EXTREMELY AMENABLE GROUPS VIA CONTINUOUS LOGIC

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ABSTRACT. These are research notes on applications of metric Fraïssé classes to the theory of extremely amenable Polish groups. We obtain a characterization of extreme amenability of any Polish group G in Fraïssé-theoretic terms, mirroring a theorem due to Kechris, Pestov and Todorcevic for subgroups of S_∞. At the moment we have no convincing examples that our characterization is useful.

1. INTRODUCTION

This paper deals with the theory of *extremely amenable* Polish groups, i.e Polish groups G with the following property: whenever G acts continuously on a compact metric space X, there exists $x \in X$ such that $g \cdot x = x$ for all $g \in G$.

The question of the existence of such topological groups was asked in the late sixties, and the first example was published in 1975 by Christensen and Herer. The group in question was a so-called "exotic" Polish group, and it was initially thought that extreme amenability itself was a pathological property, especially since it was noticed early on that there exists no locally compact extremely amenable topological group. Since then, however, numerous "natural" Polish groups have been proved to be extremely amenable; examples include the unitary group of an infinite-dimensional separable Hilbert space (Gromov-Milman), the automorphism group of a standard probability space (Giordano-Pestov), and the isometry group of the universal Urysohn metric space (Pestov). For a detailed discussion of the theory of extremely amenable Polish groups, as well as detailed bibliographical references, we refer the reader to Pestov's book [P1], from which the quick historical discussion above was taken.

In the seminal article [KPT], Kechris, Pestov and Todorcevic provided a characterization of extremely amenable subgroups of S_∞, the permutation group of an infinite countable set. These groups may naturally be seen as automorphism groups of countable first-order structures, and the characterization obtained in [KPT] is in terms of a combinatorial property of Fraïssé classes, the so-called *Ramsey property*. This characterization, which completely captures the combinatorial content of extreme amenability for subgroups of S_∞, has led to some new examples of extremely amenable Polish groups, and one may also hope to use this connection between topological dynamics and combinatorics to obtain some new Ramsey-type theorems. However, the characterization of [KPT] cannot be applied to general Polish groups.

In this article, we use the framework of *continuous logic*, which was introduced in [BYBHU], and a notion of Fraïssé class adapted to that context -which we call *metric Fraïssé classes* and describe in detail in the next section- to provide a characterization of extreme amenability for Polish groups in terms of an "approximate Ramsey property" (metric Fraïssé classes were first studied in [S1]).

A very similar characterization, in the more restrictive context of isometry groups of ultrahomogeneous Polish metric spaces, had been obtained earlier by Pestov in [P2]. We then describe a method that may be used to prove that a metric Fraïssé class satisfies the approximate Ramsey property; this method applies, for instance, to recover the (known) results that the isometry group of Urysohn's universal metric space and the automorphism group of a standard probability space are extremely amenable. It is our hope that this approach may lead to interesting new examples of extremely amenable Polish groups, or new Ramsey-type theorems, but so far we have not found any.

2. METRIC FRAÏSSÉ CLASSES

We refer the reader to [BYBHU] for an introduction to the language of continuous logic. In this section we describe quickly the terminology and explain how Fraïssé classes are defined in the metric context (these classes were first considered in [S1]).

Definition 2.1. By a *relational metric language* $\mathcal{L}$, we mean a set $\mathcal{L}$ whose elements are of the form (n, K) where n is an integer and K is a positive real number.

A $\mathcal{L}$-structure $\mathbf{X}$ is then the given of a *complete* metric space (X, d) (called the *universe* of $\mathbf{X}$) endowed for each $l = (n, K) \in \mathcal{L}$ with a n-ary predicate $f_l \colon X^n \to \mathbf{R}$ which is K-Lipschitz from X^n, endowed with the sup-metric, into $\mathbf{R}$.

A *Polish relational metric structure* is a metric structure in some relational language $\mathcal{L}$ whose universe is a Polish metric space, i.e a complete and separable metric space.

We always assume that our language contains a binary predicate which is 1-Lipschitz and is interpreted by the metric in all the $\mathcal{L}$-structures.

