§5. Canonical Linear Orderings

Definition 28. For $\alpha, \beta < \omega_1$ put

$\alpha <_{\rho_2} \beta$ iff $\rho_2(\xi, \alpha) < \rho_2(\xi, \beta)$

for $\xi = \Delta_{\rho_2}(\alpha, \beta) = \min\{\eta \leq \min\{\alpha, \beta\} : \rho_2(\eta, \alpha) \neq \rho_2(\eta,$

Lemma 29. The cartesian square of the total ordering $<_{\rho_2}$ on ω_1 can be decomposed into countably many chains

Proof: It suffices to decompose the set $\{(\alpha, \beta) \in \omega_1^2 : \alpha < \beta\}$ into countably many chains. Fix a coloring

$$c : [\omega_1]^{<\omega} \to \omega$$

of the graph $([\omega_1]^{<\omega}, \neg\Delta_\delta)$, i.e.

$c(F) = c(G)$ implies F and G are in the Δ_0-position.

For $\alpha < \beta < \omega_1$, let $t(\alpha,\beta)$ be the isomorphism type of the <u>ϱ_2-model</u>

$$(F(\alpha,\beta) \cup \{\beta\}, <, \varrho_2).$$

It suffices to prove the following fact.

<u>Claim.</u> Suppose $\alpha < \beta$ and $\alpha' < \beta'$ are such that

$$c(F(\alpha,\beta) \cup \{\beta\}) = c(F(\alpha',\beta') \cup \{\beta'\})$$

and $$t(\alpha,\beta) = t(\alpha',\beta').$$

Then

$$\alpha <_{\varrho_2} \alpha' \text{ implies } \beta <_{\varrho_2} \beta'.$$

~~Let $\xi = \Delta_{\varrho_2}(\alpha,\alpha')$ and $\eta = \Delta_{\varrho_2}(\beta,\beta')$.~~

<u>Proof</u>: Assume $\alpha <_{\varsigma_2} \alpha'$. This in particular means that $\alpha \neq \alpha'$ and since $F(\alpha,\beta) \cup \{\beta\}$ and $F(\alpha',\beta') \cup \{\beta'\}$ are in the Δ_0-position, $\beta \neq \beta'$ must hold, and in fact $\beta \notin F(\alpha',\beta')$ and $\beta' \notin F(\alpha,\beta)$. Moreover, $F(\alpha,\beta) \,\Delta\, F(\alpha',\beta') \neq \emptyset$ as α and α' are members of this set. Let

$$\xi = \min\big(F(\alpha,\beta) \,\Delta\, F(\alpha',\beta')\big)$$

and let

$$I = F(\alpha,\beta) \cap F(\alpha',\beta').$$

Then $I \sqsubseteq F(\alpha,\beta), F(\alpha',\beta')$, so

$$I = F(\alpha,\beta) \cap \xi = F(\alpha',\beta') \cap \xi.$$

Let $\xi' \in F(\alpha',\beta')$ corresponds to ξ in the isomorphism between the two ς_2-models.

Then $\xi < \xi'$ and

$$\xi' = \min\left(F(\alpha', \beta') \setminus \xi\right).$$

Then by Lemma 15,

$$\begin{aligned} \rho_2(\xi, \alpha') &= \rho_2(\xi', \alpha') + \rho_2(\xi, \xi') \\ &= \rho_2(\xi, \alpha) + \rho_2(\xi, \xi'), \end{aligned}$$

and so in particular $\rho_2(\xi, \alpha') \neq \rho_2(\xi, \alpha)$ and we conclude that

$$\Delta_{\rho_2}(\alpha, \alpha') \leq \xi.$$

Similarly,

$$\Delta_{\rho_2}(\beta, \beta') \leq \xi.$$

We claim that $\Delta_{\rho_2}(\alpha, \alpha') > \max(I)$ and $\Delta_{\rho_2}(\beta, \beta') > \max(I)$.

To see this consider $\eta \leq \max(I)$ and let

$$\bar{\eta} = \min(I \setminus \eta).$$

By Lemma 15,

$\rho_2(\eta, \alpha) = \rho_2(\bar{\eta}, \alpha) + \rho_2(\eta, \bar{\eta})$ and

$\rho_2(\eta, \alpha') = \rho_2(\bar{\eta}, \alpha') + \rho_2(\eta, \bar{\eta})$.

