A DYNAMICAL PROOF OF MATUI'S ABSORPTION THEOREM

JULIEN MELLERAY

ABSTRACT. We give a dynamical, relatively elementary proof of an absorption theorem due to Matui, in the spirit of arguments of an earlier joint work of the author and S. Robert. In an appendix we explain how to use this result to correct the dynamical proof given by Melleray–Robert of a classification theorem for orbit equivalence of minimal ample groups due to Giordano, Putnam and Skau (the original argument had a gap).

1. INTRODUCTION

This article is a continuation of [MR], and the motivation for this work is an error in one of the main arguments in that paper. We are concerned with actions of countable groups by homeomorphisms on the Cantor space X which are *minimal*, i.e. such that all orbits are dense. Given an action $\Gamma \curvearrowright X$, we denote by R_Γ the associated equivalence relation and say that two equivalence relations R, S on X are *orbit equivalent* if there exists a homeomorphism $h \colon X \to X$ such that $(h \times h)R = S$.

Denote by $M(\Gamma)$ the set of Γ-invariant Borel probability measures on X. It is not hard to see that if h realizes an orbit equivalence between the relations induced by actions of two countable groups Γ, Λ on X then one must have $h_*M(\Gamma) = M(\Lambda)$. For minimal actions of $\mathbf{Z}$ there is a converse to that statement, proved by Giordano, Putnam and Skau [GPS1]: if two minimal actions of $\mathbf{Z}$ preserve the same Borel probability measures then they are orbit equivalent. This theorem is far from being obvious; for instance, preserving the same Borel probability measures certainly does not imply that both actions have the same orbits.

Since [GPS1], this classification theorem has been re-proved in several papers, including [P1] and [HKY]; all known proofs are fairly technical. Motivated in part by the perspective of extending this classification theorem for other group actions (so far it is known for every $\mathbf{Z}^d$, see [GMPS2]), the author and S. Robert claimed to give in [MR] an elementary, purely dynamical proof of the classification theorem. I discovered recently that this proof has a gap.

Loosely speaking, the proof strategy in [MR], similarly to what is done in [GPS2] or [P1], is to first prove that a "small extension" of an equivalence relation induced by a minimal $\mathbf{Z}$-action results in a relation which is orbit equivalent to the relation one started with; then to prove that given two equivalence relations R, S induced by minimal $\mathbf{Z}$-actions which preserve the same Borel probability measures, one can produce a third equivalence relation T which is a small extension of both R and S, thereby deriving that R and S are orbit equivalent.

A theorem stating that a "small extension" of a given equivalence relation R is orbit equivalent to R is called an *absorption theorem*. The first of those was given in [GPS2], then it was improved in [GMPS1] and the strongest such theorem, due to Matui, appeared in [M]. Matui's proof builds on the proofs of earlier absorption

theorems, making for a fairly involved argument where the dynamical aspects are complicated to understand.

In [MR], instead of working with minimal **Z**-actions, we followed ideas of Giordano, Putnam and Skau and worked instead with minimal actions of certain locally finite subgroups of Homeo(X), which were called *ample groups* by Krieger [K]. As pointed out by Putnam [P1], it seems more natural to first establish a classification theorem for minimal actions of ample groups and then derive the theorem for minimal **Z**-actions.

S. Robert and the author claimed in [MR] to provide a dynamical proof of an absorption theorem that, while weaker than Matui's, was sufficient to prove the classification theorem of Giordano, Putnam and Skau, via an argument that only involved cutting-and-stacking methods. There is, however, a gap in that argument; once that gap is identified, it becomes apparent that for that approach to work one needs a stronger absorption theorem, and that Matui's absorption theorem is adequate to the task. We give here an elementary proof of such a strong absorption theorem (equivalent to Matui's, though we do not prove that equivalence, since we want to avoid going into the theory of étale equivalence relations).

To study ample group actions up to orbit equivalence, it is common to employ Bratteli diagrams, which have been instrumental in the proofs of several deep results (see e.g. [GPS1], [GPS2], [GMPS1], [M]). While those diagrams are very natural for someone with a background in operator algebras (and are nicely connected with homological invariants) and their effectiveness to tackle the type of questions we are concerned with is well established, their use can lead to proofs where the dynamical aspects are hard to grasp. Here, as in [MR], we always work directly with clopen subsets of X via cutting-and-stacking methods. Still, it must be pointed out that many of the ideas and concepts that we use are closely related to those found in the works of Giordano, Matui, Putnam and Skau mentioned above.

Let us briefly discuss the organization of the paper. After reviewing some basic notions needed for our argument, we first develop some elementary theory of what we call malleable subsets. These are analogs of the étale extensions considered in [GPS2], [GMPS1] and [M] and provide the paradigm for the "small extensions" alluded to above. We then need to develop some machinery in order to prove the absorption theorem. To that end, we extend a theorem of Krieger [K]; a consequence of that work is a homogeneity result which is instrumental in our proof of the absorption theorem (see Lemma 5.3). Using this theorem of Krieger as a step towards the classification theorem is one of the key ideas of [MR] and our strategy here is similar. Then we prove our version of Matui's absorption theorem (Theorem 5.5), using a method which is related to what Putnam calls the "Hilbert–Bratteli hotel" in [P2]. Informally, to prove that the relation R_Γ induced by the action of a minimal ample group Γ is orbit equivalent to a small extension of itself, we begin by showing (see Lemma 5.4 and how it is used to prove Theorem 5.5) that R_Γ can be obtained from the relation R_Λ induced by another minimal ample group Λ by repeating countably many times the same small extension. Thus one more small extension should not (and, as it turns out, does not) change the orbit equivalence class of R_Γ. The absorption theorem may thus be thought of as an analogue of the classical ordinal equation $1 + \omega = \omega$ (hence Putnam's analogy with the Hilbert hotel).

At the end of the paper the reader will find an appendix, which serves as a corrigendum to [MR]. There I assume that the reader is familiar with the arguments and notations of [MR] and explain how use the improved absorption theorem so as to fix the proof of the classification theorem for orbit equivalence given in [MR].

Acknowledgements. I am very grateful to F. Le Maître for making many useful comments and suggestions after reading a first version of this article.

2. Background and vocabulary

We recall some notions and terminology. The interested reader can find a much more detailed exposition in [MR], with proofs for some statements we only mention here in passing.

Assume that X is a compact, metrizable, 0-dimensional space. Given a subgroup $\Gamma \leq \mathrm{Homeo}(X)$, we denote by $M(\Gamma)$ the set of all Borel probability measures on X which are Γ-invariant. The *full group* $F(\Gamma)$ generated by Γ is the set of all $g \in \mathrm{Homeo}(X)$ such that there exists a clopen partition $X = \bigsqcup_{i=1}^{n} U_i$ and $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that $g(x) = \gamma_i(x)$ for all $x \in U_i$. We say that Γ is a full group if $\Gamma = F(\Gamma)$. Note that we always have $M(\Gamma) = M(F(\Gamma))$.

For any countable subgroup $\Gamma \leq \mathrm{Homeo}(X)$, we denote by R_Γ the associated equivalence relation on X, and let $[R_\Gamma]$ denote the subgroup of all $g \in \mathrm{Homeo}(X)$ which map each Γ-orbit to itself. Then $[R_\Gamma]$ is a full group and $M([R_\Gamma]) = M(\Gamma)$.

An *ample group over X* is a countable, locally finite full group $\Gamma \leq \mathrm{Homeo}(X)$ with the property that for all $\gamma \in \Gamma$ the set $\{x \in X : \gamma(x) = x\}$ is clopen in X.

Given a subgroup $\Gamma \leq \mathrm{Homeo}(X)$ and a Boolean subalgebra $\mathcal{A}$ of $\mathrm{Clopen}(X)$, we say that $(\mathcal{A}, \Gamma)$ is a *unit system*[1] if:

- For every $A \in \mathcal{A}$ $\gamma(A) \in \mathcal{A}$, giving us an evaluation map $e_{\mathcal{A}} \colon \Gamma \to \mathrm{Aut}(\mathcal{A})$.
- The morphism $e_{\mathcal{A}}$ is injective (equivalently, the only element of Γ mapping every element of $\mathcal{A}$ to itself is the identity).
- For every $\gamma \in \Gamma$, $\{x \colon \gamma x = x\} \in \mathcal{A}$.
- For every $g \in \mathrm{Homeo}(X)$, if there exists a partition $X = \bigsqcup_{i=1}^{n} A_i$ with $A_i \in \mathcal{A}$ such that g coincides on each A_i with some $\gamma_i \in \Gamma$, then $g \in \Gamma$.

We sometimes denote by $\Gamma_{\mathcal{A}}$ the subgroup $e_{\mathcal{A}}(\Gamma)$ of $\mathrm{Aut}(\mathcal{A})$.

One says that the unit system $(\mathcal{A}, \Gamma)$ is *finite* if $\mathcal{A}$ is finite, in which case Γ is also finite. We say that a unit system $(\mathcal{B}, \Sigma)$ *refines* another unit system $(\mathcal{A}, \Gamma)$ if $\mathcal{B}$ contains $\mathcal{A}$ and Σ contains Γ.

Krieger [K, Lemma 2.1] proved that for any ample group there exists a sequence $(\mathcal{A}_n, \Gamma_n)_n$ of finite unit systems such that $(\mathcal{A}_{n+1}, \Gamma_{n+1})$ refines $(\mathcal{A}_n, \Gamma_n)$ for all n and

$$\bigcup_n \mathcal{A}_n = \mathrm{Clopen}(X) \quad ; \quad \bigcup_n \Gamma_n = \Gamma$$

We say that $(\mathcal{A}_n, \Gamma_n)$ is an *exhaustive sequence* for (X, Γ).

From now on (throughout the paper) the letter X stands for the Cantor space.

An action $\Gamma \curvearrowright X$ by homeomorphisms is *minimal* if all of its orbits are dense; we say that $\varphi \in \mathrm{Homeo}(X)$ is minimal if the **Z**-action $n \cdot x = \varphi^n(x)$ is minimal. We denote by $F(\varphi)$ the full group generated by $\{\varphi^n \colon n \in \mathbf{Z}\}$ (it is often denoted $[\![\varphi]\!]$

---

[1]There is (again) an imprecision in [MR]: the definition given there is different from this one for unit systems which are not finite. This is not an issue since those are not used anywhere in [MR].

in the literature). This is a countable group which acts minimally on X; for any $x_0 \in X$, the subgroup

$$\Gamma_{x_0}(\varphi) = \{\gamma \in F(\varphi) \colon \gamma(O^+(x_0)) = O^+(x_0)\}$$

is an ample group (here $O^+(x_0) = \{\varphi^n(x_0) \colon n \geq 0\}$). The orbits of $\Gamma_{x_0}(\varphi) \curvearrowright X$ are the same as the φ-orbits, except for the orbit $O(x_0)$ which splits into its positive part $O^+(x_0)$ and negative part $O^-(x_0)$. It is proved in [MR] that all minimal ample groups over X can be realized in this way, so minimal ample groups and topological full groups of minimal $\mathbf{Z}$-actions are closely related.