Remark. It is customary in continuous logic to ask that the metric spaces considered are all of diameter bounded by some constant (usually, 1), and that the functions interpreting the logical symbols take values in some fixed closed subinterval of the real line (usually, $[0, 1]$); the reason why this is desirable is that one may then use the compactness theorem. Since we will do no logic with our Fraïssé classes, but simply use them as a convenient setup, we do not impose any restriction on the diameters of the metric spaces we consider.

Definition 2.2. Let $\mathcal{L}$ be a countable relational metric language and $\mathbf{X}$ be a $\mathcal{L}$-structure. An *automorphism* of $\mathbf{X}$ is a bijection σ of X which preserves all the relations - i.e whenever l is in $\mathcal{L}$ and f_l is the corresponding function from $X^n \to [0, 1]$, one must have

$$\forall x_1, \ldots, x_n \in X^n \; f_l(\sigma(x_1), \ldots, \sigma(x_n)) = f(x_1, \ldots, x_n) .$$

Note that our convention that the language always contains a symbol for the metric implies that any automorphism of $\mathbf{X}$ is an isometry of its universe (X, d).

Definition 2.3. Let $\mathcal{L}$ be a countable relational metric language and $\mathbf{X}$ be a $\mathcal{L}$-structure. We say that two tuples $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_m)$ in X^m have the *same quantifier-free type* if for all $l = (n, K) \in \mathcal{L}$ such that $n \leq m$, and all $\{j_1, \ldots, j_n\} \subseteq \{1, \ldots, m\}$, one has

$$f_l(x_{j_1}, \ldots, x_{j_k}) = f_l(y_{j_1}, \ldots, y_{j_k}) .$$

For instance, if our language only consists of a symbol for the metric, saying that two tuples $\bar{x}, \bar{y}$ in (X, d) have the same quantifier-free type simply means that $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_m\}$ are isometric enumerated metric spaces, i.e the map $x_i \mapsto y_i$ is distance-preserving.

Definition 2.4. Let $\mathbf{X}$ be a Polish metric relational structure. We say that $\mathbf{X}$ is *approximately ultrahomogeneous* if for any two tuples $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ with the same quantifier-free type, and any $\varepsilon > 0$, there exists an automorphism g of $\mathbf{X}$ such that $d(g(a_i), b_i) \leq \varepsilon$ for all i.

We now move on to the definition of Fraïssé classes in the metric context.

Definition 2.5. Consider a class of finite metric $\mathcal{L}$-structures $\mathcal{K}$. We say that $\mathcal{K}$ has:

- the *hereditary property* (HP) if for any $\mathbf{B} \in \mathcal{K}$, any $\mathcal{L}$-structure embedding in $\mathbf{B}$ belongs to $\mathcal{K}$.
- the *joint embedding property* (JEP) if for any $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ there exists $\mathbf{C} \in \mathcal{K}$ such that both $\mathbf{A}$ and $\mathbf{B}$ embed in $\mathbf{C}$.
- the *near-amalgamation property* (NAP) if the following happens: for any $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$, any embeddings $\alpha_B \colon \mathbf{A} \to \mathbf{B}$ and $\alpha_C \colon \mathbf{A} \to \mathbf{C}$ and any $\varepsilon > 0$, there exists $\mathbf{D} \in \mathcal{K}$ and embeddings $\beta_B \colon \mathbf{B} \to \mathbf{D}$ and $\beta_C \colon \mathbf{C} \to \mathbf{D}$ such that

$$\forall a \in A \; d(\beta_B \circ \alpha_B(a), \beta_C \circ \alpha_C(a)) \leq \varepsilon \, .$$

  If it is possible to take $\varepsilon = 0$ in the above definition then we say that $\mathcal{K}$ has the *exact amalgamation property*.

The two first properties are stated in exactly the same way as in the usual (discrete) setting, while the near-amalgamation property is an approximate version of the usual (exact) amalgamation property. Beware, however, that the near-amalgamation property may very well be false for a class of metric structures even if it holds for its discrete counterpart: for instance, consider the class $\mathcal{K}$ made up of all finite metric spaces with a 1-Lipschitz binary predicate (besides the metric). Intuitively, this class is simply the continuous analogue of the class of finite sets endowed with a binary relation (besides the equality relation). Clearly, the discrete class amalgamates; however, one may check that its continuous version does not amalgamate, even approximately - there exist for instance two structures of cardinality 3 with a common substructure of cardinality 2 which do not near-amalgamate over that common substructure.

