§8 The distance function ρ

Definition 58. Define $\rho : [\omega_1]^2 \to \omega$ recursively by

$$\rho(\alpha,\beta) = \max\{|C_\beta \cap \alpha|, \rho(\alpha, \min(C_\beta \setminus \alpha)), \rho(\xi,\alpha) : \xi \in C_\beta \cap \alpha\}$$

with the boundary condition $\rho(\alpha,\alpha) = 0$ for all α.

Lemma 59. For all $\alpha < \beta < \gamma < \omega_1$ and $n < \omega$,

(a) $\{\xi \le \alpha : \rho(\xi,\alpha) \le n\}$ is finite,

(b) $\rho(\alpha,\gamma) \le \max\{\rho(\alpha,\beta), \rho(\beta,\gamma)\}$,

(c) $\rho(\alpha,\beta) \le \max\{\rho(\alpha,\gamma), \rho(\beta,\gamma)\}$.

Proof: (a) This follows from the inequality

$$\rho(\alpha,\beta) \geq \rho_1(\alpha,\beta)$$

and the corresponding property of ϱ_1 proved above (Lemma 1 (a)).

We prove (b) and (c) by induction.

First we deal with (b). Let

$$n = \max\{\varrho(\alpha,\beta), \varrho(\beta,\gamma)\}$$

We need to show that $\varrho(\alpha,\gamma) \leq n$. Let $\gamma_\alpha = \min(C_\gamma \setminus \alpha)$ and $\gamma_\beta = \min(C_\gamma \setminus \beta)$.

<u>Case 1</u>^{b}. $\gamma_\alpha = \gamma_\beta$. Then by the ind. hyp.,

$$\varrho(\alpha, \gamma_\alpha) \leq \max\{\varrho(\alpha,\beta), \varrho(\beta,\gamma_\beta)\}$$

From the definition of $\varrho(\beta,\gamma)$ we get

$$n \geq \varrho(\beta,\gamma) \geq \varrho(\beta,\gamma_\beta).$$

Plugging this into the above inequality, we get $n \geq \varrho(\alpha,\gamma_\alpha)$.

Consider $\xi \in C_\mu \cap \alpha = C_\mu \cap \beta$.

By the ind. hyp

$$\rho(\xi, \alpha) \leq \max\{\rho(\xi, \beta), \rho(\alpha, \beta)\}$$

From the definition of $\rho(\beta, \mu)$ we infer that $\rho(\beta, \mu) \geq \rho(\xi, \beta)$. Plugging this into the above inequality we get that $n \geq \rho(\xi, \alpha)$. Since

$$|C_\mu \cap \alpha| = |C_\mu \cap \beta| \leq \rho(\beta, \mu) \leq n$$

we get the last factor in the definition of $\rho(\alpha, \mu)$ to be dominated by n, and therefore $\rho(\alpha, \mu) \leq n$.

<u>Case 2^b</u>. $\gamma_\alpha < \gamma_\beta$. Then

$$\gamma_\alpha \in C_\mu \cap \beta,$$

and so $\rho(\gamma_\alpha, \beta) \leq \rho(\beta, \mu) \leq n$.

similarly, for every

$$\xi \in C_\gamma \cap \alpha \subseteq C_\gamma \cap \beta,$$

$$\rho(\xi, \alpha) \le \max\{\rho(\xi,\beta), \rho(\alpha,\beta)\} \le n.$$

Finally, $|C_\gamma \cap \alpha| \le |C_\gamma \cap \beta| \le \rho(\beta,\gamma) \le n$.
Combining all these inequalities we get the desired conclusion $\rho(\alpha,\gamma) \le n$.

To prove (c), let

$$n = \max\{\rho(\alpha,\gamma), \rho(\beta,\gamma)\}.$$

We need to show that $\rho(\alpha,\beta) \le n$.
Let γ_α and γ_β be as above.