Any action $\Gamma \curvearrowright X$ also induces an equivalence relation $\sim_\Gamma$ on $\mathrm{Clopen}(X)$, where $A \sim_\Gamma B$ iff there exists $\gamma \in \Gamma$ such that $\gamma(A) = B$. If Γ is ample, then $\sim_\Gamma$ is *full* in the following sense: whenever $A, B \in \mathrm{Clopen}(X)$ are such that $A = \bigsqcup_{i=1}^n A_i$, $B = \bigsqcup_{i=1}^n B_i$ and $A_i \sim_\Gamma B_i$ for all i then $A \sim_\Gamma B$.

Definition 2.1. We say that two ample subgroups Γ, Λ *induce isomorphic relations on* $\mathrm{Clopen}(X)$ if there exists $h \in \mathrm{Homeo}(X)$ such that for any clopen A, B one has $(A \sim_\Gamma B) \Leftrightarrow (h(A) \sim_\Lambda h(B))$.

Krieger's theorem alluded to in the introduction (and which we strengthen in Section 4) implies that if two ample groups Γ, Λ induce isomorphic relations on $\mathrm{Clopen}(X)$ then Γ and Λ are conjugated in $\mathrm{Homeo}(X)$. In particular R_Γ and R_Λ are then orbit equivalent.

The following lemma, whose analogue for $\mathbf{Z}$-actions is due to Glasner and Weiss [GW], is crucial for our approach. We note that, while the proof for ample groups is essentially the same as the Glasner–Weiss proof for $\mathbf{Z}$, we falsely claim in [MR] that the argument works for topologically transitive actions (see the remark at the end of the Appendix).

Lemma 2.2 (Glasner–Weiss). *Assume that $\Gamma \leq \mathrm{Homeo}(X)$ is an ample group acting minimally. Fix $A, B \in \mathrm{Clopen}(X)$.*

(1) *If $\mu(A) < \mu(B)$ for all $\mu \in M(\Gamma)$ then there exists $\gamma \in \Gamma$ such that $\gamma(A) \subset B$.*
(2) *If $\mu'A) = \mu(B)$ for all $\mu \in M(\Gamma)$ then there exists $g \in F(R_\Gamma)$ such that $g(A) = B$.*

The first point above is true for any ample subgroup of $\mathrm{Homeo}(X)$, but the proof for the second property given in [MR] requires minimality to work.

We will use another property of $\sim_\Gamma$ (which is implicitly used in [MR].

Lemma 2.3. *Let Γ be an ample group over X, and A, B be clopen subsets of X.*

Assume that $A \sim_\Gamma B$ and that $A_1 \subseteq A$, $B_1 \subseteq B$ are clopen sets such that $A_1 \sim_\Gamma B_1$. Then $A \setminus A_1 \sim_\Gamma B \setminus B_1$.

Proof. Fix an exhaustive sequence $(\mathcal{A}_n, \Gamma_n)$ of finite unit systems for Γ. By assumption, there exists n and $\gamma, \delta \in \Gamma_n$ such that A, A_1, B, B_1 all belong to $\mathcal{A}_n$, $\gamma A = B$ and $\delta A_1 = B_1$.

Given an atom U of $\mathcal{A}_n$ and $C \in \mathcal{A}_n$ define

$$n_C(U) = |\{V \in \Gamma_n U \colon V \subseteq C\}|$$

Since $A, B \in \mathcal{A}_n$, the existence of $\gamma \in \Gamma_n$ such that $\gamma(A) = B$ amounts to saying that $n_A(U) = n_B(U)$ for every atom U of $\mathcal{A}_n$; similarly, we obtain $n_{A_1}(U) = n_{B_1}(U)$ for every U. Then for every U we have

$$n_{A \setminus A_1}(U) = n_A(U) - n_{A_1}(U) = n_B(U) - n_{B_1}(U) = n_{B \setminus B_1}(U)$$

and that proves the lemma.   □

3. Malleable subsets

Throughout this section we fix an ample Γ group over X which acts minimally. Below, we will use without mention the fact that for any closed subset K of X, any clopen subset of K is the intersection of K with a clopen subset of X.

Definition 3.1 (see [GPS2, Definition 4.11]). Let K be a closed subset of X. We say that K is Γ-*thin* if $\mu(K) = 0$ for every $\mu \in M(\Gamma)$.

The next lemma is standard.

Lemma 3.2. *Assume that K is a closed, Γ-thin subset of X. Then for every $\varepsilon > 0$ there exists a clopen subset U such that $K \subset U$ and $\mu(U) \leq \varepsilon$ for all $\mu \in M(\Gamma)$.*

Proof. Pick a decreasing sequence $(U_n)_{n \in \mathbf{N}}$ of clopen subsets such that $\bigcap_n U_n = K$. Given $n \in \mathbf{N}$, the function $\Phi_n \colon \mu \mapsto \mu(U_n)$ is continuous on $M(\Gamma)$ (endowed with its usual compact topology).

Furthermore, for every $\mu \in M(\Gamma)$ the sequence $(\Phi_n(\mu))_n$ decreases to $\mu(K) = 0$. Since $M(\Gamma)$ is compact we can apply Dini's theorem and conclude that there is n such that $\Phi_n(\mu) \leq \varepsilon$ for all $\mu \in M(\Gamma)$, equivalently $\mu(U_n) \leq \varepsilon$ for all $\mu \in M(\Gamma)$. □

Then the argument used to prove [MR, Lemma 3.13] gives the following result.

Lemma 3.3. *Assume that K is a closed Γ-thin subset. Let U be a nontrivial clopen subset of X, and A a clopen subset of K. Let $V \in \mathrm{Clopen}(X)$ be such that $A \subset V \cap K$ and $\mu(U) < \mu(V)$ for all $\mu \in M(\Gamma)$.*
Then there exists $U' \in \mathrm{Clopen}(K)$ such that $U' \cap K = A$, $U' \subset V$, and $U' \sim_\Gamma U$.

Proof. By compactness of $M(\Gamma)$ (and minimality of the action) there exists $\eta > 0$ such that both $\eta < \mu(U)$ and $\mu(U) + \eta < \mu(V)$ for all $\mu \in M(\Gamma)$.

By Lemma 3.2, there exists a clopen subset W containing K such that $\mu(W) < \eta$ for all $\mu \in M(\Gamma)$. Since A is clopen in K, we have $A = K \cap B$ with B clopen in X and contained in $W \cap V$. Since $\mu(B) < \eta < \mu(U)$ for all $\mu \in M(\Gamma)$, we may apply Lemma 2.2 and conclude that there exists a clopen $C \subseteq U$ such that $C \sim_\Gamma B$.

For all $\mu \in M(\Gamma)$ we have

$$\mu(U \setminus C) + \eta = \mu(U) - \mu(C) + \eta < \mu(V) - \mu(B) = \mu(V \setminus B)$$

Applying Lemma 3.2 again, we pick a clopen subset D of $V \setminus B$ containing $(K \cap V) \setminus A$ and such that $\mu(D) < \eta$ for all $\mu \in M(\Gamma)$. We then have $\mu(U \setminus C) < \mu(V \setminus (B \cup D))$ so by Lemma 2.2 there exists a clopen $E \subset V \setminus (B \cup D)$ and such that $E \sim_\Gamma U \setminus C$.

We conclude by setting $U' = B \cup E$. □

The way we used Lemma 2.2 above is typical of our arguments; from now on we will not mention this lemma explicitly but it is very often in the background.

Definition 3.4. Let K be a closed subset of X. We say that K is Γ-*étale* if for any $\gamma \in \Gamma$ and any clopen subset A of K the set $\gamma A \cap K$ is clopen in K.
We say that K is Γ-*malleable* if it is a closed, Γ-thin and Γ-étale subset of X.

The terminology "étale" comes from work of Giordano–Putnam–Skau which itself originates from operator algebra theory. Our definition amounts to saying that the restriction of R_Γ to K is étale for the topology induced by the topology of R_Γ, that is, K is R_Γ-étale; see [GPS2, Definition 2.1]. In [GPS2], one requires compatibility between K and the topology of the étale equivalence relation under consideration; here, analogously, we ask for compatibility with the acting ample

group. Our malleable sets correspond to the R-closed, R-étale and R-thin subsets considered in [GPS2] and later in [GMPS1] and [M].

Assume that K is Γ-étale. Given an involution $\gamma \in \Gamma$, define $\gamma_K \colon K \to K$ by setting $\gamma_K(x) = \gamma(x)$ if $\gamma(x) \in K$, $\gamma_K(x) = x$ otherwise. Denote $K_1 = \{x \in K \colon \gamma(x) \in K\}$. Then K_1 is clopen in K since K is Γ-étale, and $\gamma_K(K_1) = K_1$. From this we obtain that γ_K is a homeomorphic involution of K.

Definition 3.5. Let K be a Γ-étale subset of X, and $\gamma \in \Gamma$ an involution. We say that γ is K-compatible if $\gamma(K) = K$.

Lemma 3.6. *Let K be a Γ-étale subset of X, and $\gamma \in \Gamma$ be an involution. Then there exists a K-compatible involution $\delta \in \Gamma$ such that $\delta_K = \gamma_K$.*

Proof. The sets $\{x \in K \colon \gamma(x) \in K\}$ and $\{x \in K \colon \gamma(x) \neq x\}$ are both clopen in K; denote by L their intersection. Since γ is an involution we have $\gamma(L) = L$.

Choose a clopen subset U of X which contains L and is disjoint from $K \setminus L$, and set $V = U \cap \gamma(U)$. Then V is clopen, $V \cap K = L$ and $V = \gamma(V)$. We can then define $\delta \in \Gamma$ by setting $\delta(x) = \gamma(x)$ for all $x \in V$ and $\delta(x) = x$ for all $x \notin V$.

By construction, $\delta(x) = \gamma(x)$ for all $x \in L$, and $\delta(x) = x$ for all $x \in K \setminus L$, so that $\delta_K = \gamma_K$. $\square$

Definition 3.7. Let Γ be a minimal ample subgroup of X and K a Γ-étale subset. Let $(\mathcal{B}, \Delta)$ be a finite unit system with $\Delta \leq \Gamma$.