Now, we need to describe the analogue, in the continuous context, of a countable class of structures - it should come as no surprise that the relevant notion is that of a *separable* class of structures.

Definition 2.6. Given an integer n, we denote by $\mathcal{K}_n$ the class of *enumerated* elements of $\mathcal{K}$ of cardinality less than n. We define a pseudo-metric d_n on $\mathcal{K}$ as follows: fix enumerated structures $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ with at most n elements. For any structure $\mathbf{C} \in \mathcal{K}$ and any embeddings $\alpha \colon \mathbf{A} \to \mathbf{C}$ and $\beta \colon \mathbf{B} \to \mathbf{C}$ with $\alpha(A) = \{a_1, \ldots, a_n\}$ and $\beta(B) = \{b_1, \ldots, b_n\}$ one may consider the quantity

$$d_n(\mathbf{C}, \alpha, \beta) = \max\{d(a_i, b_i) \colon i \in \{1, \ldots, n\}\} \, .$$

We then let $d_n(\mathbf{A}, \mathbf{B})$ denote the infimum of the quantities $d_n(\mathbf{C}, \alpha, \beta)$ where $\mathbf{C}$ ranges over $\mathcal{K}$ and α, β range over all possible embeddings.

It is straightforward to check that d_n is a pseudo-metric on $\mathcal{K}_n$ as soon as $\mathcal{K}$ has both the JEP and the NAP, and that $d_n(\mathbf{A}, \mathbf{B}) = 0$ if and only if $\mathbf{A}, \mathbf{B}$ are isomorphic (as enumerated structures, of course).

Definition 2.7. Let $\mathcal{K}$ be a class of finite $\mathcal{L}$-structures. We say that $\mathcal{K}$ is a *(complete) metric Fraïssé class* if it has the following properties:

- $\mathcal{K}$ has the HP, the JEP and the NAP.
- For all n d_n is separable and complete.

Examples (besides the discrete Fraïssé classes) include, for instance, the class of all finite metric spaces, the class of finite euclidean metric spaces, etc.

The following is the continuous version of one of the famous Fraïssé theorems, and is proved similarly by using (near) amalgamation and the back-and-forth method.

Proposition 2.8. *Given a metric Fraïssé class $\mathcal{K}$ in a countable relational metric language $\mathcal{L}$, there exists a (unique, up to isomorphism) Polish metric $\mathcal{L}$-structure $\mathbf{X}$ which is approximately ultrahomogeneous and such that the finite substructures of $\mathbf{X}$ are exactly the elements of $\mathcal{K}$. $\mathbf{X}$ is called the* Fraïssé limit *of $\mathcal{K}$.*

For instance, the Fraïssé limit of the class of finite metric spaces is Urysohn's universal metric space, The Fraïssé limit of the class of metric spaces of diameter at most d is the Urysohn space of diameter d, the limit of the class of finite euclidean spaces is the (real) separable Hilbert space, etc. All these structures are ultrahomogeneous in a strong sense, i.e given any two finite tuples with the same quantifier-free type one may map one onto the other (and not merely as close to the other as desired); in the general theory of metric Fraïssé classes the ε's seem to be needed, especially if one wants to have something like Proposition 2.10 below.

Proposition 2.9. *Any approximately ultrahomogeneous Polish metric structure in a countable relational metric language is the Fraïssé limit of its finite substructures.*

The relevance of the above considerations to the theory of Polish groups comes from the following observation (see [M], Theorem 6): any Polish group is isomorphic to the automorphism group of some approximately ultrahomogeneous Polish metric structure - i.e, given what we saw above, of a Fraïssé limit.

To give a bit more detail, let $\mathbf{X}$ be a Polish relational metric structure with universe (X, d); we endow its automorphism group G with the pointwise convergence topology, i.e the topology which has a basis of neighborhoods of id given by

$$\{g \in G \colon \forall a \in A \ d(g(a), a) < \varepsilon\}$$

where $\varepsilon > 0$ and A is a finite subset of X. This turns G into a closed subgroup of the Polish group $\mathrm{Iso}(X)$, and so, when endowed with this topology, G is a Polish group in its own right.

