§ 15. The oscillation mapping and walks on ordinals

Fix an ordinal θ (typically regular and uncountable). Define

$$\mathrm{osc} : \mathcal{P}(\theta) \to \omega$$

by

$$\mathrm{osc}(x,y) = | \, x \setminus \max(\bar{x} \cap \bar{y}) + 1 \, / \sim \, | ,^{*)}$$

where $\sim$ is the equivalence relation on the set

$$x \setminus \{\xi : \xi \leq \max(\bar{x} \cap \bar{y})\}$$

defined by

$\alpha \sim \beta$ iff $[\min\{\alpha,\beta\}, \max\{\alpha,\beta\}] \cap y = \emptyset$.

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* $\bar{x}$ and $\bar{y}$ are the closures of x and y in θ

Remark 96. (a) If $\theta = \omega$ one has that $\bar{x} = x$ and $\bar{y} = y$ and one usually defines $\mathrm{osc}(x, y)$ by letting it equal to the cardinality of the quotient $x \Delta y / \sim$ with $\sim$ defined by

$$m \sim n \text{ iff } [\min\{m,n\}, \max\{m,n\}] \cap x = \emptyset \text{ or}$$
$$[\min\{m,n\}, \max\{m,n\}] \cap y = \emptyset.$$

Thus, when $\theta > \omega$ we remove more from $x \cup y$ (as we are interested only in finite oscillations) and in fact we count only classes in $x \setminus y$.

(b) Recall the oscillation mapping

$$\mathrm{osc} : (\omega^{\omega})^2 \to \mathrm{Card}$$

defined by

$$\mathrm{osc}(x, y) = |\{n < \omega : x(n) \leq y(n) \ \&\ x(n+1) > y(n+1)\}|.$$

The oscillation theory in essentially all known contexts rely on the fact that 'unbounded' families realize all possible oscillations. This is the case in this context as well.

<u>Definition 97</u>. A family $\mathcal{X} \subseteq \mathcal{P}(\Theta)$ is <u>unbounded</u> if for every closed and unbounded set $C \subseteq \Theta$ there exist an arbitrarily long finite increasing sequence $\{\delta_k : k \leq l\} \subseteq C$ such that for all $k < l$

$$\sup(x \cap \delta_{k+1}) < \delta_{k+1} \text{ and } [\delta_k, \delta_{k+1}) \cap x \neq \emptyset.$$

Here is a typical result of the oscillation theory in this context.

Lemma 98. If $\mathcal{X}$ is an unbounded family of bounded subsets of θ then for every positive integer n there exist x and y in $\mathcal{X}$ such that $\operatorname{osc}(x, y) = n$. $\square$

Definition 99. A C-sequence $(C_\alpha : \alpha < \theta)$ on θ is called nontrivial if there is no club $C \subseteq \theta$ such that for all $\alpha \in \lim(C)$ there is $\beta \geq \alpha$ such that $C \cap \alpha \subseteq C_\beta$.

Exercises (a) Show that every successor cardinal θ supports a nontrivial C-sequence.

(b) Show that the first inaccessible cardinal θ supports a nontrivial C-sequence.

(c) Show that first Mahlo cardinal θ has a nontrivial C-sequence.

Definition 100. Fix a C-sequence $(C_\alpha : \alpha < \theta)$ on θ. A partial action of $\theta^{<\omega}$ on θ $(\alpha, t) \mapsto \alpha_t$ is defined recursively as follows:

$$\alpha_{\langle\rangle} = \alpha$$

$\alpha_{\langle\xi\rangle}$ = the ξth element of C_α if $\xi < tp(C_\alpha)$

$\alpha\langle\xi\rangle$ is undefined if $\xi \geq tp(C_\alpha)$

$$\alpha_{t^\frown\langle\xi\rangle} = (\alpha_t)_{\langle\xi\rangle}.$$

Exercise. Let $\beta = \beta_0 > \cdots > \beta_n = \alpha$ be the walk from β to α along the C-sequence. Let $t = \rho_0(\alpha, \beta)$. Then $\beta_t = \alpha$ and $\beta_{t \restriction i} = \beta_i$ for $i \leq n$. Show, however, that in general $\beta_t = \alpha$ does not imply $\rho_0(\alpha, \beta) = t$.

Definition 101. To every C-sequence $(C_\alpha : \alpha < \theta)$ on θ we attach the corresponding oscillation mapping

$$o = o(C_\alpha : \alpha < \theta) : [\theta]^2 \to \omega$$

as follows. Given $\alpha < \beta < \theta$, if there is $t \sqsubseteq \varrho_0(\alpha, \beta)$ such that

(i) $osc(\alpha_t, \beta_t) > 1$, but

(ii) $osc(\alpha_s, \beta_s) = 1$ for all $s \sqsubset t$,

let $o(\alpha, \beta) = osc(C_{\alpha_t}, C_{\beta_t})$; otherwise let $o(\alpha, \beta) = 0$.