From

$$(F(\alpha, \beta) \cup \{\beta\}, <, \rho_2) \cong (F(\alpha', \beta') \cup \{\beta'\}, <, \rho_2)$$

we conclude that $\rho_2(\bar{\eta}, \alpha) = \rho_2(\bar{\eta}, \alpha')$,

so $\rho_2(\eta, \alpha) = \rho_2(\eta, \alpha')$. Similarly, we

get that $\rho_2(\eta, \beta) = \rho_2(\eta, \beta')$.

Consider an ordinal η such that

$$\max(I) < \eta < \xi.$$

By Lemma 15,

$\rho_2(\eta, \alpha) = \rho_2(\xi, \alpha) + \rho_2(\eta, \xi)$, and

$\rho_2(\eta, \alpha') = \rho_2(\xi', \alpha') + \rho_2(\eta, \xi)$

By $\rho_2(\xi, \alpha) = \rho_2(\xi', \alpha')$ follows from the isomorphism of the two ϱ_2-models

so we conclude that $\varrho_2(\eta,\alpha) = \varrho_2(\eta,\alpha')$

It follows that $\Delta_{\varrho_2}(\alpha,\alpha') \geq \xi$, and

Similarly $\Delta_{\varrho_2}(\beta,\beta') \geq \xi$. It follows that

$$\Delta_{\varrho_2}(\alpha,\alpha') = \xi = \Delta_{\varrho_2}(\beta,\beta').$$

Recall that we are working under the assumption $\alpha <_{\varrho_2} \alpha'$ and

$\xi = \min(F(\alpha,\beta) \Delta F(\alpha',\beta')) \in F(\alpha,\beta)$

and therefore $\xi < \xi'$ which by Lemma 15 transfers to

$$\varrho_2(\xi,\alpha') = \varrho_2(\xi,\alpha) + \varrho_2(\xi,\alpha') >$$
$$> \varrho_2(\xi,\alpha)$$

So the symmetric assumption $\xi \in F(\alpha',\beta')$

would not agree with $\alpha <_{\varrho_2} \alpha'$.

Applying Lemma 15 again we get

$$\begin{aligned}\varrho_2(\xi,\beta') &= \varrho_2(\xi',\beta') + \varrho(\xi,\xi') \\ &= \varrho_2(\xi,\beta) + \varrho(\xi,\xi') \\ &> \varrho_2(\xi,\beta).\end{aligned}$$

From this we conclude that $\beta <_{\varrho_2} \beta'$. $\Box$

__Definition 30__. For $\alpha,\beta < \omega_1$, let

$$\alpha <_{\varrho_0} \beta \text{ iff } \varrho_0(\xi,\alpha) <^r_{lex} \varrho_0(\xi,\beta),$$

for $\xi = \Delta_{\varrho_0}(\alpha,\beta) = \min\{\eta \leq \min\{\alpha,\beta\} : \varrho_0(\eta,\alpha) \neq \varrho_0(\eta,\beta)\}$.

Then working as above we get

__Theorem 31__. The cartesian square of the total ordering $<_{\varrho_0}$ on ω_1 can be decomposed into countably many chains.

<u>Definition 32</u>. For $\alpha, \beta < \omega_1$, set

$$\alpha <_{\rho_1} \beta \text{ iff } \rho_1(\xi, \alpha) < \rho_1(\xi, \beta) \text{ for}$$

$$\xi = \Delta_{\rho_1}(\alpha, \beta) = \min\{\eta \leq \min\{\alpha, \beta\} : \rho_1(\eta, \alpha) \neq \rho_1(\eta, \beta)\}$$

<u>Theorem 33</u>. The cartesian square of the total ordering $<_{\rho_1}$ on ω_1 can be decomposed into countably many chains. $\square$.

<u>Question 34</u>. Are the ordering $<_{\rho_0}$, $<_{\rho_1}$ and $<_{\rho_2}$ any different?