Whenever we consider the automorphism group of a Polish relational metric structure $\mathbf{X}$ as a Polish group, we endow it with the topology described above (which we call the *natural* Polish topology on $\mathrm{Aut}(\mathbf{X})$).

Let us conclude this crash-course on metric Fraïssé classes by reformulating Theorem 6 of [M] using the language of Fraïssé limits.

Proposition 2.10. *Let G be a Polish group. Then there exists a countable relational language $\mathcal{L}$, and a Fraïssé metric class $\mathcal{K}$ in the language $\mathcal{L}$, such that G is (isomorphic to, as a topological group) the automorphism group of the Fraïssé limit of $\mathcal{K}$.*

In theory, this means that whenever we state a general result about automorphism groups of metric Fraïssé limits, we are talking about all Polish groups. For practical purposes, this is far from true - for example, we know of no way to turn the homeomorphism group of the Hilbert cube into the automorphism group of a Fraïssé limit in such a way that combinatorial properties of the corresponding Fraïssé class may be understood.

3. Approximate Ramsey property

In this section, $\mathcal{K}$ is a metric Fraïssé class with limit $\mathbf{K}$. We let G denote the automorphism group of $\mathbf{K}$, endowed with its natural Polish topology. To give some intuition of what is going on, we will use the same vocabulary as in the discrete setting (most notably, we will speak of colorings); part of what follows in this section was already done by Pestov in [P2], in the context of ultrahomogeneous metric spaces.

Definition 3.1. Given $\mathbf{A}, \mathbf{B} \in \mathcal{K}$, we denote by ${}^{\mathbf{A}}\mathbf{B}$ the set of embeddings of $\mathbf{A}$ in $\mathbf{B}$. We endow it with the metric ρ_A defined by

$$\rho_A(\alpha, \beta) = \sup\{d(\alpha(a), \beta(a)) \colon a \in A\}$$

A *coloring* of ${}^{\mathbf{A}}\mathbf{B}$ is a 1-Lipschitz map $\gamma \colon ({}^{\mathbf{A}}\mathbf{B}, \rho_A) \to [0,1]$.

We define similarly ${}^{\mathbf{A}}\mathbf{K}$, a metric (still denoted by ρ_A) on ${}^{\mathbf{A}}\mathbf{K}$, and colorings of ${}^{\mathbf{A}}\mathbf{K}$.

Definition 3.2. Let $\mathbf{A} \leq \mathbf{B} \in \mathcal{K}$, $\mathbf{C} \in \mathcal{K}$ and $\beta \in {}^{\mathbf{B}}\mathbf{C}$. We denote by ${}^{\mathbf{A}}\mathbf{C}(\beta)$ the subset of ${}^{\mathbf{A}}\mathbf{C}$ made up of all α that factor through β, i.e

$$\alpha \in {}^{\mathbf{A}}\mathbf{C}(\beta) \;\Leftrightarrow\; \exists \delta \in {}^{\mathbf{A}}\mathbf{B} \quad \alpha = \beta \circ \delta \,.$$

Similarly, if $\beta \in {}^{\mathbf{B}}\mathbf{K}$, we denote by ${}^{\mathbf{A}}\mathbf{K}(\beta)$ the subset of ${}^{\mathbf{A}}\mathbf{K}$ made up of all α that factor through β.

Definition 3.3. We say that $\mathcal{K}$ has the *approximate Ramsey property* if the following happens: for all $\mathbf{A} \leq \mathbf{B} \in \mathcal{K}$, and for all $\varepsilon > 0$, there exists $\mathbf{C} \in \mathcal{K}$ such that for any coloring γ of ${}^{\mathbf{A}}\mathbf{C}$ there exists $\beta \in {}^{\mathbf{B}}\mathbf{C}$ such that the oscillation of γ restricted to ${}^{\mathbf{A}}\mathbf{C}(\beta)$ is less that ε.