<u>Case 1°</u>. $\gamma_\alpha = \gamma_\beta$. By the inductive hypothesis

$$\rho(\alpha,\beta) \le \max\{\rho(\alpha,\gamma_\alpha), \rho(\beta,\gamma_\beta)\}$$

Since $\rho(\alpha, \gamma_\alpha) \le \rho(\alpha, \gamma) \le n$ and $\rho(\beta, \gamma_\beta) \le \rho(\beta, \gamma) \le n$ we get the desired inequality $\rho(\alpha, \beta) \le n$.

<u>Case 2°</u>: $\gamma_\alpha < \gamma_\beta$. By ind. hyp.

$$\rho(\alpha, \beta) \le \max\{\rho(\alpha, \gamma_\alpha), \rho(\gamma_\alpha, \beta)\}.$$

As before $\rho(\alpha, \gamma_\alpha) \le \rho(\alpha, \gamma) \le n$. Since $\gamma_\alpha \in C_\gamma \cap \beta$ from the definition of $\rho(\beta, \gamma)$ we get that

$$\rho(\gamma_\alpha, \beta) \le \rho(\beta, \gamma) \le n.$$

Combining the two inequalities, we get that $\rho(\alpha, \beta) \le n$, as required. $\square$.

Lemma 60

$\alpha < \beta < \gamma$ and $\rho(\alpha,\beta) > \rho(\beta,\gamma)$ implies

$$\rho(\alpha,\gamma) = \rho(\alpha,\beta).$$

Proof: Exercise. $\square$

Corollary 61. $T(\rho) = \{\rho(\cdot,\beta)\restriction\alpha : \alpha \leq \beta < \omega_1\}$ is also a coherent tree. $\square$

§9. The injective ρ

Definition 62. Define $\bar{\rho} : [\omega_1]^2 \to \omega$ by

$$\bar{\rho}(\alpha,\beta) = 2^{\rho(\alpha,\beta)} \cdot (2 \cdot |\{\xi \leq \alpha : \rho(\xi,\alpha) \leq \rho(\alpha,\beta)\}| + 1).$$

Lemma 63. For $\alpha < \beta < \gamma < \omega_1$,

(a) $\bar{\rho}(\alpha,\gamma) \neq \bar{\rho}(\beta,\gamma)$

(b) $\bar{\rho}(\alpha,\gamma) \leq \max\{\bar{\rho}(\alpha,\beta), \bar{\rho}(\beta,\gamma)\}$

(c) $\bar{\rho}(\alpha,\beta) \leq \max\{\bar{\rho}(\alpha,\gamma), \bar{\rho}(\beta,\gamma)\}$.

<u>Proof</u>: Execise. □

<u>Lemma 64</u>. $\bar{\rho}(\alpha,\beta) \neq \bar{\rho}(\beta,\gamma)$ for all $\alpha < \beta < \gamma < \omega_1$.

<u>Proof</u>: Suppose the conclusion fails for some triple $\alpha < \beta < \gamma$. Let

$$n = \bar{\rho}(\alpha,\beta) = \bar{\rho}(\beta,\gamma).$$

Write $n = 2^i(2j+1)$ for some integers i and j. Then

$$i = \rho(\alpha,\beta) = \rho(\beta,\gamma)$$

and

$$|\{\xi \leq \alpha : \rho(\xi,\alpha) \leq i\}| = j = |\{\xi \leq \beta : \rho(\xi,\beta) \leq i\}|$$

Since $\rho(\alpha,\beta) = i$,

$$\alpha \in \{\xi \leq \beta : \rho(\xi,\beta) \leq i\},$$

So the set

$$\{\xi \leq \alpha : \rho(\xi, \alpha) \leq i\}$$

is an initial segment of the set

$\{\xi \leq \beta : \rho(\xi, \beta) \leq i\}$ (EXERCISE).