We say that $(\mathcal{B}, \Delta)$ is K-compatible if for all atoms A, B of $\mathcal{B}$ and every $\delta \in \Delta$ such that $\delta(A) = B$, if $K \cap A$ and $K \cap B$ are both nonempty then we have $\delta(K \cap A) = K \cap B$.

This extends a definition given in [MR, Definition 3.10], where we only considered closed sets K such that for all $x \neq y \in K$ one has $y \notin \Gamma x$, which we called Γ-sparse sets. The assumption of K-compatibility amounts to saying that, if A, B are two elements of the same Δ-orbit that both intersect K, then the involution in Δ which maps A to B and is equal to the identity outside of $A \cup B$ is K-compatible.

Lemma 3.8. *Assume that K is Γ-étale and let $(\mathcal{A}, \Delta)$ be a finite unit system with $\Delta \leq \Gamma$. Then there exists $\mathcal{A}'$ refining $\mathcal{A}$ and Δ' such that $(\mathcal{A}', \Delta')$ is a K-compatible finite unit system and $\Delta \leq \Delta' \leq \Gamma$.*

Since any two finite unit systems $(\mathcal{A}, \Delta)$ and $(\mathcal{B}, \Lambda)$ with $\Delta, \Lambda \leq \Gamma$ have a common refinement (C, Σ) with $\Sigma \leq \Gamma$, it follows from Lemma 3.8 and the existence of an exhaustive sequence of finite unit systems for Γ that if K is Γ-étale then there exists an exhaustive sequence of K-compatible finite unit systems for Γ.

Proof. The argument proceeds by cutting the Δ-orbit of each atom of $\mathcal{A}$. First, fix some such orbit τ.

Let $U \neq V$ be two elements of τ and $\delta \in \Delta$ the involution such that $\delta U = V$ and $\delta(x) = x$ for every $x \notin U \sqcup V$. We first consider $K_1(\delta) = \{x \in K \cap U \colon \delta(x) \in K\}$, which is clopen in K; we pick a clopen set $U' \subseteq U$ such that $U' \cap K = K_1(\delta)$. Then $\delta(K_1(\delta)) = \delta(U' \cap K)$ is clopen in K and contained in V, so we can find a clopen subset V' of V such that $V' \cap K = \delta(K_1(\delta))$. Set $U_1 = U' \cap \delta V'$, $V_1 = \delta(U_1)$.

By definition, $K_1(\delta) = U_1 \cap K$ and $\delta(U_1 \cap K) = \delta(K_1(\delta)) = V_1 \cap K$.

If $(U \setminus U_1) \cap K = \emptyset$ or $(V \setminus V_1) \cap K = \emptyset$ we let $U_2 = U \setminus U_1$, $V_2 = \delta(U_2)$ and stop partitioning U, V. Else, let $U'' = U \setminus U_1$, $V'' = V \setminus V_1$. Then $\delta(U'' \cap K) \cap K = \emptyset$. Since these two sets are closed, we can find a clopen set $U_2 \subset U''$ containing $U'' \cap K$ and

such that $\delta(U_2) \cap K = \emptyset$, and set $V_2 = \delta(U_2)$. Finally we define $U_3 = U \setminus (U_1 \sqcup U_2)$ and $V_3 = \delta(U_3)$.

We now have $\delta(U_1 \cap K) = V_1 \cap K$; $V_2 = \delta(U_2)$ and $V_2 \cap K = \emptyset$; $V_3 = \delta(U_3)$ and $U_3 \cap K = \emptyset$.

We do this for every pair of distinct U, V in τ and choose a finite Δ-invariant partition of τ refining all those partitions. We obtained our desired refinement of τ. Doing this for every τ gives us $\mathcal{A}'$. We then let Δ' be the group of all homeomorphisms of X which preserve $\mathcal{A}'$ and coincide on each atom U of $\mathcal{A}$ with some $\delta_U \in \Delta$. □

Given a Γ-étale $K \subset X$, we denote by Γ_K the smallest full subgroup of $\mathrm{Homeo}(K)$ which contains γ_K for every involution $\gamma \in \Gamma$. For every $x \in X$ and every $\gamma \in \Gamma$ there exists an involution $\delta \in \Gamma$ such that $\delta(x) = \gamma(x)$; hence the restriction of R_Γ to some malleable K is induced by Γ_K.

The group Γ_K can equivalently be described as follows.

Lemma 3.9. *Assume that K is Γ-étale. Then $\Gamma_K = \{\gamma_{|K} : \gamma \in \Gamma$ and $\gamma(K) = K\}$. In particular, Γ_K is an ample group over K.*

Proof. Fix an exhaustive sequence $(\mathcal{A}_n, \Gamma_n)$ of K-compatible finite unit systems for Γ (see the remark following the statement of Lemma 3.8).

Pick $g \in \Gamma_K$. There exists a clopen partition $(U_i)_{i=1,\dots,p}$ of K such that for each i the restriction of g to U_i coincides on U_i with a product of K-compatible involutions. Let A be a finite set of K-compatible involutions witnessing that fact, and define $\Delta = \langle A \rangle$ (a finite subgroup of Γ); note that $\delta(K) = K$ for all $\delta \in \Delta$, in particular restrictions of elements of Δ to K form a finite subgroup of $\mathrm{Homeo}(K)$.

Next, pick a finite subalgebra $\mathcal{B}$ of $\mathrm{Clopen}(K)$ containing every U_i and such that $\delta(B) \in \mathcal{B}$ for all $\delta \in \Delta$. Choose a Boolean subalgebra $\tilde{\mathcal{B}}$ of $\mathrm{Clopen}(X)$ such that $\{\tilde{B} \cap K : \tilde{B} \in \tilde{\mathcal{B}}\} = \mathcal{B}$ and $\tilde{\mathcal{B}}$ is Δ-invariant. For n large enough we have that $\mathcal{A}_n$ refines $\tilde{\mathcal{B}}$ and Γ_n contains Δ. For every atom U of $\mathcal{A}_n$ which intersects K, there exists $\delta \in \Delta$ such that δ and g coincide on $U \cap K$. We must then have $\delta(U \cap K) = g(U \cap K)$, and such a δ is unique since Δ is contained in Γ_n and $(\mathcal{A}_n, \Gamma_n)$ is a unit system.

Denoting this δ by δ_U, we define $\gamma \in \mathrm{Homeo}(X)$ by setting $\gamma(x) = \delta_U(x)$ for every atom U of $\mathcal{A}_n$ which intersects K and every $x \in U$, $\gamma(x) = x$ elsewhere. By definition we have both that $\gamma \in \Gamma$ and $\gamma_{|K} = g$.

Conversely, assume that $\gamma \in \Gamma$ is such that $\gamma(K) = K$. Then find $n \in \mathbf{N}$ such that $\gamma \in \Gamma_n$. Since $(\mathcal{A}_n, \Gamma_n)$ is K-compatible, for each atom U of $\mathcal{A}_n$ such that $U \cap K \neq \emptyset$ there exists a unique involution γ_U in Γ_n such that $\gamma_U(U \cap K) = \gamma(U \cap K)$ and $\gamma_U(x) = x$ for all $x \notin U \cup \gamma(U)$. By definition of a unit system, γ and γ_U coincide on U, and γ_U is K-compatible. Let λ be the homeomorphism which coincides with γ_U on each $U \cap K$ whenever $U \cap K \neq \emptyset$ and coincides with the identity everywhere else. Then $\lambda \in \Gamma_K$ and $\lambda = \gamma_{|K}$. □

Definition 3.10. Assume that K is Γ-étale and that $(\mathcal{A}, \Lambda)$ is a K-compatible finite unit system with $\Lambda \leq \Gamma$. We let $\mathcal{A}_K$ denote the Boolean subalgebra of $\mathrm{Clopen}(K)$ induced by $\mathcal{A}$, and $\Lambda_K = \{\lambda_{|K} : \lambda \in \Lambda$ and $\lambda(K) = K\}$.

Note that by K-compatibility we have that $(\mathcal{A}_K, \Lambda_K)$ is a finite unit system in K.

Lemma 3.11. *Assume that $(\mathcal{A}_n, \Gamma_n)$ is an exhaustive sequence of K-compatible finite unit systems for Γ. Then $(\mathcal{A}_{n,K}, \Gamma_{n,K})$ is an exhaustive sequence of finite unit systems for Γ_K.*

Proof. Clearly $\bigcup_n \mathcal{A}_{n,K} = \mathrm{Clopen}(K)$. Given $\sigma \in \Gamma_K$, by Lemma 3.9 there exists $\gamma \in \Gamma$ such that $\sigma = \gamma_{|K}$, whence $\sigma \in \Gamma_{n,K}$ for n large enough. $\qquad\square$

4. A further strengthening of Krieger's theorem

Our aim now is to establish the following version of Krieger's theorem. This is a close cousin of the "Fundamental Lemma" [GPS2, Lemma 4.15] (each statement follows readily from the other, though the proofs are completely different) and a further strengthening of the version established in [MR, Theorem 3.11]

Theorem 4.1. *Let Γ, Λ be two minimal ample groups over X, and K (resp. L) be a Γ-malleable (resp. Λ-malleable) subset of X. Assume also that Γ, Λ induce isomorphic relations on $\mathrm{Clopen}(X)$.*

Then every homeomorphism $h\colon K \to L$ such that $h\Gamma_K h^{-1} = \Lambda_L$ extends to a homeomorphism of X such that $h\Gamma h^{-1} = \Lambda$.

For the remainder of this section, we fix two minimal ample groups Γ, Λ which induce isomorphic relations on $\mathrm{Clopen}(X)$; by conjugating Λ if necessary, we reduce to the case where $\sim_\Gamma$ and $\sim_\Lambda$ coincide, and we denote this relation by $\sim$. Explicitly, for any two clopen subsets U, V of X we have

$$(\exists \gamma \in \Gamma \; \gamma U = V) \Leftrightarrow (U \sim V) \Leftrightarrow (\exists \lambda \in \Lambda \; \lambda U = V)$$

We also fix a Γ-malleable subset K of X, a Λ-malleable subset L of X, and $h\colon K \to L$ a homeomorphism such that $h\Gamma_K h^{-1} = \Lambda_L$.

Definition 4.2. Assume that $\Delta \leq \Gamma$ and $(\mathcal{A}, \Delta)$ is a K-compatible finite unit system, $\Sigma \leq \Lambda$ and $(\mathcal{B}, \Sigma)$ is a L-compatible finite unit system.