Let us point out that, for us, the empty function on the empty set is 1-Lipschitz, hence the assumptions above imply that ${}^{\mathbf{B}}\mathbf{C}$ is nonempty.

Proposition 3.4. *The following properties are equivalent:*

(i) *$\mathcal{K}$ has the approximate Ramsey property.*
(ii) *For any finite substructures $\mathbf{A} \leq \mathbf{B}$ of $\mathbf{K}$, for any $\varepsilon > 0$ and any coloring γ of ${}^{\mathbf{A}}\mathbf{K}$, there exists $\beta \in {}^{\mathbf{B}}\mathbf{K}$ such that the oscillation of γ on ${}^{\mathbf{A}}\mathbf{K}(\beta)$ is less than ε.*

Proof. $(i) \Rightarrow (ii)$: Pick finite substructures $\mathbf{A} \leq \mathbf{B} \leq \mathbf{K}$, and fix $\varepsilon > 0$. We may find $\mathbf{C} \in \mathbf{K}$ witnessing the approximate Ramsey property and assume that $\mathbf{C} \leq \mathbf{K}$; since any coloring of ${}^{\mathbf{A}}\mathbf{K}$ restricts to a coloring of ${}^{\mathbf{A}}\mathbf{C}$, it is clear that $\mathbf{C}$ has the desired property.

$(ii) \Rightarrow (i)$: We proceed by contraposition, and assume that (i) is false. Thus, we may find $\mathbf{A} \leq \mathbf{B} \in \mathcal{K}$ and $\varepsilon > 0$ such that for any $\mathbf{C} \in \mathcal{K}$ there exists a bad coloring γ of ${}^{\mathbf{A}}\mathbf{C}$, i.e for any $\beta \in {}^{\mathbf{B}}\mathbf{C}$ the oscillation of γ on ${}^{\mathbf{A}}\mathbf{C}(\beta)$ is larger than ε (note that if there is no embedding from $\mathbf{B}$ into $\mathbf{C}$ the preceding property holds for any γ). For any finite substructure C of $\mathbf{K}$, we pick such a bad coloring γ_C of ${}^{\mathbf{A}}\mathbf{C}$.

We fix an ultrafilter $\mathcal{U}$ on the set of finite subsets of $\mathbf{K}$ with the property that for any finite subset D of $\mathbf{K}$ one has

$$\{E \subseteq \mathbf{K} \colon E \text{ is finite and } D \subseteq E\} \in \mathcal{U} .$$

We define a mapping $\gamma \colon {}^{\mathbf{A}}\mathbf{K} \to [0,1]$ by setting $\gamma = \lim_{\mathcal{U}} \gamma_C$. In other words, for any $\alpha \in {}^{\mathbf{A}}\mathbf{K}$, one has

$$\gamma(\alpha) = t \Leftrightarrow \forall \delta > 0 \; \{C \colon \gamma_C(\alpha) \in [t-\delta, t+\delta]\} \in \mathcal{U} .$$

Note that the set $\{C \colon \alpha(A) \subseteq C\}$ belongs to $\mathcal{U}$ so the above definition makes sense (for any such C, $\gamma_C(\alpha)$ is well-defined, and the ultrafilter thinks that the set made up of all the other C's is negligible). Also, since each γ_C is 1-Lipschitz it is immediate that γ is also 1-Lipschitz, in other words γ is a coloring of ${}^{\mathbf{A}}\mathbf{K}$. We simply have to check that γ witnesses the fact that $\mathbf{K}$ fails to have property (ii).

To that end, pick $\beta \in {}^{\mathbf{B}}\mathbf{K}$. Then $U_\beta = \{D \colon \beta(B) \subseteq D\}$ belongs to $\mathcal{U}$. Denoting by $\alpha_1, \ldots, \alpha_n$ the elements of ${}^{\mathbf{A}}\mathbf{K}(\beta)$ (which are also the elements of ${}^{\mathbf{A}}\mathbf{D}(\beta)$ for any $D \in U_\beta$), we know that for any $D \in U_\beta$ there exist $i, j \in \{1, \ldots, n\}$ such that

$$\left|\gamma_D(\alpha_i) - \gamma_D(\alpha_j)\right| \geq \varepsilon .$$

Hence there exist $i, j \in \{1, \ldots, n\}$ such that

$$\left\{D \colon \left|\gamma_D(\alpha_i) - \gamma_D(\alpha_j)\right| \geq \varepsilon\right\} \in \mathcal{U} .$$

Fixing such a pair i, j, we obtain that $|\gamma(\alpha_i) - \gamma(\alpha_j)| \geq \varepsilon$, so the oscillation of γ on ${}^{\mathbf{A}}\mathbf{K}(\beta)$ is larger than ε. Since β was arbitrary, this shows that $\mathbf{K}$ does not have property (ii).