Since the two sets have the same cardinality, they must be equal, a contradiction since clearly β does not belong to $\{\xi \leq \alpha : \rho(\xi, \alpha) \leq i\}$. $\square$

<u>Lemma 65.</u>

$$\eta_\alpha \neq \eta_\beta < \min\{\alpha, \beta\} \text{ and}$$

$$\bar{\rho}(\eta_\alpha, \alpha) = \bar{\rho}(\eta_\beta, \beta) = n$$

imply not

$$\bar{\rho}(\eta_\alpha, \beta), \bar{\rho}(\eta_\beta, \alpha) > n. \quad \square$$

§10. A Souslin tree from ϱ

Definition 66. For $p \in \omega^{<\omega}$ define $<_p \subseteq \omega_1 \times \omega_1$ by letting

$\alpha <_p \beta$ iff (a) $\alpha < \beta$,
(b) $\bar{\varrho}(\alpha, \beta) \in |p|$, and
(c) $(\forall \xi < \alpha)[\bar{\varrho}(\xi, \alpha) < p \implies \overleftarrow{p}(\bar{\varrho}(\xi,\alpha)) = p(\bar{\varrho}(\xi,\beta)).]$

Lemma 67.

(a) $<_p$ is a tree ordering on ω_1 of height $\leq |p|+1$

(b) $p \subseteq q$ implies $<_p \subseteq <_q$.

Definition 68. For $x \in \omega^\omega$, set

$$<_x = \bigcup_{n<\omega} <_{x \restriction n}$$

Lemma 69. For every infinite $\Gamma \subseteq \omega_1$, the set

$$U_\Gamma = \{x \in \omega^\omega : (\exists \alpha, \beta \in \Gamma)\ \alpha <_x \beta\}$$

is dense open subset of ω^ω. $\square$

Lemma 70. For every infinite $\Gamma \subseteq \omega_1$ that is an antichain in $T(\bar{\varrho})$ (i.e. $\varrho(\cdot,\alpha) \not\subseteq \varrho(\cdot,\beta)$ for all $\alpha \neq \beta$ in Γ), the set

$$V_\Gamma = \{x \in \omega^\omega : \exists \alpha, \beta \in \Gamma \ \ \alpha \not<_x \beta\}$$

is a dense open subset ~~of~~ of ω^ω.

Corollary 71. If $c \in \omega^\omega$ is a Cohen real, $(\omega_1, <_c)$ is a Souslin tree. □

§11 A Hausdorff Gap From ϱ

Definition 72. A function $a : [\omega_1]^2 \to \omega$ is transitive whenever

$$a(\alpha,\gamma) \leq \max\{a(\alpha,\beta), a(\beta,\gamma)\}$$

for all $\alpha < \beta < \gamma < \omega_1$.

Example 73 Suppose

$$\{A_\alpha : \alpha < \omega_1\} \subseteq [\omega]^\omega$$

is such that $A_\alpha \subseteq^* B_\beta$ for $\alpha < \beta < \omega_1$.

Then $a : [\omega_1]^2 \to \omega$ defined by

$$a(\alpha, \beta) = \min\{n : A_\alpha \setminus n \subseteq A_\beta\}$$

is transitive.

Definition 74. Fix transitive

$$a : [\omega_1]^2 \to \omega.$$

Define $\rho_a : [\omega_1]^2 \to \omega$ recursively by

$$\rho_a(\alpha, \beta) = \max\{|C_\beta \cap \alpha|, a(\min(C_\beta \setminus \alpha), \beta), \rho_a(\alpha, \min(C_\beta \setminus \alpha)),$$
$$\rho_a(\xi, \alpha) : \xi \in C_\beta \cap \alpha\}$$

where the boundary condition is that

$\rho_a(\alpha, \alpha) = 0$ for all α.

Lemma 75. For all $\alpha < \beta < \gamma < \omega_1$ and $n < \omega$,

(a) $\{\xi \leq \alpha : \rho_a(\xi, \alpha) \leq n\}$ is finite,

(b) $\rho_a(\alpha, \gamma) \leq \max\{\rho_a(\alpha, \beta), \rho_a(\beta, \gamma)\}$,

(c) $\rho_a(\alpha, \beta) \leq \max\{\rho_a(\alpha, \gamma), \rho_a(\beta, \gamma)\}$

(d) $\rho_a(\alpha, \beta) \geq a(\alpha, \beta)$. $\square$

Lemma 76 $\rho_a(\alpha, \beta) \geq \rho_a(\alpha+1, \beta)$ for $0 < \alpha < \beta$, $\alpha \in \Lambda$. $\square$