A Boolean algebra isomorphism $\Phi\colon \mathcal{A} \to \mathcal{B}$ is said to be *h-compatible* if:

- $\Phi(A) \sim A$ for every $A \in \mathcal{A}$.
- $\Phi\Delta_{\mathcal{A}}\Phi^{-1} = \Sigma_{\mathcal{B}}$ (we then say that Φ conjugates $(\mathcal{A}, \Delta)$ on $(\mathcal{B}, \Sigma)$).
- For every $A \in \mathcal{A}$ we have $\Phi(A) \cap L = h(A \cap K)$.

In the second bullet point above, recall that $\Delta_{\mathcal{A}}$ is the subgroup of $\mathrm{Aut}(\mathcal{A})$ induced by the action of Δ (and similarly for $\Sigma_{\mathcal{B}}$).

The proof of Theorem 4.1 goes through a back-and-forth argument. To make this argument work, it is enough to establish the following lemma (the proof of which is essentially the same as in [MR, Lemma 3.17] once one has Lemma 3.3 in hand, though we repeat it here for the sake of completeness).

Lemma 4.3. *Assume that $(\mathcal{A}, \Delta)$, $(\mathcal{B}, \Sigma)$ are respectively K- and L-compatible finite unit systems with $\Delta \leq \Gamma$, $\Sigma \leq \Lambda$, and $\Phi\colon \mathcal{A} \to \mathcal{B}$ is a h-compatible Boolean algebra isomorphism.*

Let $(\mathcal{A}', \Delta')$ be a K-compatible finite unit system refining $(\mathcal{A}, \Delta)$ with $\Delta' \leq \Gamma$.

Then one can find a L-compatible finite unit system $(\mathcal{B}', \Sigma')$ refining $(\mathcal{B}, \Sigma)$, with $\Sigma' \leq \Lambda$ and a h-compatible isomorphism $\Phi'\colon \mathcal{A}' \to \mathcal{B}'$ which extends Φ.

Proof. For every orbit ρ of the action of Δ on the atoms of $\mathcal{A}$, we choose a representative A_ρ. If ρ intersects K, we choose A_ρ so that $A_\rho \cap K \neq \emptyset$.

For every $A \in \rho$, we denote by $\delta(\rho, A)$ the element of Δ which maps A to A_ρ, A_ρ to A, and is the identity everywhere else. This is an involution (and it is uniquely defined by definition of a unit system); in the particular case where $A = A_\rho$ we have $\delta(\rho, A_\rho) = \mathrm{id}$. Similarly, we denote $\sigma(\rho, A)$ the involution of Σ exchanging $\Phi(A)$ and $\Phi(A_\rho)$ and which is the identity everywhere else.

For every ρ we have

$$A_\rho = \bigsqcup_{C \in \text{atoms}(\mathcal{A}'):\, C \subseteq A_\rho} C$$

Let $C_1, \ldots, C_q$ denote the atoms of $\mathcal{A}'$ contained in A_ρ. If $q = 1$ we let $U(C) = \Phi(C)$.

Assume that $q \geq 2$. Applying Lemma 3.3, we find a clopen $U(C_1) \sim C_1$ contained in $\Phi(A_\rho)$ and such that $U(C_1) \cap L = h(C_1 \cap K)$; then a clopen $U(C_2) \sim C_2$ contained in $\Phi(A_\rho)$, disjoint from $U(C_1)$ and such that $U(C_2) \cap L = h(C_2 \cap K)$; and so on until $q - 1$.

We have no choice but to set $U(C_q) = \Phi(A_\rho) \setminus \bigsqcup_{i=1}^{q-1} U(C_i)$. Since h is bijective we have $U(C_q) \cap L = h(C_q \cap K)$; and by Lemma 2.3 we also have $U(C_q) \sim C_q$.

We now have

$$\Phi(A_\rho) = \bigsqcup_{C \in \text{atoms}(\mathcal{A}'):\, C \subseteq A_\rho} U(C)$$

where $U(C) \sim C$, and $U(C) \cap L = h(C \cap K)$ for all C.

We define the algebra $\mathcal{B}'$ by setting as its atoms all $U(C)$, for C an atom of $\mathcal{A}'$ contained in some A_ρ, as well as all $\sigma(\rho, A)(U(C))$ for $A \in \rho$ and C contained in A_ρ. We obtain an isomorphism $\Phi' : \mathcal{A}' \to \mathcal{B}'$ by setting $\Phi'(C) = U(C)$ for every atom C of $\mathcal{A}'$ contained in some A_ρ; and then for any atom C of $\mathcal{A}'$ contained in some $A \in \mathcal{A}$ whose Δ-orbit is ρ,

$$\Phi'(C) = \sigma(\rho, A)(U(\delta(\rho, A)(C)))$$

For every atom C of $\mathcal{A}'$ we have $\Phi'(C) \cap L = h(C \cap K)$ by choice of $U(C)$, K-compatibility of $(\mathcal{A}, \Delta)$ and L-compatibility of $(\mathcal{B}, \Sigma)$.

We now need to construct the group Σ'. In the remainder of the proof, the letter τ always stands for an orbit of the action of Δ on the atoms of $\mathcal{A}'$, and the letter π for an orbit of the action of Δ' on the atoms of $\mathcal{A}'$. For any τ there exists a unique π which contains τ.

For any τ we choose a representative B_τ, which intersects K if some element of τ intersects K. Among all B_τ contained in a given π we choose one B_π, and ask again that B_π intersects K if some element of π intersects K. For every τ contained in π, we choose an involution $\lambda(\tau, \pi) \in \Lambda$ mapping $\Phi'(B_\pi)$ to $\Phi'(B_\tau)$, and equal to the identity elsewhere. We also require that $\lambda(\tau, \pi)(\Phi'(B_\pi) \cap L) = \Phi'(B_\tau) \cap L$ if $B_\tau \cap K \neq \emptyset$ (which is possible thanks to Lemma 3.3).

Let Σ' be the group generated by Σ and $\{\lambda(\tau, \pi) : \tau \subset \pi\}$. Then $(\mathcal{B}', \Sigma')$ is a finite unit system (because we have added at most one link between any two Σ-orbits) and Φ' conjugates $(\mathcal{A}', \Delta')$ on $(\mathcal{B}', \Sigma')$.

We still need to show that $(\mathcal{B}', \Sigma')$ is L-compatible; so let U, V be two atoms in $\mathcal{B}'$ belonging to the same Σ'-orbit and such that $U \cap L, V \cap L$ are both nonempty.

There exists a Δ'-orbit π, two Δ-orbits τ_1, τ_2 contained in π and involutions $\sigma_1, \sigma_2 \in \Sigma$ such that :

$$U = \sigma_1 \Phi'(B_{\tau_1}) = \sigma_1 \lambda(\tau_1, \pi) \Phi'(B_\pi) \text{ and } V = \sigma_2 \Phi'(B_{\tau_2}) = \sigma_2 \lambda(\tau_2, \pi) \Phi'(B_\pi)$$

Any element σ of Σ' mapping U to V must coincide on U with $\sigma_2 \lambda(\tau_2, \pi) \lambda(\tau_1, \pi) \sigma_1$.

Using that $(\mathcal{B}, \Sigma)$ is L-compatible we then have:

$$\begin{aligned}
\sigma(U \cap L) &= \sigma_2 \lambda(\tau_2, \pi) \lambda(\tau_1, \pi) \sigma_1(U \cap L) \\
&= \sigma_2 \lambda(\tau_2, \pi) \lambda(\tau_1, \pi) \sigma_1((\sigma_1 \lambda(\tau_1, \pi) \Phi'(B_\pi)) \cap L) \\
&= \sigma_2 \lambda(\tau_2, \pi) \lambda(\tau_1, \pi) \sigma_1(\sigma_1 \lambda(\tau_1, \pi)(\Phi'(B_\pi) \cap L)) \\
&= \sigma_2 \lambda(\tau_2, \pi)(\Phi'(B_\pi) \cap L)) \\
&= (\sigma_2 \lambda(\tau_2, \pi) \Phi'(B_\pi)) \cap L = V \cap L \qquad \qquad \square
\end{aligned}$$

We then obtain Theorem 4.1 by using the same back-and-forth argument as in [K, Theorem 3.5], which is also detailed in the proof of [MR, Theorem 3.11].

5. A version of Matui's absorption theorem

We again fix an ample group Γ over X which acts minimally. We need to recall some more vocabulary from [MR].

Definition 5.1. We say that a clopen partition $\mathcal{A} = (A_{i,j})_{(i,j) \in I}$ is a $\sim_\Gamma$-*partition* if:

$$\forall i, j, k \quad ((i,j) \in I \text{ and } (i,k) \in I) \Rightarrow (A_{i,j} \sim_\Gamma A_{i,k})$$

Denoting by $I_i = \{j : (i,j) \in I\}$ we say that $\{A_{i,j} : j \in I_i\}$ is an $\mathcal{A}$-*orbit*.

A *fragment* of an $\mathcal{A}$-orbit $(A_{i,j})_{j \in I_i}$ is a family $(B_{i,j})_{j \in I_i}$ of nonempty clopen subsets of X such that $B_{i,j} \subseteq A_{i,j}$ for all $j \in I_i$ and $B_{i,k} \sim_\Gamma B_{i,j}$ for all $j, k \in I_i$.

Note that whenever $(\mathcal{A}, \Delta)$ is a finite unit system with $\Delta \leq \Gamma$, one can view $\mathcal{A}$ as a $\sim_\Gamma$-partition by grouping together atoms of $\mathcal{A}$ which belong to the same Δ-orbit.

We say that a $\sim_\Gamma$-partition $\mathcal{B}$ *refines* another $\sim_\Gamma$-partition $\mathcal{A}$ if every $\mathcal{B}$-orbit can be written as a disjoint union of fragments of $\mathcal{A}$-orbits (intuitively, $\mathcal{B}$ has been obtained from $\mathcal{A}$ by cutting some $\mathcal{A}$-orbits, then grouping some fragments together). We note that we sometimes identify a clopen partition and the Boolean algebra it generates, which should cause no risk of confusion.

Definition 5.2. Let K be a closed subset of X, and Δ be a subgroup of Homeo(K). We denote by $R_\Gamma(K, \Delta)$ the coarsest equivalence relation S which refines R_Γ and is such that $(x, \delta x) \in S$ for all $x \in K$ and all $\delta \in \Delta$.

Informally, $R_\Gamma(K, \Delta)$ is obtained from R_Γ by joining the Γ-orbits of Δ-equivalent elements of K, and leaving untouched the Γ-orbits which do not intersect K.

Lemma 5.3. *Let K, K' be two Γ-malleable subsets, and Δ, Δ' two ample groups over K. Assume that there exists a homeomorphism $h \colon K \to K'$ such that $h\Gamma_K h^{-1} = \Gamma_{K'}$ and $(h \times h)R_\Delta = R_{\Delta'}$.*

Then $R_\Gamma(K, \Delta)$ and $R_\Gamma(K', \Delta')$ are orbit equivalent.