□

Definition 3.5. Let $\mathbf{A}, \mathbf{B}$ be in $\mathcal{K}$, and $\alpha \in {}^{\mathbf{A}}\mathbf{B}$. Let also $B' \in \mathcal{K}$ and $\alpha' \in {}^{\mathbf{A}}\mathbf{B}'$, and fix $\varepsilon > 0$.

We say that (B', α') *ε-approximates* (B, α) if there exists $\mathbf{C} \in \mathcal{K}$ and embeddings $\beta \in {}^{\mathbf{B}}\mathbf{C}$ and $\beta' \in {}^{\mathbf{B}'}\mathbf{C}$ such that the following properties hold:

- $\forall a \in A \; d(\beta \circ \alpha(a), \beta' \circ \alpha'(a)) \leq \varepsilon$
- For any partial automorphism of $\mathbf{B}$ with domain $\alpha(A)$, there exists a partial automorphism g' of B' with domain $\alpha'(A)$ such that

$$\forall a \in A \; d(\beta \circ g \circ \alpha(a), \beta' \circ g' \circ \alpha'(a)) \leq \varepsilon .$$

(we then say that g' ε-approximates g).

Note that, if $\mathbf{A} \leq \mathbf{B}$ are finite substructures of $\mathbf{K}$, and if $(\mathbf{B}', \alpha')$ ε-approximates $(\mathbf{B}, id_{|A})$, then we may apply the universality and approximate ultrahomogeneity of $\mathbf{K}$ to find for any $\delta > 0$ some $\mathbf{B}'' \leq \mathbf{K}$ isomorphic to $\mathbf{B}'$ and an embedding $\alpha'' \in {}^{\mathbf{A}}\mathbf{B}''$ such that:

- (B'', α'') $(\varepsilon + \delta)$-approximates $(B, id_{|A})$;
- $d(a, \alpha''(a)) \leq \varepsilon + \delta$ for all $a \in A$.

In applications we may take $\delta = \varepsilon$, the important point being that we can control the error incurred when moving (inside **K**) from **B** to an approximating substructure.

Definition 3.6. We say that $\mathcal{K}$ has the *weak approximate Ramsey property* (WARP) if for any $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ and any $\alpha \in {}^{\mathbf{A}}\mathbf{B}$ there exists $B' \in \mathcal{K}$ and $\alpha' \in {}^{\mathbf{A}}\mathbf{B}'$ such that:

- (B', α') ε-approximates (B, α); we denote by $g_1, \dots, g_n$ the partial automorphisms of $\mathbf{B}$ with domain $\alpha(A)$, and let $g'_1, \dots, g'_n$ denote partial automorphisms of $\mathbf{B}'$ with domain $\alpha'(A)$ which approximate $g_1, \dots, g_n$.
- there exists $\mathbf{C} \in \mathcal{K}$ such that for any coloring γ of ${}^{\mathbf{A}}\mathbf{C}$ there exists $\beta \in {}^{\mathbf{B}'}\mathbf{C}$ such that the oscillation of γ on $\{\beta \circ g'_1 \circ \alpha', \dots, \beta \circ g'_n \circ \alpha'\}$ is less than ε.

The above definition is somewhat messy - what it means intuitively is the following: we relax the condition defining the approximate Ramsey property by allowing us to move B around (and possibly enlarge it or make it smaller), in such a way that we keep track of the original embeddings from **A** into **B**, before looking for a structure **C** witnessing the approximate Ramsey property.