Fix a strictly increasing $\subseteq^*$-sequence $\{A_\alpha : \alpha < \omega_1\} \subseteq [\omega]^\omega$ and let $a : [\omega_1]^2 \to \omega$ be the corresponding transitive map

$$a(\alpha, \beta) = \min\{n : A_\alpha \setminus n \subseteq A_\beta\}.$$

Let $\rho_a : [\omega_1]^2 \to \omega$ be the corresponding ρ-function and for $\alpha < \omega_1$, let

$$D_\alpha = A_{\alpha+1} \setminus A_\alpha.$$

Lemma 77.

$$(D_\alpha \setminus \rho_a(\alpha,\gamma)) \cap (D_\beta \setminus \rho_a(\beta,\gamma)) = \emptyset$$

for all $0 < \alpha < \beta < \gamma$ with $\alpha, \beta \in \Delta$. $\square$

Definition 78. Define ~~partial~~ $m : [\omega_1]^2 \to \omega$ by

$$m(\alpha,\beta) = \min(D_\alpha \setminus \rho_a(\alpha,\beta)).$$

Lemma 77*. $m(\alpha,\gamma) \neq m(\beta,\gamma)$ for $\alpha \neq \beta$ in Δ and $\gamma > \alpha, \beta$. $\square$

Lemma 79. m is coherent on Δ, i.e.,

$$\{\xi < \min\{\alpha,\beta\} : m(\xi,\alpha) \neq m(\xi,\beta)\} \cap \Delta$$

is finite for all $\alpha, \beta < \omega_1$. $\square$.

Definition 80. For $\beta < \omega_1$, set
$$B_\beta = \{ m(\alpha,\beta) : \alpha \in \Lambda \cap \beta \}.$$

Lemma 81. $B_\beta =^* B_\gamma \cap A_\alpha$ for $\beta < \gamma$. $\square$

Lemma 82. $m(\alpha,\beta) = \max(B_\beta \cap D_\alpha)$ for $\alpha \in \Lambda \cap \beta$. $\square$

Lemma 83. There is <u>no</u> $B \subseteq \omega$ such that $B \cap A_\alpha =^* B_\beta$ for all β. $\square$

Theorem 84. For every strictly $\subseteq^*$-increasing chain $\{A_\alpha : \alpha < \omega_1\} \subseteq [\omega]^\omega$ there is a sequence $\{B_\alpha : \alpha < \omega_1\} \subseteq [\omega]^\omega$ such that:

(a) $B_\alpha =^* B_\beta \cap A_\alpha$ for $\alpha < \beta$

(b) There is no B such that $B_\alpha =^* B \cap A_\alpha$ for all α. $\square$

Problem 85. Investigate the structure of ω_1-gaps as in Theorem 84. What are the ~~cammonical~~ canonical objects?

§12. Some high-dimensional theory

Fix a positive integer n. Then for every $0 < m \leq n$, we can find a C-sequence

$$(C^m_\alpha : \alpha < \omega_m)$$

such that

(1) $C_{\alpha+1} = \{\alpha\}$

(2) C_α is a club in α

(3) $\operatorname{otp} C_\alpha = \operatorname{cf}(\alpha)$ for α limit

(4) $C_\alpha \setminus \lim(C_\alpha) \subseteq \{\xi+1 : \xi < \omega_m\}$.

and consider the corresponding walks and their characteristics.