Proof. Since K and K' are Γ-malleable, Theorem 4.1 allows us to extend h to an homeomorphism of X such that $h\Gamma h^{-1} = \Gamma$. Then $(h \times h)R_\Gamma(K, \Delta) = R_\Gamma(K', \Delta')$. $\square$

The following lemma is the key step of our proof of the absorption theorem (this is what allows the "Hilbert–Bratteli hotel" argument to go through).

Lemma 5.4. *Let Y be a compact, 0-dimensional metric space and $\Delta \leq \Sigma$ two ample groups over Y.*

There exists a minimal ample group $\Lambda \leq \Gamma$, a closed subset K which is both Λ- and Γ-malleable, and a homeomorphism $h \colon Y \to K$ such that $h\Delta h^{-1} = \Lambda_K$, $h\Sigma h^{-1} = \Gamma_K$ and $R_\Gamma = R_\Lambda(K, \Gamma_K)$.

Proof. We fix an exhaustive sequence $(\mathcal{A}_n, \Sigma_n)$ of finite unit systems for (Y, Σ) and let $\Delta_n = \Sigma_n \cap \Delta$.

Denote by h_n the number of atoms of $\mathcal{A}_n$. We pick an exhaustive sequence $(\mathcal{B}_n, \Gamma_n)$ of finite unit systems for (X, Γ); for all n we view $\mathcal{B}_n$ as a $\sim_\Gamma$-partition so that the $\mathcal{B}_n$- and Γ_n-orbits of every atom of $\mathcal{B}_n$ coincide.

We also assume that for any n there are more than h_n distinct $\mathcal{B}_n$-orbits (which may be achieved by cutting them if necessary). Denoting by k_n the number of atoms of $\mathcal{B}_n$, we may also ensure that every $\mathcal{B}_{n+1}$-orbit contains more than $(k_n + 1)h_{n+1}$ disjoint fragments of every $\mathcal{B}_n$-orbit and that $k_n \geq nh_{n+1}$.

To initialize the constructions, we add the assumption that $\mathcal{A}_0$, $\mathcal{B}_0$, Σ_0 and Γ_0 are all trivial.

Step 1. **Building a copy of** Y.

By first choosing some elements of $\mathcal{B}_n$, then grouping them together with other elements of $\mathcal{B}_n$, we build a sequence of maps $\Phi_n \colon \mathcal{A}_n \to \mathcal{B}_n$ and a sequence of equivalence relations $\sim_n$ on $\mathcal{B}_n$; intuitively, $\Phi_n(\mathcal{A}_n)$ is our approximation (at step n) of the domain of the copy of (Y, Δ, Σ) that we are trying to build, and the restriction of $\sim_n$ to $\Phi_n(\mathcal{A}_n)$ mimics $\sim_{\Delta_n}$.

Explicitly, our construction proceeds by enforcing the following conditions:

(1) $\Phi_n \colon \mathcal{A}_n \to \mathcal{B}_n$ is injective, and for any $A, B \in \mathcal{A}_n$ we have

$$(\Phi_n(A) \sim_{\Gamma_n} \Phi_n(B)) \Leftrightarrow (A \sim_{\Sigma_n} B) \,; \, (\Phi_n(A) \sim_n \Phi_n(B)) \Leftrightarrow (A \sim_{\Delta_n} B)$$

(2) For any $U, V \in \mathcal{B}_n$, $U \sim_n V \Rightarrow U \sim_{\Gamma_n} V$.

(3) If $U \in \mathcal{B}_n$ is such that the $\mathcal{B}_n$-orbit τ_U of U does not intersect the image of Φ_n, then all elements of τ_U are $\sim_n$-equivalent. Else, every element of τ_U is $\sim_n$-equivalent to some $\Phi_n(A)$ for $A \in \mathcal{A}_n$.

(4) If $A \in \mathcal{A}_n$, $B \in \mathcal{A}_{n+1}$ are such that $B \subseteq A$ then $\Phi_{n+1}(B) \subseteq \Phi_n(A)$.

Denote by C_n the $\sim_\Gamma$-partition whose atoms are the same as those of $\mathcal{B}_n$ and whose orbits are the $\sim_n$-classes. Each $\mathcal{B}_n$-orbit is a union of C_n-orbits.

(5) Given $A \in \mathcal{A}_{n+1}$, the $\sim_{n+1}$-class of $\Phi_{n+1}(A)$ is obtained by joining one fragment of the C_n-orbit of every $\Phi_{n+1}(A')$ for $A' \in \Delta_{n+1}A$ and fragments of the other $\mathcal{B}_n$-orbits, so that a fragment of every $\mathcal{B}_n$-orbit appears at least once. In particular, C_{n+1} refines C_n and each C_{n+1}-orbit contains a fragment of every C_n-orbit.

To see that this is indeed possible, assume that this construction has been carried out up to some n (for $n = 0$ there is nothing to do). Below, for A an atom of $\mathcal{A}_{n+1}$ we denote by A' the unique atom of $\mathcal{A}_n$ which contains A.

We first define an injective $\Phi_{n+1} \colon \mathcal{A}_{n+1} \to \mathcal{B}_{n+1}$. For any $A \in \mathcal{A}_{n+1}$ we ask that $\Phi_{n+1}(A) \subseteq \Phi_n(A')$, and require also that, for any $A, B \in \mathcal{A}_{n+1}$, $\Phi_{n+1}(A)$ and $\Phi_{n+1}(B)$ belong to the same $\mathcal{B}_{n+1}$-orbit iff A and B belong to the same Σ_{n+1}-orbit. This is feasible because the number of $\mathcal{B}_{n+1}$-orbits is greater than h_{n+1} (hence larger than the number of Σ_{n+1}-orbits of atoms of $\mathcal{A}_{n+1}$) and, for any $A' \in \mathcal{A}_n$, in any $\mathcal{B}_{n+1}$-orbit there are more than h_{n+1} atoms contained in A', so that we can guarantee the injectivity of Φ_{n+1}.

Next, fix a $\mathcal{B}_{n+1}$-orbit τ which intersects $\Phi_{n+1}(\mathcal{A}_{n+1})$. We start building $\sim_{n+1}$ on τ, by choosing for each $A \in \mathcal{A}_{n+1}$ such that $\Phi_{n+1}(A) \in \tau$ a fragment of the C_n-orbit of $\Phi_n(A')$ which contains $\Phi_{n+1}(A)$ and is contained in τ. We do this in such a way that $\Phi_{n+1}(A)$ and $\Phi_{n+1}(B)$ belong to the same fragment iff $A' \in \Delta_n B'$. Again, this is feasible because τ contains many fragments of every $\mathcal{B}_n$-orbit, hence of every

C_n-orbit. Then we include in the $\sim_{n+1}$-class of $\Phi_{n+1}(A)$ the union of the fragments we have chosen for all $B \in \Delta_{n+1}A$ (and nothing else at this point).

The union of fragments of C_n-orbits on which we have so far defined $\sim_{n+1}$ in τ is actually a union of fragments of $\mathcal{B}_n$-orbits: indeed, for each $\mathcal{B}_n$-orbit ρ which intersects $\Phi_n(\mathcal{A}_n)$ we have selected whole fragments, because ρ is the union of $\sim_n$-classes of all $\Phi_n(A')$ contained in ρ (and the fragments of other $\mathcal{B}_n$-orbits have not yet been taken under consideration). Thus, at this point, each partial $\sim_{n+1}$-equivalence class in τ consists of a union of a most h_{n+1} fragments of $\mathcal{B}_n$-orbits. Inside τ, we have at least $(k_n + 1)h_{n+1}$ such fragments of each $\mathcal{B}_n$-orbit at our disposal, hence we can distribute the remaining fragments among the (at most h_{n+1}) disjoint partial $\sim_{n+1}$-classes contained in τ in such a way that each $\sim_{n+1}$-class contains at least one fragment of every $\mathcal{B}_n$-orbit, and every atom of τ is in the $\sim_{n+1}$-class of some $\Phi_{n+1}(A)$.

Performing the above procedure for every τ which intersects $\Phi_{n+1}(\mathcal{A}_{n+1})$, we obtain the desired relation $\sim_{n+1}$.

For all $n \in \mathbf{N}$ we let $K_n = \bigsqcup_{A \in \mathcal{A}_n} \Phi_n(A)$. This is a clopen set, $K_{n+1} \subseteq K_n$ for all n and we define $K = \bigcap_{n \in \mathbf{N}} K_n$.

The sequence $(\Phi_n)_n$ induces an isomorphism $\Phi \colon \mathrm{Clopen}(Y) \to \mathrm{Clopen}(K)$, and we let $h \colon Y \to K$ be the corresponding homeomorphism.

Step 2. Defining Λ.

Denote by Λ_n the group of permutations of the atoms of $\mathcal{B}_n$ which map each $\sim_n$-class to itself. We define embeddings $i_n \colon \Lambda_n \to \Lambda_{n+1}$ so that (among other properties) for any $\lambda \in \Lambda_n$ and any $B \in \mathcal{B}_n$, if $C_1, \ldots, C_p$ are the atoms of $\mathcal{B}_{n+1}$ such that $B = \bigsqcup_{i=1}^{p} C_i$ then $\lambda(\Phi_n(B)) = \bigsqcup_{i=1}^{p} i_n(\lambda)(\Phi_{n+1}(C_i))$.

To explain how i_n is defined, we fix a $\sim_n$-orbit τ and assume first that there is $A \in \mathcal{A}_n$ such that $\Phi_n(A) \in \tau$. Given another atom U of τ, denote λ_U the unique involution in Λ_n which maps $\Phi_n(A)$ to U and leaves all other elements of $\mathcal{B}_n$ fixed.

First, we consider the case where $U = \Phi_n(\delta A)$ with $\delta \in \Delta_n$. We can write $A = \bigsqcup_{i=1}^{p} A_i$ with $A_i \in \mathcal{A}_{n+1}$ and set

$$i_n(\lambda_U)(\Phi_{n+1}(A_i)) = \Phi_{n+1}(\delta A_i)$$

We still have to define $i_n(\lambda_U)$ on some atoms of $\mathcal{B}_{n+1}$ contained in $\Phi_n(A), \Phi_n(\delta A)$. In each C_{n+1}-orbit there remain as many atoms contained in $\Phi_n(\delta A)$ where $i_n(\lambda_U)$ has not been defined as there are such atoms contained in $\Phi_n(A)$ (because C_{n+1} refines C_n, every C_{n+1}-orbit has as many atoms contained in $\Phi_n(A)$ and $\Phi_n(\delta A)$). We may thus match those atoms arbitrarily to define $i_n(\lambda_U)$ so that it maps each of these atoms to an atom in the same $\sim_{n+1}$-class.