We are now almost ready to state, and prove, the main result of this section, which is the characterization of extreme amenability of G in terms of the approximate Ramsey property for $\mathcal{K}$ (extending Pestov's results from [P2] to the context of metric Fraïssé classes). Before this, we need to recall a criterion for extreme amenability, and set our notations.

Definition 3.7. If A is a finite subset of **K**, we let d_A denote the pseudometric on G defined by

$$\forall g, h \in G \; d_A(g, h) = \max\{d(g(a), h(a)) \colon a \in A\} \,.$$

Slightly abusing notation, we will still denote by (G, d_A) the metric space obtained by identyfing elements g, h such that $d_A(g, h) = 0$.

The pseudometric d_A is obviously related to the metric ρ_A on ${}^{\mathbf{A}}\mathbf{K}$ - actually, it is almost the same thing, as witnessed by the following lemma.

Lemma 3.8. *Let* $\mathbf{A}$ *be a finite substructure of* $\mathbf{K}$*, and denote by* $\Phi_A \colon G \to {}^{\mathbf{A}}\mathbf{K}$ *the mapping defined by* $\Phi_A(g) = g_{|A}$*. Then* Φ_A *is a distance-preserving map from* (G, d_A) *into* $({}^{\mathbf{A}}\mathbf{K}, \rho_A)$ *and* $\Phi_A(G)$ *is dense in* ${}^{\mathbf{A}}\mathbf{K}$*.*

Proof. The first part of the statement is obvious from the definitions of d_A, ρ_A. Since embeddings preserve the quantifier-free type, the fact that $\Phi_A(G)$ is dense in ${}^{\mathbf{A}}\mathbf{K}$ is equivalent to saying that **K** is approximately ultrahomogeneous, which is true since **K** is the Fraïssé limit of $\mathcal{K}$. □

In particular, any 1-Lipschitz map $f \colon (G, d_A) \to [0, 1]$ uniquely extends to a coloring γ_f of ${}^{\mathbf{A}}\mathbf{K}$ (by identifying (G, d_A) and $\Phi_A(G)$), while any coloring γ of ${}^{\mathbf{A}}\mathbf{K}$ restricts to a 1-Lipschitz map $f_\gamma \colon (G, d_A) \to [0, 1]$. One may as well consider that both notions are the same, but to avoid possible confusions we decided to use different notations (f, γ) for the two notions.

Finally, we state a criterion of extreme amenability for a Polish group; this is essentially Theorem 2.1.11 in Pestov's book [P1], reformulated to fit the fact that we consider G as the automorphism group of a Fraïssé limit, and so we have a natural directed collection of left-invariant pseudometrics defining the topology of G: the metrics d_A we introduced above, where A ranges over all finite subsets of the Fraïssé limit.

Proposition 3.9. *G is extremely amenable if, and only if, for any finite subset A of K the left-translation action of G on (G, d_A) is finitely oscillation stable, i.e:*

For any finite subset F of G, any $\varepsilon > 0$ and any 1-Lipschitz map $f \colon (G, d_A) \to [0,1]$, there exists $g \in G$ such that the oscillation of f on gF is less than ε.

Theorem 3.10. *The following are equivalent, for a Fraïssé metric class $\mathcal{K}$ and G the automorphism group of its limit, endowed with its natural Polish topology:*

(i) G is extremely amenable.
(ii) $\mathcal{K}$ has the approximate Ramsey property.
(iii) $\mathcal{K}$ has the weak approximate Ramsey property.

Proof. $(i) \Rightarrow (ii)$: We assume that G is extremely amenable, and we want to show that $\mathbf{K}$ satisfies condition (ii) of Proposition 3.4. To that end, we fix finite substructures $\mathbf{A} \leq \mathbf{B}$ of $\mathbf{K}$ and $\varepsilon > 0$, and consider a coloring γ of ${}^{\mathbf{A}}\mathbf{K}$. We also let $\alpha_1, \ldots, \alpha_n$ enumerate the elements of ${}^{\mathbf{A}}\mathbf{B}$, and use the approximate ultrahomogeneity of $\mathbf{K}$ to find $g_1, \ldots, g_n \in G$ such that $\rho_A(\alpha_i, g_{i|A}) \leq \varepsilon$.