For example, we can consider

$$\rho^{(m)} : [\omega_m]^2 \longrightarrow \omega_{m-1}$$

defined by

$$\rho^{(m)}(\alpha,\beta) = \sup\{\mathrm{otp}(C_\beta \cap \alpha),\ \rho^{(m)}(\alpha, \min(C_\beta \setminus \alpha)),\ \rho^{(m)}(\xi, \alpha) : \xi \in C_\beta \cap \alpha\}$$

where $\rho^{(m)}(\alpha,\alpha) = 0$ is the boundary condition. We can also consider the injective version

$$\bar\rho^{(m)} : [\omega_m]^2 \longrightarrow \omega_{m-1}$$

and prove the following:

Lemma 86. For all $\alpha < \beta < \gamma < \omega_m$,

(a) $\bar\rho^{(m)}(\alpha,\gamma) \leq \max\{\bar\rho^{(m)}(\alpha,\beta), \bar\rho^{(m)}(\beta,\gamma)\}$,

(b) $\bar\rho^{(m)}(\alpha,\beta) \leq \max\{\bar\rho^{(m)}(\alpha,\gamma), \bar\rho^{(m)}(\beta,\gamma)\}$,

(c) $\bar\rho^{(m)}(\alpha,\gamma) \neq \bar\rho^{(m)}(\beta,\gamma)$;

(d) $\bar\rho^{(m)}(\alpha,\beta) \neq \bar\rho^{(m)}(\beta,\gamma)$.

Definition 87. For each $0 \le i \le n$ define recursively

$$f_i^{(n)} : [\omega_n]^{i+1} \to \omega_{n-i}$$

as follows:

(1) $f_0^{(n)} = Id_{\omega_n}$

(2) for $0 < i \le n$ and $\alpha_0 < \dots < \alpha_i$, set

$$f_i^{(n)}(\alpha_0, \alpha_1, \dots, \alpha_i) = \bar{\varrho}^{(n-i+1)}\left(f_{i-1}^{(n)}(\alpha_0, \dots, \alpha_{i-1}),\ f_{i-1}^{(n)}(\alpha_1, \dots, \alpha_i)\right).$$

Finally, let $f_n = f_n^{(n)} : [\omega_n]^{n+1} \to \omega$.

We study some properties of these higher-dimensional ϱ-functions.

Lemma 88. Suppose that

$$\alpha, \bar{\alpha} < \alpha_0 < \dots < \alpha_{i-1}$$

are such that

(a) $$f_j^{(n)}(\alpha, \alpha_0, \dots, \alpha_{j-1}), f_j^{(n)}(\bar\alpha, \alpha_0, \dots, \alpha_j) <$$
$$< f_j^{(n)}(\alpha_0, \dots, \alpha_j)$$

for all $j < i$, and

(b) $$f_i^{(n)}(\alpha, \alpha_0, \dots, \alpha_{i-1}) = f_i^{(n)}(\bar\alpha, \alpha_0, \dots, \alpha_{i-1}).$$

Then $\alpha = \bar\alpha$.

<u>Proof</u>: Induction on i. The case $i=0$ is trivial so we assume $i>0$. Then

$$\bar\rho^{(n-i+1)}(f_{i-1}^{(n)}(\alpha, \alpha_0, \dots, \alpha_{i-2}), f_{i-1}^{(n)}(\alpha_0, \dots, \alpha_i)) =$$
$$f_i^{(n)}(\alpha, \alpha_0, \dots, \alpha_{i-1}) = f_i^{(n)}(\bar\alpha, \alpha_0, \dots, \alpha_{i-1}) =$$
$$\bar\rho^{(n-i+1)}(f_{i-1}^{(n)}(\bar\alpha, \alpha_0, \dots, \alpha_{i-2}), f_{i-1}^{(n)}(\alpha_0, \dots, \alpha_{i-1}))$$

From this and the injectivity of $\bar\rho^{(n-i+1)}$

(Lemma 86(c)) we conclude that

$$f^{(n)}_{i-1}(\alpha, \alpha_0, \ldots, \alpha_{i-2}) = f^{(n)}_{i-1}(\bar{\alpha}, \alpha_0, \ldots, \alpha_{i-2}),$$

~~so we are done by the ind. hyp~~ (i.e., we get the desired conclusion $\alpha = \bar{\alpha}$). □

<u>Definition 89.</u> (a) For $s, t \in [\omega_n]^{<\omega}$ we let

$s \vartriangleleft t$ iff $s \setminus \{\min(s)\} \sqsubseteq t$.