The second case is when U is not in the image of $\Phi_n(\mathcal{A}_n)$. Again, each C_{n+1}-orbit has as many atoms contained in U and in $\Phi_n(A)$, and we match those arbitrarily to define $i_n(\lambda_U)$.

If our $\sim_n$-orbit τ does not intersect the image of Φ_n, then we choose $V \in \tau$ at random, define λ_U as above (for U another element of τ) and define $i_n(\lambda_U)$ as in the previous paragraph (any involution which extends λ_U and fixes setwise each $\sim_{n+1}$-class will do the job).

Once all this is done, we have completely defined $i_n \colon \Lambda_n \to \Lambda_{n+1}$.

The inductive limit Λ of the sequence $(\Lambda_n, i_n)_n$ naturally acts on $\mathrm{Clopen}(X)$, and we view it as a subgroup of $\mathrm{Homeo}(X)$. By construction, this group is ample.

Denoting also by i_n the embedding of Λ_n in Λ, the $i_n(\Lambda_n)$-orbit of every element of C_n coincides with its C_n-orbit. Given a nonempty $U \in \mathrm{Clopen}(X)$, let n be such that U is a union of atoms of C_n. Since every C_{n+1}-orbit contains a fragment of every C_n-orbit, hence an atom contained in U, we have $i_{n+1}(\Lambda_{n+1})U = X$. We conclude that Λ acts minimally on X.

In each C_{n+1}-orbit there are at least k_n elements, and at most h_{n+1} of those belong to K_{n+1}; since $h_{n+1}/k_n \xrightarrow[n \to +\infty]{} 0$, we conclude that $\mu(K) = 0$ for every $\mu \in M(\Lambda)$, so K is Λ-thin.

By construction, if $A, B \in \mathscr{A}_n$ and $\lambda \in \Lambda_n$ are such that $\lambda\Phi_n(A) = \Phi_n(B)$ we can write $A = \bigsqcup_{i=1}^{p} A_i$, $B = \bigsqcup_{i=1}^{p} B_i$ with $A_i, B_i \in \mathscr{A}_{n+1}$ and $i_n(\lambda)(\Phi_{n+1}(A_i)) = \Phi_{n+1}(B_i)$. By induction, it follows that $i_n(\lambda)(\Phi_n(A) \cap K) = \Phi_n(B) \cap K$. This implies that K is Λ-étale. Finally, K is Λ-malleable.

Our definition of i_n also ensures that $h\Delta h^{-1} = \Lambda_K$ (where $h \colon Y \to K$ is the homeomorphism defined at the end of the first step).

Unfortunately we are not quite done yet: at this stage there is no reason why Λ would be contained in Γ. To remedy this, we replace Γ by a conjugate which we construct now.

Step 3. **Defining a conjugate of Γ.**

For all n, we let $\tilde{\Gamma}_n$ denote the group of all permutations of $\mathscr{B}_n$ which map each $\mathscr{B}_n$-orbit to itself; note that Λ_n is a subgroup of $\tilde{\Gamma}_n$. Similarly to what we did above, we build embeddings $j_n \colon \tilde{\Gamma}_n \to \tilde{\Gamma}_{n+1}$ to form an adequate inductive limit.

First, we ask that j_n coincide with i_n on Λ_n. Next, let τ be a $\mathscr{B}_n$-orbit; if τ is also a C_n-orbit then we already know how to extend elements of $\tilde{\Gamma}_n$ whose support is contained in τ (those belong to Λ_n so we simply apply i_n). So assume that τ intersects the image of Φ_n and let $A_1, \ldots, A_p \in \mathscr{A}_n$ be such that $p \geq 2$ and τ is the disjoint union of the C_n-orbits of $\Phi_n(A_1), \ldots, \Phi_n(A_p)$.

The Σ_n-orbit of A_1 is the disjoint union of the Δ_n-orbits of $A_1, \ldots, A_p$. For $i \in \{2, \ldots, p\}$ we let σ_i be the involution in Σ_n which maps A_1 to A_i and fixes all other atoms, and denote γ_i the element of $\tilde{\Gamma}_n$ which maps $\Phi_n(A_1)$ to $\Phi_n(A_i)$ and fixes all other atoms. Defining j_n amounts to defining $j_n(\gamma_i)$ for every $i \in \{2, \ldots, p\}$ (and doing this for every τ which contains at least 2 C_n-orbits).

Fix $i \in \{2, \ldots, p\}$. Write $A_1 = \bigsqcup_{k=1}^{q} A_{1,k}$, $A_i = \bigsqcup_{k=1}^{q} \sigma_i(A_{1,k})$ with $A_{1,k} \in \mathscr{A}_{n+1}$, then set

$$j_n(\gamma_i)(\Phi_{n+1}(A_{1,k})) = \Phi_{n+1}(\sigma_i A_{1,k})$$

On each atom of $\mathscr{B}_{n+1}$ which does not intersect $\Phi_n(A_1) \sqcup \Phi_n(A_i)$ we must have $j_n(\gamma_i)$ coincide with the identity. Now, we observe that in each C_{n+1}-orbit there are as many atoms contained in $\Phi_n(A_1)$ on which $j_n(\gamma_i)$ has not yet been defined as there are such atoms in $\Phi_n(A_i)$. This follows from the way $\sim_{n+1}$ has been defined above: so far we have defined what $j_n(\gamma_i)$ does to atoms contained in the C_n-fragments intersecting $\Phi_{n+1}(\mathscr{A}_{n+1})$, and the remainder consists of unions of fragments of $\mathscr{B}_n$-orbits. We then match those atoms arbitrarily to define $j_n(\gamma_i)$ so that, on each atom U which is not in $\Phi_{n+1}(\mathscr{A}_{n+1})$, $j_n(\gamma_i)U$ is $\sim_{n+1}$-equivalent to U.

This finally defines j_n, and we obtain an ample group $\tilde{\Gamma}$ by considering the inductive limit of $(\tilde{\Gamma}_n, j_n)$.

Denoting j_n the embedding of $\tilde{\Gamma}_n$ into $\tilde{\Gamma}$, the $j_n(\tilde{\Gamma}_n)$-orbit of every element of $\mathscr{B}_n$ coincides with its Γ_n-orbit, and $(\mathscr{B}_n, j_n(\tilde{\Gamma}_n))_n$ is an exhaustive sequence of unit systems for $\tilde{\Gamma}$. It follows from Krieger's theorem that Γ and $\tilde{\Gamma}$ are conjugate.

By definition, $\tilde{\Gamma}$ contains Λ. Since $\mathcal{B}_n$-orbits are unions of C_n-orbits, and we already noted that $\mu(K) = 0$ for all $\mu \in M(\Lambda)$, we also have $\mu(K) = 0$ for all $\mu \in M(\tilde{\Gamma})$.

Similarly to Step 2, the definition of each j_n also ensures that for $A, B \in \mathcal{A}_n$ and $\gamma \in \tilde{\Gamma}_n$ such that $\gamma\Phi_n(A) = \Phi_n(B)$ we have $j_n(\gamma)(\Phi_n(A) \cap K) = \Phi_n(B) \cap K$. Hence K is $\tilde{\Gamma}$-malleable. We also have $h\Sigma h^{-1} = \tilde{\Gamma}_K$ by construction.

Since $\Lambda \leq \tilde{\Gamma}$ we have that $R_\Lambda(K, \tilde{\Gamma}_K)$ is contained in $R_{\tilde{\Gamma}}$. To see the converse inclusion, let $\gamma \in \tilde{\Gamma}$ and pick n such that $\gamma \in j_n(\tilde{\Gamma}_n)$. Let U be an atom of $\mathcal{B}_n$. If γU and U belong to the same C_n-orbit then γ coincides on U with an element of Λ, so $\gamma x \in \Lambda x$ for every $x \in U$. If γU and U belong to different C_n-orbits, then there exist $\lambda_1, \lambda_2 \in i_n(\Lambda_n)$, $A_1, A_2 \in \Phi_n(\mathcal{A}_n)$ and $\sigma \in \Sigma_n \setminus \Delta_n$ such that $\sigma(A_1) = A_2$ and $\lambda_1 U = \Phi_n(A_1)$, $\lambda_2\Phi_n(A_2) = \gamma U$. Let γ' be the involution of $\tilde{\Gamma}_n$ which maps $\Phi_n(A_1)$ to $\Phi_n(A_2)$ and fixes all other atoms. By construction, we have that $j_n(\gamma')(K) = K$, and for every $x \in X \setminus K$ we have $j_n(\gamma')(x) \in \Lambda x$. Since γ coincides on U with $i_n(\lambda_2)j_n(\gamma')i_n(\lambda_1)$, we see that $(x, \gamma x) \in R_\Lambda(K, \tilde{\Gamma}_K)$ for every $x \in U$. We finally conclude that $R_{\tilde{\Gamma}} = R_\Lambda(K, \tilde{\Gamma}_K)$.

We have obtained the desired result for $\tilde{\Gamma}$ instead of Γ; since Γ and $\tilde{\Gamma}$ are conjugate in $\mathrm{Homeo}(X)$ this is enough. $\square$

We are finally ready to prove the main result. Informally, the idea is to first use Lemma 5.5 to see R_Γ as having been obtained by absorbing countably many "copies" of (K, Δ); intuitively, absorbing one more copy which is independent from the previous ones should not change the orbit equivalence class of R_Γ.

Theorem 5.5 (The absorption theorem). *Let Γ be a minimal ample group, K a Γ-malleable subset and Σ an ample group over K which contains Γ_K. Then $R_\Gamma(K, \Sigma)$ is orbit equivalent to R_Γ.*

Proof. Form a compact metric space $Y = (\bigsqcup_{n \in \mathbf{N}} K_n) \sqcup \{y\}$, where each K_n is a clopen subset of Y homeomorphic to K via some homeomorphism $h_n \colon K \to K_n$, and $(K_n)_n$ converges to $\{y\}$.

For $n \in \mathbf{N}$ let $\Delta_n = h_n\Gamma_K h_n^{-1}$, which we view as a subgroup of $\mathrm{Homeo}(Y)$ (whose elements act trivially outside K_n); denote by Δ the full subgroup of $\mathrm{Homeo}(Y)$ generated by $\bigcup_{n \in \mathbf{N}} \Delta_n$. It is ample because every element of Δ fixes pointwise a neighborhood of y, and for any n any $\delta \in \Delta$ coincides on K_n with an element of Δ_n.