Theorem 102. If $(C_\alpha : \alpha < \theta)$ is a nontrivial C-sequence on θ and if $o : [\theta]^2 \to \omega$ is the corresponding oscillation mapping then for every unbounded $\Gamma \subseteq \theta$ and integer $n \geq 2$ there exist $\alpha < \beta$ in Γ such that $o(\alpha, \beta) = n$.

<u>Proof:</u> Fix $n \geq 2$ and unbounded $\Gamma \subseteq \theta$. Choose a continuous $\in$-chain $\mathcal{M}$ of elementary submodels M of H_{θ^+} such that $\delta_M = M \cap \theta \in \theta$ and such that $\langle C_\alpha : \alpha < \theta \rangle, \Gamma \in M$. Let

$$C = \{ \delta_M : M \in \mathcal{M} \}$$

Then C is a club in θ.

Choose now $N \prec H_{(2^\theta)^+}$ containing all these objects such that $\delta = N \cap \theta \in \theta$. Pick $\beta \in \Gamma$ such that $\beta \geq \delta$. Let

$$\beta = \beta_0 > \beta_1 > \dots > \beta_k \geq \delta$$

be the part of the walk from β to δ such that

$$\sup (C_{\beta_k} \cap \delta) = \delta .$$

Thus either $\beta_k = \delta$ or $\beta_{k+1} = \delta$ is the last step of the walk from β to δ. Note that this in particular means that for all $i < k$

$$\xi_i = \max(C_{\beta_i} \cap \delta) < \delta .$$

Let $t = \rho_0(\beta_k, \beta)$. Then $\beta_i = \beta_{t+i}$ for $i \leq k$.

Since $(C_\alpha : \alpha < \theta)$ is nontrivial, applying this to the club $C = \{\delta_M : M \in \mathcal{M}\}$ and using the elementarity of N, we conclude that

$$\sup((C \cap \delta) \setminus C_{\beta_k}) = \delta .$$

So, we can pick an $\in$-chain M_i $(i \leq n)$ from $\mathcal{M} \cap N$ such that:

(1) $\delta_i = \delta_{M_i} \notin C_{\beta_k}$ for all $i \leq n$

(2) $[\delta_i, \delta_{i+1}) \cap C_{\beta K} \neq \emptyset$ for all $i < n$

(3) $\delta_0 > \xi_i$ for all $i < K$.

Let $J_0 = [0, \max(C_{\beta K} \cap \delta_1)]$,

$J_i = [\delta_i, \max(C_{\beta K} \cap \delta_{i+1}]$ $(0 < i < n)$

$J_n = [\delta_n, \beta]$.

Then $J_0 < J_1 < \dots < J_n$ is a block sequence of closed intervals which covers $C_{\beta K}$ with the property that for every $0 < j \leq n$,

(*) $(J_i : i < j) \in M_j$ and $J_j \geq M_j \cap \theta = \delta_j$.

Let $\mathcal{F}$ be the family of all block-sequences $(I_i : i \leq n)$ of closed intervals of θ for which we can find $\alpha \in \Gamma$ such that:

(4) $C_{\alpha_t} \subseteq \bigcup_{i \leq n} I_i$,

(5) $\xi_i = \max C_{\alpha_{t+r_i}} \cap \alpha_t$ for $i<k$

(6) $\max I_i \in C_{\alpha_t}$ for all $i<k$

(6) $\max I_n = \alpha$ and $C_{\alpha_t} \cap I_n \neq \emptyset$

(i.e., α_t is a limit ordinal from the interior of I_n)

(7) $I_0 = J_0$

Clearly $(J_i : i \leq n) \in \mathcal{F}$ and $\mathcal{F} \in M_i$ for all $0 < i \leq n$.

Let $\partial\mathcal{F}$ be the collection of all sequences $(I_i : i < n)$ of intervals such that

$$\forall \gamma < \theta \ \exists \text{ interval } I \geq \gamma \ \ (I_i : i<n)^\frown I \in \mathcal{F}$$

Then using $(*)$ for $j=n$ and the elementarily of M_n we conclude that

$$(J_i : i<n) \in \partial\mathcal{F}.$$

Let $\partial^2 \mathcal{F}$ be the collection of all block sequences $(I_i : i<n-1)$ of closed interval ~~for which~~ such that

$$\forall \mu < \omega \ \exists \text{ interval } I \geq \mu \quad (I_i : i<n-1)^\frown I \in \partial \mathcal{F}.$$