Let

$$C(\rho_i) = (\omega_1, <_{\rho_i}) \quad (i = 0, 1, 2, 3).$$

Theorem 35. Assume $\mathfrak{m} > \omega_1$. Then

(a) For every $i \leq 3$, the ordering $C(\varrho_i)$ is a minimal uncountable ordering i.e. for every uncountable linear ordering L such that $L \leq C(\varrho_i)$ we have $C(\varrho_i) \leq L$.

(b) Fix $i \leq 3$. Then for every uncountable ordering L whose square $L \times L$ can be decomposed into countably many chains, we have not either $C(\varrho_i) \leq L$ or $C(\varrho_i)^* \leq L$. ∎

Corollary 36. If $\mathfrak{m} > \omega_1$ then $C(\varrho_i) \equiv C(\varrho_j)$ for all $i, j \leq 3$.

Theorem (Baumgartner, Moore). Assume MM > ...
If B is any uncountable set of reals of cardinality $\aleph_1$ then

$$\omega_1,\ \omega_1^*,\ B,\ C(\S_2),\ C(\S_2)^*$$

forms a basis for the class of all uncountable linear orderings, i.e., for every uncountable linearly ordered set L there is a member K of the basis such that $K \leq L$.

Problem 38. Determine the consistency strength of this conclusion. Does it involve large cardinals at all?

§6. Lipschitz trees

Definition 39. A partial map

$$g : S \to T$$

from a tree S into a tree T is *Lipshitz* if g is *level-preserving* and if

$$\Delta(g(x), g(y)) \geq \Delta(x, y)$$

for all $x, y \in \mathrm{dom}(g)$. [Recall, that for a tree T, $\Delta : T^2 \to \mathrm{Ord}$ is defined by

$$\Delta(s,t) = \mathrm{otp}\{x \in T : x <_T s \ \& \ x <_T t\}]$$

<u>Definition 40.</u> A <u>Lipschitz tree</u> is any A-tree T with the property that every level preserving map from an uncountable subset A of T is Lipschitz on an uncountable subset of A.

~~Examples $T(\varrho_0)$, $T(\varrho_1)$, $T(\varrho_2)$ and $T(\varrho_3)$ are all Lipschitz~~

<u>Definition 41.</u> A <u>coherent tree</u> is any A-tree T that is isomorphic to a downward closed subset S of $I^{<\omega_1}$ for some countable set I such that for all $\alpha < \omega_1$ and $s, t \in S \cap I^{\alpha}$,

$$\{\xi < \alpha : s(\xi) \neq t(\xi)\} \text{ is finite.}$$

<u>Lemma 4.2</u> Suppose T is a coherent tree with the property that every uncountable subset of T contains an uncountable antichain. Then T is Lipschitz.

<u>Proof</u>: Consider uncountable $A \subseteq T$ and $g : A \to T$ such that

$(\forall s \in A)\ ht(s) = ht(g(s))$.

For each limit ordinal $\delta < \omega_1$ pick $t_\delta \in A$ of height $\geq \delta$ and a node $s_\delta \in T$ of height $= \delta$. Let

$$D_\delta = \{\xi < \delta : s_\delta(\xi) \neq t_\delta(\xi) \text{ or } s_\delta(\xi) \neq g(t_\delta)(\xi)\}$$

Then D_δ is a finite subset of δ for every countable limit ordinal δ, so applying the Pressing Down Lemma we get stationary $E \subseteq \Delta = \{\delta < \omega_1 : \delta \text{ limit}\}$ and $D \subseteq \omega_1$ such that

$$\forall \delta \in E \quad D_\delta = D.$$

Shrinking S, we may assume that for some $s, t \in T$ of height $\gamma_D = \max(D)+1$, we have that

$$\forall \delta \in E \quad s_\delta \restriction \gamma_D = s \ \& \ t_\delta \restriction \gamma_D = t$$

Applying our assumption about T, we now find uncountable $F \subseteq E$ such that both sets

$$\{t_\delta \restriction \delta : \delta \in F\} \text{ and } \{g(t_\delta) \restriction \delta : \delta \in F\}$$

are antichains. It follows that for $\gamma \neq \delta$ in F

$$\Delta(t_\gamma, t_\delta) = \Delta(g(t_\gamma), g(t_\delta)),$$

so $g \restriction \{t_\delta : \delta \in F\}$ is Lipschitz. $\square$

<u>Lemma 43.</u> If $\mathfrak{m} > \omega_1$ then every Lipschitz tree is coherent. $\square$

<u>Question 44</u> Are $T(\rho_0)$ and $T(\rho_2)$ coherent without the assumption of $\mathfrak{m} > \omega_1$?