(b) We call a function*) $g : [\omega_n]^{<\omega} \to \mathrm{Ord}$

<u>shift-increasing</u> whenever

$s \vartriangleleft t$ implies $g(s) < g(t)$.

(c) We call a (possibly partial) function

$g : [\omega_n]^{<\omega} \to X$ <u>min-dependent</u>

whenever

$g(s) = g(t)$ implies $\min(s) = \min(t)$.

---

* possibly partial

Lemma 90.

$\forall A \in [\omega_n]^\omega \; \exists B \in [A]^\omega \; \forall i \leq n$

$$f_i^{(n)} \restriction [B]^{i+1} \text{ is shift-increasing.}$$

Proof: Define $\chi : [A]^{n+2} \to 3^{n+1}$ by

$$\chi(\alpha_0, \dots, \alpha_{n+1})(i) = \begin{cases} 0 & \text{if } f_i^{(n)}(\alpha_0, \dots, \alpha_i) < f_i^{(n)}(\alpha_1, \dots, \alpha_{i+1}) \\ 1 & \text{if } f_i^{(n)}(\alpha_0, \dots, \alpha_i) = f_i^{(n)}(\alpha_1, \dots, \alpha_{i+1}) \\ 2 & \text{if } f_i^{(n)}(\alpha_0, \dots, \alpha_i) > f_i^{(n)}(\alpha_1, \dots, \alpha_{i+1}). \end{cases}$$

By Ramsey's theorem there is $B \in [A]^\omega$ such that $\chi \restriction [B]^{n+1}$ is constant. Let $(\varepsilon_i : i \leq n) \in 3^{n+1}$ be the constant value.

Claim. $\forall i \leq n \quad \varepsilon_i = 0$

Proof: First of all note that $\varepsilon_i \neq 2$ for all $i \leq n$. Otherwise, fixing such $i \leq n$, choose $\lhd$-increasing sequence $\{s_k : k < \omega\} \subseteq [B]^{n+2}$

and conclude that

$$f_i^{(n)}(s_k \restriction i+1) > f_i^{(n)}(s_{k+1} \restriction i+1),$$

a contradiction.

Now we prove that $\varepsilon_i \neq 2$ for all $i \leq n$. We do this by induction on i. If $i = 0$ we know that $f_0^{(n)}(\alpha) = \alpha$, and this is clearly $\triangleleft$-increasing.

Suppose $i > 0$. Let $\alpha_0 < \dots < \alpha_{i+1}$ are in B. By the ind. hyp.,

$$f_{i-1}^{(n)}(\alpha_0, \dots \alpha_{i-1}) < f_{i-1}^{(n)}(\alpha_1, \dots, \alpha_i) < f_{i-1}^{(n)}(\alpha_2, \dots, \alpha_{i+1})$$

Applying the property (d) of Lemma 86 of $\bar{g}^{(n-i+1)}$ we get that

$$f_i^{(n)}(\alpha_0,\dots,\alpha_i) = \bar{\rho}^{(n-i+1)}\left(f_{i-1}^{(n)}(\alpha_0,\dots,\alpha_{i-1}), f_{i-1}^{(n)}(\alpha_1,\dots,\alpha_i)\right) \neq$$

$$\neq \bar{\rho}^{(n-i+1)}\left(f_{i-1}^{(n)}(\alpha_1,\dots,\alpha_i), f_{i-1}^{(n)}(\alpha_2,\dots,\alpha_{i+1})\right) =$$

$$= f_i^{(n)}(\alpha_1,\dots,\alpha_{i+1}).$$

So $\varepsilon_i \neq 1$.

<u>Lemma 91.</u>

$\forall A \in [\omega_n]^\omega\ \exists B \in [A]^\omega\ \forall i \leq n$

$f_i^{(n)} \restriction [B]^{i+1}$ is min-dependent.

<u>Proof</u>: By Lemma 90 we find $C \in [A]^\omega$ such that $f_i^{(n)} \restriction [C]^{i+1}$ is shift-increasing for all $i \leq n$.