Define $\Pi_0 = \Delta_0 = h_0\Gamma_K h_0^{-1}$ and, for $n \geq 1$, $\Pi_n = h_n\Sigma h_n^{-1}$ (viewed as subgroups of $\mathrm{Homeo}(Y)$ as above) then let Π be the full subgroup of $\mathrm{Homeo}(Y)$ generated by $\bigcup_{n \in \mathbf{N}} \Sigma_n$. Again, it is ample.

Then use Lemma 5.4 to find a minimal ample group Λ, a closed subset Z which is both Λ- and Γ-malleable and a homeomorphism $g \colon Y \to Z$ such that $g\Delta g^{-1} = \Lambda_Z$, $g\Pi g^{-1} = \Gamma_Z$ and $R_\Gamma = R_\Lambda(Z, \Gamma_Z)$.

Write $Z = \bigcup_{n \in \mathbf{N}} Z_n \cup \{z\}$ where for each n $Z_n = g(K_n) = g \circ h_n(K)$.

Denote $Z' = Z \setminus Z_0$. It is Λ-malleable since it is clopen in Z and Λ_Z-invariant. Note that $\Gamma_{Z_0} = g\Pi_0 g^{-1} = g\Delta_0 g^{-1} = \Lambda_{Z_0}$. So R_Λ and R_Γ coincide on Z_0 and it follows that $R_\Gamma = R_\Lambda(Z, \Gamma_Z) = R_\Lambda(Z', \Gamma_{Z'})$.

Similarly, Z_0 is Γ-malleable (again because Z_0 is clopen in Z and Γ_Z-invariant). Define $\Theta_0 = (gh_0)\Sigma(gh_0)^{-1}$. This is an ample group over Z_0.

Denote by Θ the ample subgroup of $\mathrm{Homeo}(Z)$ generated by $\Theta_0 \cup \bigcup_{n \geq 1} \Gamma_{Z_n}$. Let $f\colon Z \to Z'$ be such that for all n and all $x \in K$ one has $f(gh_n(x)) = gh_{n+1}(x)$. This is a homeomorphism.

Then $f \Lambda_Z f^{-1} = \Lambda_{Z'}$ and $f \Theta f^{-1} = \Gamma_{Z'}$. Since Z' is Λ-malleable, we can apply Lemma 5.3 and obtain that $R_\Lambda(Z, \Theta)$ and $R_\Lambda(Z', \Gamma_{Z'})$ are orbit equivalent. In other words, $R_\Gamma(Z_0, \Theta_0)$ and R_Γ are orbit equivalent.

We have that Z_0 is Γ-malleable, $(gh_0)\Gamma_K(gh_0)^{-1} = \Gamma_{Z_0}$ and $(gh_0)\Sigma(gh_0)^{-1} = \Theta_0$. Applying Lemma 5.3 once more, we get that $R_\Gamma(Z_0, \Theta_0)$ and $R_\Gamma(K, \Sigma)$ are orbit equivalent. Hence R_Γ is orbit equivalent to $R_\Gamma(K, \Sigma)$, and the absorption theorem is proved.   □

Note that, if φ is a minimal homeomorphism of X, $x_0 \in X$ and $\Gamma = \Gamma_{x_0}(\varphi)$ then the relation R_φ associated to the $\mathbf{Z}$-action induced by φ is obtained as $R_\Gamma(K, \Delta)$, where $K = \{x_0, \varphi^{-1}(x_0)\}$ and Δ is the permutation group of K. In particular, it immediately follows from the absorption theorem that R_φ is orbit equivalent to a relation induced by a minimal action of an ample group.

Corrigendum to the paper "From invariant measures to orbit equivalence, via locally finite groups", by J. Melleray and S. Robert

As explained in the introduction, the proof of the classification theorem of Giordano, Putnam and Skau given in [MR] has a gap. The same issue occurs twice, so that the proofs of the absorption theorem [MR, Theorem 5.1] as well as the proof of the classification theorem [MR, Theorem 5.2] are incorrect. It is possible to recover [MR, Theorem 5.1] (or even its improvement [MR, Theorem 6.5]) with an elementary argument based on the results and ideas of [MR] but that does not appear to be enough to fix the proof of the classification theorem.

Let us explain the issue briefly; here and below we reuse the notations and terminology of [MR]. The idea of our proof of the classification theorem is to start from a minimal ample group Γ and build a Γ-sparse K (the set of "singular" points in that proof), an involution $\pi\colon K \to K$ and a minimal ample group Λ, generated as a full group by Γ and π, so that R_Λ is obtained from R_Γ by gluing together the orbits of x and $\pi(x)$ for every $x \in K$. Unfortunately, the proof given in [MR] does not achieve this; the reason is that there are redundancies in the definition so that the full group Λ generated by Γ and π as constructed in that proof need not be ample. To fix this and ensure that Λ is ample one needs to avoid those redundancies; but then one cannot force the singular points to belong to different Γ-orbits. One can however guarantee that the set K of singular points is Γ-malleable, and then using Matui's absorption theorem (i.e. Theorem 5.5) allows the argument to go through.

We now explain how to combine Theorem 5.5 with the argument of [MR] to obtain a proof of the classification theorem for minimal ample groups.

We fix a minimal ample group Γ, and assume that $\sim_\Gamma$ and $\sim_\Gamma^*$ do not coincide. We want to prove that R_Γ is orbit equivalent to an action induced by a minimal ample group Λ with the property that $\sim_\Lambda$ and $\sim_\Lambda^*$ coincide, so as to employ Krieger's theorem one last time to prove the classification theorem.

We choose an enumeration $(U_n, V_n)_n$ of all pairs of $\sim_\Gamma^*$-equivalent clopen subsets of X and assume that $U_0 \not\sim_\Gamma V_0$.

First, we slightly modify [MR, Lemma 5.3] to obtain the following:

Lemma A.1. *We may build a sequence of $\sim_\Gamma$-partitions $(\mathcal{A}_n)$, with distinguished orbit pairs $O(\alpha_1^n), \ldots, O(\alpha_{k_n}^n), O(\beta_1^n), \ldots O(\beta_{k_n}^n)$ $(1 \le k_n$ for all $n)$ satisfying the following conditions.*

(1) $k_0 = 1$, $\alpha_1^0 = U_0$, $\beta_1^0 = V_0$ and $\mathcal{A}_0 = \{\alpha_1^0, \beta_1^0, X \setminus (\alpha_1^0 \cup \beta_1^0)\}$ *(three orbits of cardinality 1).*

For all n one has:

(2) $\mathcal{A}_{n+1}$ *refines* $\mathcal{A}_n$.
(3) *If $U_n \sim_\Gamma V_n$ then U_n and V_n are $\mathcal{A}_n$-equivalent.*
(4) $(\alpha_1^n, \ldots, \alpha_{k_n}^n, U_{n+1})$ *and* $(\beta_1^n, \ldots, \beta_{k_n}^n, V_{n+1})$ *are almost $\mathcal{A}_{n+1}$-equivalent, as witnessed by the exceptional orbits*

$$O(\alpha_1^{n+1}), \ldots, O(\alpha_{k_{n+1}}^{n+1}) \, , \, O(\beta_1^{n+1}), \ldots, O(\beta_{k_{n+1}}^{n+1})$$

(5) *For all i $\alpha_i^n \not\sim_\Gamma \beta_i^n$.*
(6) *For all i, α_i^{n+1} is contained in α_1^n and β_i^{n+1} is contained in β_1^n.*

(7) Let h_n be the number of atoms of $\mathcal{A}_n$; denote

$$N_i^n = \max\{|n_O(\alpha_i^n) - n_O(\beta_i^n)| : O \text{ is a } \mathcal{A}_{n+1} - orbit\} \quad (i \leq k_n)$$

$$N^{(n)} = \sum_{i=1}^{k_n} N_i^n$$

Then every exceptional $\mathcal{A}_{n+1}$-orbit contains more than $(n + 1)h_n(N^{(n)} + 2)$ fragments of every $\mathcal{A}_n$-orbit.

The slight modification alluded to above is twofold (and of a purely technical nature). First, the penultimate point in the Lemma's statement is not present in [MR] (but should be, given how the construction proceeds). It is not hard to enforce this condition: since every $\mathcal{A}_{n+1}$-orbit contains a fragment of every $\mathcal{A}_n$-orbit, we may choose each α_i^{n+1} so that it is contained in any prescribed nonempty clopen U, and similarly for β_i^{n+1}. The numerical constant in the last point also changed, but this is purely cosmetic, see the end of the proof of [MR, Proposition 4.10].

As in [MR] we define a sequence of $\sim_\Gamma^*$-partitions $\mathcal{B}_n$ by joining together the $\mathcal{A}_n$-orbits of α_i^n and β_i^n for each $i \in \{1, \ldots, k_n\}$ and leaving the other orbits unchanged. Then $\mathcal{B}_{n+1}$ refines $\mathcal{B}_n$ for all n.

We replace [MR, Lemma 5.4] with the following result.

Lemma A.2. *Denote by $\tilde{\Gamma}_n$ the group of permutations of $\mathcal{A}_n$ which map each $\mathcal{A}_n$-orbit to itself, and by Λ_n the group of permutations of $\mathcal{B}_n$ which map each $\mathcal{B}_n$-orbit to itself. Then we can build embeddings $i_n : \tilde{\Gamma}_n \to \tilde{\Gamma}_{n+1}$ and $j_n : \Lambda_n \to \Lambda_{n+1}$ such that, for all n:*

(1) *The action of $i_n(\tilde{\Gamma}_n)$ on $\mathcal{A}_{n+1}$ extends the action of $\tilde{\Gamma}_n$ on $\mathcal{A}_n$, and the action of $j_n(\Lambda_n)$ on $\mathcal{B}_{n+1}$ extends the action of Λ_n on $\mathcal{A}_n$.*

(2) *j_n coincides with i_n on $\tilde{\Gamma}_n$.*

(3) *Let π_0 be the involution in Λ_0 mapping α_1^0 to β_1^0 and define for $n \geq 1$ $\pi_n = j_{n-1} \circ \ldots \circ j_0(\pi_0)$. Then $\pi_n(\alpha_j^n) = \beta_j^n$ for all j.*

(4) *Say that an atom α of $\mathcal{A}_n$ is* singular *if $\pi_n(\alpha) \notin \tilde{\Gamma}_n \alpha$ and let K_n denote the union of all singular atoms of $\mathcal{A}_n$. Then $\mu(K_{n+1}) \leq \frac{1}{n+1}$ for all $\mu \in M(\tilde{\Gamma})$ and all n.*

Proof. To build the sequence of embeddings i_n, one can use an argument similar to the one we used in the second step of the proof of Lemma 5.4. So we assume that this sequence has been constructed, and focus on the definition of j_n; assume that all our conditions are satisfied up to rank $n - 1$ and we have to define j_n.