<u>Theorem 45</u> (Martinez-Ranero) $T(\rho_0)$ is coherent.

<u>Theorem 46</u> (Peng) $T(\rho_2)$ is coherent.

<u>Proof:</u> Define $a : [\omega_1]^2 \to \mathbb{Z}$ by

$$a_2(\alpha,\beta) = \rho_2(\alpha,\beta) - \rho_2(\alpha \dot{-} 1, \beta) + \rho_2(\alpha \dot{-} 1, \alpha)$$

Note that since $\alpha \dot{-} 1 = \alpha$ for α a limit ordinal we get that $a_2(\alpha,\beta) = 0$.

It follows that the tree

$$T(a_2) = \{ a_2(\cdot,\beta) \restriction \alpha : \alpha \leq \beta < \omega_1 \}$$

does not split at limit levels. As we know this is also true for $T(\varrho_2)$ (and $T(\varrho_1)$). Note also that $\Delta_{a_2} = \Delta_{\varrho_2}$ i.e., for all $\alpha < \beta < \omega_1$,

$$\min\{\xi \le \alpha : \varrho_2(\xi,\alpha) \neq \varrho_2(\xi,\beta)\} = \min\{\xi \le \alpha : a_2(\xi,\alpha) \neq a_2(\xi,\beta)\}$$

It follows that

$$\varrho_2(\cdot,\beta) \restriction \alpha \mapsto a_2(\cdot,\beta) \restriction \alpha$$

is an isomorphic embedding. So it remains to prove that $T(a_2)$ is coherent i.e. that for all $\beta < \gamma < \omega_1$

$$D = \{\xi < \beta : a_2(\xi,\beta) \neq a_2(\xi,\gamma)\}$$

is a finite set. Assume otherwise and let α be its first limit point. Choose $\xi \in D \cap \alpha$ such that

$\xi - 1 > \max(F(\alpha,\beta) \cap \alpha)$, $\max(F(\alpha,\gamma) \cap \alpha)$

(Recall that each member of D must be a successor ordinal). Applying Lemma 15, we get

$$a_2(\xi,\beta) = \varrho_2(\xi,\beta) - \varrho_2(\xi-1,\beta) + 1$$

$$= \cancel{\varrho_2(\alpha,\beta)} + \varrho_2(\xi,\alpha) - \cancel{\varrho_2(\alpha,\beta)} - \varrho_2(\xi-1,\alpha) + 1$$

$$= \varrho_2(\xi,\alpha) - \varrho_2(\xi-1,\alpha) + 1$$

Similarly

$$a_2(\xi,\gamma) = \varrho_2(\xi,\gamma) - \varrho_2(\xi-1,\gamma) + 1 =$$

$$= \varrho_2(\alpha,\gamma) + \varrho_2(\xi,\alpha) - \varrho_2(\alpha,\gamma) - \varrho_2(\xi-1,\alpha) + 1$$

$$= \varrho_2(\xi,\alpha) - \varrho_2(\xi-1,\alpha) + 1$$

So, $a_2(\xi,\beta) = a_2(\xi,\gamma)$, a contradiction!

It follows that $T(a_2)$ is coherent. $\square$

<u>Proof of Theorem 45</u>. The proof uses a similar idea. Let P be the sequence of primes $p_0, p_1, p_2, \ldots$ $(2, 3, 5, \ldots)$

For a sequence $t = (n_i : i < \ell)$ of integers, let

$$P^t = \prod_{i<\ell} p_i^{n_i}.$$

For $t = (n_i : i < \ell) \in \mathbb{Z}^{<\omega}$ let

$$-t = (-n_i : i < \ell) \in \mathbb{Z}^{<\omega}.$$

Finally, we are ready to define

$$a_0 : [\omega_1]^2 \to \omega$$

by

$$a_0(\alpha, \beta) = P^{\rho_0(\alpha,\beta)} \cdot P^{-\rho_0(\alpha-1,\beta)} \cdot P^{\rho_0(\alpha-1,\alpha)}$$