Then using $(*)$ for $j = n-1$ and the elementarily of M_{n-1} we conclude that $(J_i : i<n-1) \in \partial^2 \mathcal{F}$. Proceeding this way we arrive at the conclusion that

$$(J_0) \in \partial^n \mathcal{F}$$

Using the definition of $\partial^n \mathcal{F}$ and the elementarily of M_1 we can find $I_1 \in M_1$ such that $I_1 > J_0$ and $(J_0, I_1) \in \partial^{n-1} \mathcal{F}$. Now, using this and the elementarily of M_2

we can find interval $I_2 \in M_2$ such that $I_2 > J_1$ and $(J_0, I_1, I_2) \in \partial^{n-2} \mathcal{F}$, and so on. Proceeding this way we arrive at

$$(J_0 I_1, \dots I_n) \in M_n \cap \mathcal{F}$$

such that $I_{i+1} > J_i$ for all $i < n$. Pick an $\alpha \in \Gamma$ witnessing the fact that $(J_0, I_1, \dots, I_n)$ belongs to $\mathcal{F}$ i.e., satisfying the conditions (4) — (7). It follows that the intersections

$$C_{\alpha t} \cap I_i \qquad (0 < i \leq n)$$

are the convex pieces the set

$$C_{\alpha t} \setminus \max(C_{\alpha t} \cap C_{\beta t}) + 1$$

is split by the set $C_{\beta t}$ i.e., $\operatorname{osc}(C_{\alpha t}, C_{\beta t}) = n$. On the other hand $\operatorname{osc}(C_{\alpha t + i}, C_{\beta t + i}) = 1$ for all $i < k$. It follows that $o(\alpha, \beta) = n$, as required. $\square$

<u>Definition 103.</u> For $\alpha < \beta < \theta$ let

$$[\alpha\beta] = \beta_t$$

where $t \sqsubseteq \varrho_0(\alpha, \beta)$ is minimal for which α_t is defined and if $\xi = tp(C_{\beta_t} \cap \alpha)$, then ξth element of $C_{\alpha_t} \neq \xi^{th}$ element of C_{β_t}.

<u>Definition 104.</u> A C-sequence $(C_\alpha : \alpha < \theta)$ on θ avoids a subset S of θ whenever $C_\alpha \cap S = \emptyset$ for all limit ordinals $\alpha < \theta$.

<u>Theorem 105.</u> Suppose $(C_\alpha : \alpha < \theta)$ is a C-sequence on θ which avoids a set $S \subseteq \theta$. Then for every unbounded $\Gamma \subseteq \theta$,

$$S \setminus \{[\alpha, \beta] : \alpha, \beta \in \Gamma, \alpha < \beta\}$$

is not stationary in θ. $\square$

§16. Initial Motivations

Definition 106. For a cardinal (structure) θ and positive integers r, k, t let

$$\theta \to (\theta)^r_{k,t}$$

denote the statement

$$\forall f : [\theta]^r \to \{0, 1, .., k-1\} \ \exists \Gamma \in [\theta]^\theta \ |f[\Gamma]^r| \leq t.$$

The minimal such t that works for all k (if it exists) is called (big) Ramsey degree of r, $t_r = t_r(\theta)$.

Let

$$\theta \to [\theta]^r_k \text{ iff } \theta \to (\theta)^r_{k,k-1}$$

Examples 107

(1) (Erdös-Hajnal-Rado, 1965)

$$2^\omega \not\to [2^\omega]^r_{2^{r-1}} \text{ but } \forall \ell \; 2^\omega \to (2^\omega)^r_{\ell, 2^{r-1}}$$

(2) (D. Devlin, 1979)

$$\mathbb{Q} \not\to [\mathbb{Q}]^r_{\tan^{2r-1}(0)} \text{ but } \forall \ell \; \mathbb{Q} \to (\mathbb{Q})^r_{\ell, \tan^{(2r-1)}(0)}$$

Proposition 108. Suppose

$$\theta \not\to [\theta]^r_t \text{ but } \forall \ell \; \theta \to (\theta)^r_{\ell, t}$$

Let E_t be the equivalence relation on $[\theta]^r$ witnessing $\theta \not\to [\theta]^r_t$. Then for every other equivalence relation E on $[\theta]^r$ ~~there is $\Gamma \in [\theta]^\theta$~~ such that $[\theta]^r/E$ is finite there is $\Gamma \in [\theta]^\theta$ such that $E \restriction [\Gamma]^r$ is coarser than $E_t \restriction [\Gamma]^r$.

<u>Example 109</u> (Galvin-Shelah 1973)

Fix orderings $<_S$ and $<_A$ on ω_1 such that $(\omega_1, <_S)$ is separable while $(\omega_1, <_A)$ contains no ω_1, ω_1^* nor an uncountable separable ordering.

Define E_{GS} on $[\omega_1]^2$ by letting

$$\{\alpha,\beta\}_{<} E \{\gamma,\delta\}_{<} \text{ iff } \forall R \in \{<, <_S, <_A\} \; [\alpha R \beta \leftrightarrow \gamma R \delta].$$

<u>Question 110</u> (Galvin-Shelah 1973)

Does $\omega_1 \to (\omega_1)^2_{4,\ell}$ for all $\ell < \omega$?

In other words, does for every equivalence relation E on $[\omega_1]^2$ we can find $\Gamma \in [\omega_1]^{\omega_1}$ such that

$$E_{GS} \restriction [\Gamma]^2 \leq E \restriction [\Gamma].$$