Using the Erdös-Rado canonical Ramsey theorem, we find $B \in [C]^\omega$ and for each $i \leq n$ a set $J_i \subseteq i+1$ such that for all $s, t \in [B]^{i+1}$

$$f_i^{(n)}(s) = f_i^{(n)}(t) \text{ iff } \{s(l) : l \in J_i\} = \{t(l) : l \in J_i\}$$

The following Claim finishes the proof.

Claim. $0 \in J_i$ for all $i \leq n$.

Proof: Fix $i \leq n$. It suffices to prove that

$$f_i^{(n)}(\alpha_0, \alpha_1, \dots, \alpha_i) = f_i^{(n)}(\bar{\alpha}_0, \alpha_1, \dots, \alpha_i)$$

implies $\alpha_0 = \bar{\alpha}_0$. Since $f_j^{(n)} \upharpoonright [B]^{j+1}$ are shift-increasing for all $j < i$, this conclusion follows from Lemma 88.

□

§13. Positional graphs again

Recall the definitions from §4 above, two finite sets F and G of ordinals are in the $\underline{\Delta_k\text{-position}}$ for some integer k whenever there is a partition

$$F \cap G = I \cup J$$

such that $I \sqsubseteq F$, $I \sqsubseteq G$ and $|J| \leq k$.

For a family $\mathcal{V}$ of finite sets and an integer k, the *positional graph* spaned by $\mathcal{V}$ is the graph

$$\mathcal{G}_k(\mathcal{V}) = (\mathcal{V}, \neg\Delta_k).$$

Fix a positive integer n and recall the higher-dimensional ρ-functions $f_i^{(n)}$ from the previous section.

<u>Definition 92.</u> Let $\mathcal{V}_n$ be the set of all finite subsets F of ω_n such that

(a) $f_n = f_n^{(n)}$ is min-dependent on $[F]^{n+1}$,

(2) $f_i^{(n)} \restriction [F]^{i+1}$ is shift-increasing for all $i < n$.

<u>Theorem 93.</u>

(a) $\forall A \in [\omega_n]^{\omega}\ \exists B \in [A]^{\omega}\ [B]^{<\omega} \subseteq \mathcal{V}_n$

(b) The positional graph

$$\mathcal{G}_{2n-1}(\mathcal{V}_n) = (\mathcal{V}_n, \neg \Delta_{2n-1})$$

is countably chromatic.

<u>Proof:</u> (a) follows from lemmas 90 and 91.

To prove (b) define

$$\chi : \mathcal{V}_n \to HF$$

by letting

$$\chi(F) = \text{the isomorphism type of the structure } (F, <, f_n \restriction [F]^{n+1}).$$

The proof is finished once we establish the following.

<u>Claim</u>. $\chi(s) = \chi(t)$ implies that s and t are in the Δ_{2n-1}-position.

<u>Proof</u>: If $|s \cap t| \leq 2n-1$ the conclusion is trivial, so let us consider the case $|s \cap t| \geq 2n$. Let

$$\{\gamma_0, \gamma_1, \dots, \gamma_{n-1}, \gamma_n, \dots \gamma_{2n-1}\}_<$$

be the last $2n$-elements of $s \cap t$. Let

$$\Phi : s \to t$$

be the increasing bijection.

<u>Subclaim 1.</u> $\forall i<n \quad \Phi(\gamma_i)=\gamma_i$.

<u>Proof</u>: Fix $i<n$. Then

$$f_n(\gamma_i, \gamma_n, \dots, \gamma_{2n-1}) = f_n(\Phi(\gamma_i), \Phi(\gamma_n), \dots, \Phi(\gamma_{2n-1}))$$

Since $\Phi'' s\cap t \subseteq t$ and since $f_n \restriction [t]^{n+1}$ is min-dependent we conclude that $\Phi(\gamma_i)=\gamma_i$. $\square$

<u>Subclaim 2.</u> $s\cap\gamma_0 = t\cap\gamma_0 = (s\cap t)\cap\gamma_0$.