Working as above one checks that

$$\rho_0(\cdot,\beta) \restriction \alpha \longmapsto a_0(\cdot,\beta) \restriction \alpha$$

is an isomorphism between $T(\rho_0)$ and $T(a_0)$ and that $T(a_0)$ is coherent. $\square$

Definition 47. For two trees S and T we let $S \leq T$ if there is a strictly increasing (equivalently, Lipschitz) map $f : S \to T$. Let $S < T$ whenever $S \leq T$ and $T \not\leq S$ and let $S \equiv T$ whenever $S \leq T$ and $T \leq S$; we call S and T equivalent whenever $S \equiv T$.

Lemma 48 If $m > \omega_1$ then every coherent trees is equivalent to its homogeneous closure and two homogeneous coherent trees are equivalent iff they are isomorphic. $\square$

<u>Theorem 49.</u> Assume $\mathrm{mm} > \omega_1$.

(a) Every Lipshitz tree is comparable to every Aronszajn tree.

(b) If $\mathcal{L}$ denotes the class of Lipschitz trees then $(\mathcal{L}, \leq)$ is a discrete chain ~~and~~ every $T \in \mathcal{L}$ has an <u>immediate successor</u> $T^{(1)} \in \mathcal{L}$.

(c) If $\mathcal{A}$ denotes the class of Aronszajn trees then $\mathcal{L}$ is both cofinal and coinitial in $(\mathcal{A}, \leq)$.

d) (Martinez-Ranero – T.) $(\mathcal{L}/\mathbb{Z}, \leq)$ is isomorphic to the $\aleph_2$-saturated linear ordering of cardinality $\aleph_2$. ∎

<u>Definition 50</u>. Let g be a partial map from ω_1 into ω_1 and T a downward closed subset of some $I^{<\omega_1}$. Then the <u>g-shift</u>, $T^{(g)}$, is the downward closure of

$$\{t^{(g)} : t \in T \upharpoonright \Omega\}, \text{ where}$$

$$\Omega = \{\delta < \omega_1 : g''\delta \subseteq \delta\},$$

and where $t^{(g)}$ is defined by

$$t^{(g)}(\xi) = t(g(\xi))$$

if $\xi \in \mathrm{dom}(g)$ and $t^{(g)}(\xi) = 0$, otherwise.

<u>Theorem 51</u>. Assuming $\mathrm{MM} \geq \omega_1$, for every pair S and T of Lipschitz trees there is strictly increasing partial map $g : \omega_1 \to \omega_1$ such that $S \equiv T^{(g)}$. $\square$

<u>Corollary 52.</u> Assuming $mm > \omega_1$,

$$\mathcal{L}/_{\equiv} = \{ T(\varrho_0)^{(g)} : g: \omega_1 \to \omega_1 \text{ partial increasing} \}.$$

<u>Conclusion:</u> So under $mm > \omega_1$ there is really only one $\mathcal{L}$-tree, the one obtained from a characteristic of walks on ω_1.

§7. <u>Canonical filters on ω</u>

<u>Definition 53.</u> Fix a Lipschitz tree T. For $X \subseteq T$, let

$$\Delta_T(X) = \{ \Delta(s,t) : s,t \in X,\ s,t \text{ incomparable} \}$$

Let

$$\mathcal{U}_{\omega_1}(T) = \{ \Gamma \subseteq \omega_1 : (\exists \text{ uncountable } X \subseteq T)\ \Delta_T(X) \subseteq \Gamma \}$$

<u>Lemma 54</u>. For every Lipschitz tree T, $\mathcal{U}_{\omega_1}(T)$ is a uniform filter on ω_1.

<u>Proof</u>: Given uncountable $X, Y \subseteq T$ we need to find uncountable $Z \subseteq T$ such that

$$\Delta_T(Z) \subseteq \Delta_T(X) \cap \Delta_T(Y).$$

For each $\alpha < \omega_1$ pick $x_\alpha \in X$, $y_\alpha \in Y$ of height $\geq \alpha$. Note that in any Lipschitz tree any uncountable set can be refined by an uncountable antichain. So we can find uncountable $\Gamma \subseteq \omega_1$ such that the sets

$$x_\delta \restriction \delta \ (\delta \in \Gamma) \quad \text{and} \quad y_\delta \restriction \delta \ (\delta \in \Gamma)$$

are antichains of T.