Given $i \in \{1, \ldots, k_n\}$, let σ_i be the element of Λ_n mapping α_i^n to β_i^n. Defining j_n amounts to defining $j_n(\sigma_i)$ for all i; we first deal with σ_1. Let Ω be a $\mathcal{B}_n$-orbit.

If Ω not an exceptional orbit, we may list all atoms of Ω contained in α_1^n as $\{U_1, \ldots, U_p\}$ and all atoms of Ω contained in β_1^n as $\{V_1, \ldots, V_p\}$ with U_i, V_i belonging to the same $\mathcal{A}_{n+1}$-orbit for all i, then set $j_n(\sigma_1)(U_k) = V_k$. Note that on Ω $j_n(\sigma_1)$ coincides with an element of $\tilde{\Gamma}_{n+1}$.

Else, Ω is the disjoint union of the $\mathcal{A}_{n+1}$-orbit Ω_1 of some α_j^{n+1} and the $\mathcal{A}_{n+1}$-orbit Ω_2 of β_j^{n+1}. Note that $\alpha_j^{n+1} \subseteq \alpha_1^n$ and $\beta_j^{n+1} \subseteq \beta_1^n$ so we may set $j_n(\sigma_1)(\alpha_j^{n+1}) = \beta_j^{n+1}$ and $j_n(\sigma_1)(\beta_j^{n+1}) = \alpha_j^{n+1}$.

There exists $p \leq N_1^n + 1$ atoms $U_1, \ldots, U_p \sim_\Gamma \alpha_j^{n+1}$ and $V_1, \ldots, V_p \sim_\Gamma \beta_j^{n+1}$ such that $\Omega_1 \setminus \{U_1, \ldots, U_p\}$ and $\Omega_2 \setminus \{V_1, \ldots, V_p\}$ each have as many atoms contained in α_1^n and β_1^n (the term "+1" comes from the fact that by asking that α_j^{n+1} be mapped

to β_j^{n+1} we may have slightly increased the imbalance of Ω_1 and Ω_2). We may then pair these other atoms to define $j_n(\sigma_1)$ so that $j_n(\sigma_1)$ sends each of them to an atom belonging to the same $\mathcal{A}_{n+1}$-orbit. We also agree that

$$j_n(\sigma_1)(U_i) = V_i \quad \text{and } j_n(\sigma_1)(V_i) = U_i$$

We have now defined $j_n(\sigma_1)$. There are at most $N_1^n + 2$ atoms $U \in \mathcal{A}_n$ such that $j_n(\sigma_1)(U)$ and U belong to different $\mathcal{A}_{n+1}$-orbits. By induction, we see that we have ensured $\pi_{n+1}(\alpha_j^{n+1}) = \beta_j^{n+1}$ for each j, since we had $\pi_n(\alpha_1^n) = \beta_1^n$.

To define $j_n(\sigma_i)$ for $i \geq 2$, we proceed similarly: if Ω is (α_i^n, β_i^n)-balanced, then we match atoms of Ω as above so that on Ω $j_n(\sigma_i)$ coincides with an element of $\tilde{\Gamma}_{n+1}$. Else, there must again exist some j such that Ω is the disjoint union of the $\mathcal{A}_{n+1}$-orbit Ω_1 of some α_j^{n+1} and the $\mathcal{A}_{n+1}$-orbit Ω_2 of β_j^{n+1}, and we again find $U_1, \ldots, U_p \sim_\Gamma \alpha_j^{n+1}$, $V_1, \ldots, V_p \sim_\Gamma \beta_j^{n+1}$ such that $p \leq N_j^n$ and $\Omega_1 \setminus \{U_1, \ldots, U_p\}$, $\Omega_2 \setminus \{V_1, \ldots, V_p\}$ each have as many atoms contained in α_i^n and β_i^n. We can pair these atoms to define $j_n(\sigma_i)$ there (and on those atoms it coincides with an element of $\tilde{\Gamma}_n$). We again set $j_n(\sigma_i)(U_r) = V_r$, $j_n(\sigma_i)(V_r) = U_r$.

This completes the definition of j_n.

Denote by l_n the number of singular atoms in $\mathcal{A}_n$. Since j_n coincides with i_n on $\tilde{\Gamma}_n$, any singular atom U of $\mathcal{A}_{n+1}$ is contained in a singular atom U' of $\mathcal{A}_n$; and there exists some $\gamma_1, \gamma_2 \in \tilde{\Gamma}_n$ and $i \in \{1, \ldots, k_n\}$ such that $\pi_n(U) = \gamma_1 j_n(\sigma_i)\gamma_2(U)$. Given that fewer than $2(N^{(n)} + 2)$ singular atoms have been created from the two singular atoms constituting the support of σ_i when defining $j_n(\sigma_i)$, a very coarse estimate gives:

$$l_{n+1} \leq l_n(N^{(n)} + 2) \leq h_n(N^{(n)} + 2)$$

Since there are more than $(n + 1)h_n(N^{(n)} + 2)$ fragments of each $\mathcal{A}_n$-orbit in every exceptional $\mathcal{A}_{n+1}$-orbit, $\mu(K_{n+1}) \leq \frac{1}{n+1}$ for all $\mu \in M(\tilde{\Gamma})$. $\square$

With these definitions in hand, let $\tilde{\Gamma}$ be the inductive limit of $(\tilde{\Gamma}_n, i_n)$ and Λ be the inductive limit of (Λ_n, j_n). Denote also by j_n the embedding of Λ_n in Λ and let $\pi = j_0(\pi_0)$. Since $\pi(\alpha_j^n) = \beta_j^n$ for all j and all n, Λ is generated as a full group by $\tilde{\Gamma}$ and π.

Then $\tilde{\Gamma}$ is a minimal ample group and $\sim_{\tilde{\Gamma}}$ coincides with $\sim_\Gamma$; Λ is a saturated minimal ample group and $\sim_\Lambda$ coincides with $\sim_\Gamma^*$ (this is straightforward to check, for details see the arguments of [MR]). We thus have $M(\Lambda) = M(\tilde{\Gamma}) = M(\Gamma)$.

Let $K = \bigcap_n K_n$. It is closed, and $\mu(K) = 0$ for every $\mu \in M(\Lambda)$. Let U be clopen in K and $\gamma \in \Lambda$. Then for some n we have both that $\lambda \in j_n(\Lambda_n)$ and that there exist singular atoms $U_1, \ldots, U_p$ of $\mathcal{A}_n$ such that $U \cap K = \bigsqcup_i(U_i \cap K)$. To prove that K is Λ-étale, we may as well assume that λ is the involution of $j(\Lambda_n)$ with support $U_i \sqcup \lambda(U_i)$ for some $i \in \{1, \ldots, p\}$. Then either $\lambda(U_i) \cap K = \emptyset$ if $\lambda(U_i)$ is not a singular atom; or $\lambda(U_i \cap K) = \lambda(U_i) \cap K$ if $\lambda(U_i)$ is singular. This proves that K is Λ-malleable (hence also $\tilde{\Gamma}$-malleable).

For any $x \notin K$ we have $\pi(x) \in \tilde{\Gamma}x$. Since Λ is generated, as a full group, by $\tilde{\Gamma}$ and π, this implies that $R_\Lambda = R_{\tilde{\Gamma}}(\tilde{\Gamma}, K)$.

We finally conclude, thanks to Theorem 5.5, that R_Γ is orbit equivalent to R_Λ; so every minimal ample group is orbit equivalent to a saturated minimal ample group and (by Krieger's theorem) this concludes the proof of the classification theorem for minimal ample groups.

Writing this appendix offers an opportunity to discuss another imprecision in [MR] (already mentioned in the main body of the paper, when we stated Lemma 2.2). In part (2) of [MR, Lemma 3.4] (which is Lemma 2.2 in the current paper), the assumption that Γ acts topologically transitively is not sufficient for the argument to go through (despite our claim that "it is the natural hypothesis to make the argument work"...). The reason is that the proof uses implicitly that for any nonempty clopen set A one has $\inf_{\mu \in M(\Gamma)} \mu(A) > 0$. That condition is in fact equivalent to assuming that Γ acts minimally on X. I do not know in which generality point (2) of [MR, Lemma 3.4] holds. Fortunately, this does not affect the arguments of [MR] that use Lemma 3.4 since we are everywhere concerned with minimal actions.

References

[GMPS1] T. Giordano, H. Matui, I. F. Putnam, and C. F. Skau, *The absorption theorem for affable equivalence relations*, Ergodic Theory Dynam. Systems **28** (2008), no. 5, 1509–1531.

[GMPS2] ———, *Orbit equivalence for Cantor minimal Z^d-systems*, Inventiones Mathematicae **179** (2010), no. 1, 119–158. MR2563761 (2011d:37013)

[GPS1] T. Giordano, I. F. Putnam, and C. F. Skau, *Topological orbit equivalence and C^*-crossed products*, J. Reine Angew. Math. **469** (1995), 51–111.

[GPS2] ———, *Affable equivalence relations and orbit structure of Cantor dynamical systems*, Ergodic Theory Dynam. Systems **24** (2004), no. 2, 441–475.

[GW] E. Glasner and B. Weiss, *Weak orbit equivalence of Cantor minimal systems*, Internat. J. Math. **6** (1995), no. 4, 559–579.

[HKY] T. Hamachi, M. S. Keane, and H. Yuasa, *Universally measure-preserving homeomorphisms of Cantor minimal systems*, J. Anal. Math. **113** (2011), 1–51.

[K] W. Krieger, *On a dimension for a class of homeomorphism groups*, Math. Ann. **252** (1979/80), no. 2, 87–95.

[M] H. Matui, *An absorption theorem for minimal AF equivalence relations on Cantor sets*, J. Math. Soc. Japan **60** (2008), no. 4, 1171–1185.

[MR] J. Melleray and S. Robert, *From invariant measures to orbit equivalence, via locally finite groups*, Ann. H. Lebesgue **6** (2023), 259–295. MR4648084

[P1] I. F. Putnam, *Orbit equivalence of Cantor minimal systems: a survey and a new proof*, Expo. Math. **28** (2010), no. 2, 101–131.

[P2] ———, *Cantor minimal systems*, University Lecture Series, vol. 70, American Mathematical Society, Providence, RI, 2018.

Université Claude Bernard – Lyon 1, Institut Camille Jordan, CNRS UMR 5208, 43 boulevard du 11 novembre 1918, 69622 Villeurbanne Cedex, France