so

$$\ast\ |C_{\gamma_j} \cap \alpha| < |C_{\gamma_j} \cap \beta|$$

$$\varrho_0(\alpha,\gamma)(j) \quad \varrho_0(\beta,\gamma)(j)$$

and $\varrho_0(\alpha,\gamma)(i) = \varrho_0(\beta,\gamma)(i)$ for $i<j$

It follows that $\varrho_0(\alpha,\gamma) <^r_{lex} \varrho_0(\beta,\gamma)$. □

__Definition__ 20 (Kurepa 1935) Let $\sigma\mathbb{Q}$ be the tree of all increasing transfinite sequences of rationals.

__Corollary__ 21. $T(\varrho_0)$ is a downward closed subtree of $\sigma\mathbb{Q}$. □

__Theorem__ 22 (Kurepa 1935, 1953)
There is a strictly increasing $f: \sigma\mathbb{Q} \to \mathbb{R}$ but there is no str. increasing $f: \sigma\mathbb{Q} \to \mathbb{Q}$. □

Lemma 23 $T(\rho_0)$ is a special A-tree.

Proof: In $T(\rho_0)$ nodes of limit height have at most one immediate successor. This follows from Lemma 18. □

Lemma 24 (Peng) $T(\rho_2)$ is also special.

§3 ρ-models

Proof: For each $t = \rho_2(\cdot, \beta) \restriction \alpha$ we assume β is minimal ordinal witnessing this and set

$f(t)$ = the isomorphism type of the ρ_2-model $(F(\alpha, \beta) \cup \{\beta\}, \leq, \rho_2)$.

It suffices to show that if α and α' are limit ordinals and $t = \rho_2(\cdot,\beta)\restriction\alpha$ and $t' = \rho_2(\cdot,\beta')\restriction\alpha'$ are nodes of $T(\rho_2)$ such that

$$(F(\alpha,\beta)\cup\{\beta\},\leq,\rho_2) \cong (F(\alpha',\beta')\cup\{\beta'\},\leq,\rho_2)$$

then t and t' can't be comparable (and different). Assume $\alpha<\alpha'$.

<u>Case 1</u>. $F(\alpha,\beta)\cap\alpha \neq F(\alpha',\beta')\cap\alpha$.

Let $\xi = \min(F(\alpha,\beta)\,\Delta\,F(\alpha',\beta'))$. Then $\xi<\alpha$. May assume $\alpha\in F(\alpha,\beta)\setminus F(\alpha',\beta')$; the other subcase is considered similarly.

Let $\xi'\in F(\alpha',\beta')$ be the ordinal that corresponds to ξ.

$\xi' \in F(\alpha',\beta')$

ξ

$F(\alpha,\beta)\cap\xi = F(\alpha',\beta')\cap\xi$

Then from Lemma 15 we get

$$\rho_2(\xi,\beta') = \rho_2(\xi',\beta') + \rho_2(\xi,\xi') > \rho_2(\xi',\beta')$$

From $(F(\alpha,\beta)\cup\{\beta\},\rho_2) \cong (F(\alpha',\beta')\cup\{\beta'\},\rho_2)$ we infer that

$$\rho_2(\xi,\beta) = \rho_2(\xi',\beta') < \rho_2(\xi,\beta')$$

so $t(\xi) \neq t'(\xi)$ so t and t' are incomparable.

<u>Case 2</u>: $F(\alpha,\beta)\cap\alpha = F(\alpha',\beta')\cap\alpha$.

Then α and α' correspond to each other in the isomorphism between $(F(\alpha,\beta)\cup\{\beta\},\rho_2)$ and $(F(\alpha',\beta')\cup\{\beta'\},\rho_2)$. So by Lemma 15

$$\rho_2(\alpha,\beta') = \rho_2(\alpha',\beta') + \rho_2(\alpha,\alpha') > \rho_2(\alpha',\beta') = \rho_2(\alpha,\beta)$$

It follows that the two functions $\varrho_2(\cdot,\beta)$ and $\varrho_2(\cdot,\beta')$ disagree at the limit ordinal α, so they must disagree bellow α, and therefore t and t' are incomparable. This follows from the following analogue of Lemma 18.

Lemma 18'. For all $\alpha < \beta < \omega_1$, the set $\{\xi < \beta : \varrho_2(\xi,\beta) = \varrho_2(\xi,\gamma)\}$ is a closed subset of β. □

§4. Positional Graphs

Definition 25. Two finite sets $F, G \subseteq \mathrm{Ord}$ are in Δ_n-position for some $n < \omega$

If their intersection admits a disjoint decomposition

$$F \wedge G = I \cup J$$

such that $I \sqsubseteq F$, $I \sqsubseteq G$ and $|J| \leq n$. A *positional graph* is a graph of the form

$$(\mathcal{V}, \neg\Delta_n)$$

where $\mathcal{V}$ is a family of finite sets of ordinals.

Theorem 26. $Chr([\omega_1]^{<\omega}, \neg\Delta_0) = \aleph_0$.

Proof. Defered. ∎

Exercise 27. Show that for all $n<\omega$,

$$K_{\aleph_1} \leq ([\omega_2]^{<\omega}, \neg\Delta_n),$$

and therefore $Chr([\omega_2]^{<\omega}, \neg\Delta_n) > \aleph_0$.