Applying the definition that T is Lipschitz successively first for the function $x_\delta \restriction \delta \mapsto y_\delta \restriction \delta$ and then to its inverse, we obtain an uncountable set $\Sigma \subseteq \Gamma$ such that

$$\Delta_T(x_\gamma \restriction \gamma, x_\delta \restriction \delta) = \Delta_T(y_\gamma \restriction \gamma, y_\delta \restriction \delta)$$

for all $\gamma, \delta \in \Sigma$, $\gamma \neq \delta$. Since $\Delta_T(x_\gamma \restriction \gamma, x_\delta \restriction \delta) = \Delta_T(x_\gamma, x_\delta)$ and $\Delta_T(y_\gamma \restriction \gamma, y_\delta \restriction \delta) = \Delta_T(y_\gamma, y_\delta)$, taking $Z = \{x_\gamma : \gamma \in \Sigma\}$ we get the desired conclusion $\Delta_T(Z) \subseteq \Delta_T(X) \cap \Delta_T(Y)$. $\square$

<u>Theorem 55.</u> If $m > \omega_1$ then $\mathcal{U}_{\omega_1}(T)$ is an ultrafilter for every Lipschitz tree T.

<u>Proof</u>: Let $\Gamma \subseteq \omega_1$ be a given set. We need to find uncountable $X \subseteq T$ such that either

$$\Delta_T(X) \subseteq \Gamma \text{ or } \Delta_T(X) \cap \Gamma = \emptyset.$$

Let $\mathcal{P}_\Gamma$ be the poset of all finite subsets p of T that take no more than one point from a given level of T such that $\Delta_T(p) \subseteq \Gamma$. If $\mathcal{P}_\Gamma$ satisfies the countable chain condition then an application of $m > \omega_1$ will give us an uncountable filter $\mathcal{G} \subseteq \mathcal{P}_\Gamma$, and therefore an uncountable set

$$X = \bigcup \mathcal{G}$$

such that $\Delta_T(X) \subseteq \Gamma$. So we analize the alternative that $\mathcal{P}_\Gamma$ is not ccc. Fix a sequence p_δ ($\delta<\omega_1$) of pairwise incompatible members of $\mathcal{P}_\Gamma$. Applying the Δ-system Lemma we may assume p_δ's are pairwise disjoint and all of some fix size n. So, re-enumerating, we may assume that for each δ, the nodes in p_δ have all height $\geq \delta$. Applying the Lipschitz condition on T successively n^2 times we arrive at an uncountable set $\Gamma \subseteq \omega_1$ such that for all $i,j<n$,

$$p_\delta(i) \upharpoonright \delta \longmapsto p_\delta(j) \upharpoonright \delta \quad (\delta \in \Gamma)$$

is a Lipshitz map. (Here $p_\delta(i)$ is the ith element of p_δ in some fixed enumeration.)

Refining Γ we may assume that for all $i < n$,

$$p_\gamma(i) \restriction \gamma \quad (\gamma \in \Gamma)$$

is a sequence of pairwise incompatible elements and that for some fixed ordinal $\bar{\gamma} < \omega_1$ and all $\gamma, \delta \in \Gamma$ and $i, j < n$:

(a) $p_\gamma(i) \restriction \gamma \neq p_\gamma(j) \restriction \gamma$ implies $p_\gamma(i) \restriction \bar{\gamma} \neq p_\gamma(j) \restriction \bar{\gamma}$,

(b) $p_\gamma(i) \restriction \bar{\gamma} = p_\delta(i) \restriction \bar{\gamma}$.

Combining all these properties we get that for $\gamma \neq \delta$ in Γ

$$\Delta_T(p_\gamma \cup p_\delta) = \Delta_T(p_\gamma) \cup \Delta_T(p_\delta) \cup \{\Delta_T(p_\gamma(0), p_\delta(0))\}$$

Since $\Delta_T(p_\gamma) \cup \Delta_T(p_\delta) \subseteq \Gamma$ and since p_γ and p_δ are incompatible we conclude that $\Delta_T(p_\gamma(0), p_\delta(0)) \notin \Gamma$ for all $\gamma \neq \delta$ in Γ.