<u>Proof</u>: Consider $\gamma\in s\cap\gamma_0$. Then

$$f_n(\gamma, \gamma_0, \dots, \gamma_{n-1}) = f_n(\Phi(\gamma), \Phi(\gamma_0), \dots, \phi(\gamma_{n-1})) =$$
$$= f_n(\Phi(\gamma), \gamma_0, \dots, \gamma_{n-1})$$

Since $f_i^{(n)} \restriction [s]^{i+1}$ and $f_i^{(n)} \restriction [t]^{i+1}$ are shif-increasing we conclude that

$$f_i^{(n)}(\gamma, \gamma_0, \dots, \gamma_i),\ f_i^{(n)}(\phi(\gamma), \gamma_0 \dots \gamma_i) <$$
$$< f_i^{(n)}(\gamma_0, \dots, \gamma_{i+1})$$

holds for all $i \leq n$. This means that the hypotheses of Lemma 88 are satisfied. It follows that $\Phi(\gamma) = \gamma$, as required.

Let

$$I = s \cap \gamma_0 + 1 = t \cap \gamma_0 + 1 = (s \cap t) \cap \gamma_0 + 1$$

and let

$$J = \{\gamma_1, \dots \gamma_n, \gamma_{n+1}, \dots \gamma_{2n-1}\}.$$

Then $s \cap t = I \cup J$ is the decomposition witnessing that s and t are in the Δ_{2n-1}-position. $\square$

§14. Conditional weakly null sequences

Recall that a (semi) normalized sequence $(x_\alpha)_{\alpha < \mu}$ in some normed space $(X, \|\cdot\|)$, indexed by some <u>ordinal</u> Γ, is called a (Schauder) <u>basic sequence</u> if there is a constant $K \geq 1$ such that

$$\| \textstyle\sum_{\alpha<\beta} a_\alpha x_\alpha \| \leq K \cdot \| \textstyle\sum_{\alpha<\Gamma} a_\alpha x_\alpha \|$$

for all $\beta < \Gamma$ and all sequences $(a_\alpha)_{\alpha<\Gamma}$ of scalars. Such a basic sequence is called <u>unconditional</u> whenever there is a constant $K \geq 1$ such that

$$\| \textstyle\sum_{\alpha \in A} a_\alpha x_\alpha \| \leq K \| \textstyle\sum_{\alpha<\Gamma} a_\alpha x_\alpha \|$$

for all $A \subseteq \Gamma$ and all sequences $(a_\alpha)_{\alpha<\Gamma}$ of scalars. The <u>unconditional basic sequence problem</u> asking for conditions that guarantee the existence of <u>infinite</u> unconditional basic sequences has been one of the most fruitful problems in this area of mathematics.

Lemma 94. Suppose that Γ is an infinite ordinal for which we can find an integer $n \neq 0$ and a family $\mathcal{V} \subseteq [\Gamma]^{<\omega}$ such that

(a) $\forall A \in [\Gamma]^{\omega} \ \exists B \in [A]^{\omega} \ [B]^{<\omega} \subseteq \mathcal{V}$

(b) the postulated graph $\mathcal{G}_n(\mathcal{V}) = (\mathcal{V}, \neg\Delta_n)$ is countably chromatic

Then there is a norm $\|\cdot\|$ on $c_{00}(\Gamma)$ such that $(e_\gamma)_{\gamma<\Gamma}$ is weakly null normalized sequence in $(c_{00}(\Gamma), \|\cdot\|)$ but it contains no infinite unconditional subsequence. $\square$

Notation: $c_{00}(\Gamma)$ the family of all finitely supported mappings from Γ to $\mathbb{R}$. $e_\gamma : \Gamma \longrightarrow \mathbb{R}$ is the mapping with support $\{\gamma\}$ such that $e_\gamma(\gamma) = 1$.

Theorem 95 (Lopez Abad – T., 2011)
For every $n < \omega$ there is a weakly-null sequence $(x_\gamma)_{\gamma < \omega_n}$ of length ω_n with no infinite unconditional basic subsequence. $\square$

Theorem 96 (Dodos - Lopez Abad - T., 2009)
It is consistent that every weakly null sequence of length ω_ω contains an infinite unconditional basic subsequence. $\square$