Putting $X = \{ p_\gamma^{-1}(0) : \gamma \in \Gamma \}$ we get an uncountable subset of T such that

$$\Delta_T(X) \cap \Gamma = \emptyset,$$

as required. $\square$

<u>Theorem 56</u>. Assume $m > \omega_1$.

(a) For every pair S and T of Lipschitz trees,

$$S \equiv T \quad \text{iff} \quad \mathcal{U}_{\omega_1}(S) = \mathcal{U}_{\omega_1}(T)$$

(b) For every pair S and T of Lipschitz trees

$$\mathcal{U}_{\omega_1}(S) \equiv_{RK} \mathcal{U}_{\omega_1}(T). \ \square$$

<u>Conclusion</u>. Under $m > \omega_1$, there is a canonical ultrafilter $\mathcal{U}_{\omega_1}(\rho_3) = \mathcal{U}_{\omega_1}(T(\rho_3))$ that is Σ_1-definable in $(H(\omega_2), \in)$.

<u>Theorem 57</u>. Assume $\mathfrak{m} > \omega_1$.

(a) For every Lipschitz tree T and every $f: \omega_1 \to \omega$ the image

$$f[\mathcal{U}_{\omega_1}(T)] = \{M \subseteq \omega_1 : f^{-1}(M) \in \mathcal{U}_{\omega_1}(T)\}$$

is a <u>selective</u> ultrafilter on ω.

(b) For every pair S and T of Lipschitz trees and any pair of mappings $f: \omega_1 \to \omega$ and $g: \omega_1 \to \omega$, if the ultrafilters

$$f[\mathcal{U}_{\omega_1}(S)] \text{ and } g[\mathcal{U}_{\omega_1}(T)]$$

are nonprincipal then they are RK-equivalent.

Recall that an ultrafilter $\mathcal{U}$ on ω is <u>selective</u> if for every sequence $\{M_n : n<\omega\} \subseteq \mathcal{U}$ there is $N \in \mathcal{U}$ such that $N \setminus \{0,1,\dots,n\} \subseteq M_n$ for all $n \in N$; equivalently if for every $f:\omega \to \omega$ there is $M \in \mathcal{U}$ such that either $f \upharpoonright M$ is 1-1 or constant.

<u>Conclusion</u>. Under $m > \omega_1$ there is a canonical ultrafilter $\mathcal{U}_\omega(\varrho_3)$ on ω Σ_1-definable in $(H(\omega_2), \in)$ using the structure $(C_\alpha : \alpha < \omega_1)$ and characteristics of walks. It is rather remarkable

that this RK-unique ultrafilter is actually <u>selective</u> i.e., <u>generic</u>.

<u>Remark</u>: The ultrafilter $\mathcal{U}_{\omega_1}(\varrho_3)$ has a canonical map $d_\Delta : \omega_1 \to \omega$ such that

$$\mathcal{U}_\omega(\varrho_3) := d_\Delta[\mathcal{U}_{\omega_1}(\varrho_3)]$$

is nonprincipal. The map d_Δ is simply the distance map from the set Δ of all countable limit ordinals. More precisely

$$d_\Delta(\alpha) = \alpha - \lambda(\alpha),$$

where $\lambda(\alpha) = \max\{\lambda \in \Delta : \lambda \leq \alpha\}$.

<u>Summary</u>: The basic structure

$(\omega_1, C_\alpha\ (\alpha<\omega_1))$

allows walks to be defined and their characteristics such as

ρ_0, ρ_1, ρ_2 and ρ_3

to be analysed. They lead to structures such as

$C(\rho_i),\ T(\rho_i),\ \mathcal{U}_{\omega_1}(\rho_i),\ \mathcal{U}_\omega(\rho_i)$

that are at the same time <u>critical</u> for ~~their~~ own classes of structures and <u>unique</u>, and so, in particular, independent of the base structure $(\omega_1, C_\alpha(\alpha<\omega_1))$.