We set $F = \{g_1, \ldots, g_n\}$, and consider the 1-Lipschitz function $f_\gamma \colon (G, d_A) \to [0,1]$ induced by γ. By Proposition 3.9, we know that there exists $g \in G$ such that the oscillation of f_γ on gF is less than ε.

Note that we have for all $i \in \{1, \ldots, n\}$ that $\rho_A(gg_{i|A}, g\alpha_i) = \rho_A(g_{i|A}, \alpha_i) \leq \varepsilon$; the triangle inequality implies that, for any i, j:

$$\left|\gamma(g\alpha_i) - \gamma(g\alpha_j)\right| \leq \left|\gamma(g\alpha_i) - \gamma(gg_{i|A})\right| + \left|\gamma(gg_{i|A}) - \gamma(gg_{j|A})\right| + \left|\gamma(gg_{j|A}) - \gamma(g\alpha_j)\right|$$

The first and third term above are both smaller than ε since γ is 1-Lipschitz, and the term in the middle is $|f_\gamma(gg_i) - f_\gamma(gg_j)|$, which is less than ε by definition of g. Denoting by β the restriction of g to B, we just proved that the oscillation of γ on ${}^{\mathbf{A}}\mathbf{K}(\beta)$ is less than 3ε, and this proves that $\mathcal{K}$ has the approximate Ramsey property.

$(ii) \Rightarrow (iii)$ is obvious.

$(iii) \Rightarrow (i)$: assume that $\mathcal{K}$ has the WARP, let $A \subset K$ be finite, $f \colon (G, d_A) \to [0,1]$ be a 1-Lipschitz map, and fix $\varepsilon > 0$. Recall that γ_f denotes the coloring of ${}^{\mathbf{A}}\mathbf{K}$ induced by f. We fix $g_1, \ldots, g_n \in G$ and assume without loss of generality that $g_1 = id$. We want to find some $g \in G$ such that the oscillation of f on $\{gg_1, \ldots, gg_n\}$ is less than ε.

We define

$$B = \bigcup_{i=1}^{n} g_i(A) .$$

Using the WARP, and the remark following Definition 3.5 (with $\delta = \varepsilon$), we may find $\mathbf{B}' \leq \mathbf{K}$ and embeddings $\alpha_1, \ldots, \alpha_n \colon \mathbf{A} \to \mathbf{B}'$ such that:

- $\forall a \in A \; d(\alpha_i(a), g_i(a)) \leq 2\varepsilon$;
- There exists a finite $\mathbf{C} \in \mathcal{K}$ such that for any coloring γ of ${}^{\mathbf{A}}\mathbf{C}$ there exists $\beta \in {}^{\mathbf{B}'}\mathbf{C}$ such that the oscillation of γ on $\{\beta \circ \alpha_1, \ldots, \beta \circ \alpha_n\}$ is less than ε.

We may, and do, assume that $\mathbf{C} \leq \mathbf{K}$, and apply the above property to γ_f; this yields an embedding $\beta \in {}^{\mathbf{B}'}\mathbf{C}$ such that γ_f has oscillation less than ε on $\{\beta \circ \alpha_1, \dots, \beta \circ \alpha_n\}$. Using the approximate ultrahomogeneity of $\mathbf{K}$, we may find $g_\beta \in G$ such that for all $b' \in B'$ one has $d(g_\beta(b'), \beta(b')) \leq \varepsilon$.

Using the triangle inequality as in the proof of $(i) \Rightarrow (ii)$, we obtain, using straightforward computations, that for any $i, j \in \{1, \dots, n\}$ one has

$$\left| f(g_\beta g_i) - f(g_\beta g_j) \right| \leq 7\varepsilon \,.$$

Since ε was arbitrary, this is enough to show that G is extremely amenable. □

A very similar statement had been obtained by Pestov in [P2] in the case of ultrahomogeneous metric spaces. He was of course not using the vocabulary of metric Fraïssé classes, so the characterization he obtained was similar to point (ii) in our Proposition 3.4 characterizing the approximate Ramsey property.

We should probably point out here that there are (usually) many ways to turn a given Polish group G into the automorphism group of the Fraïssé limit of some class $\mathcal{K}$; the properties of the class $\mathcal{K}$ may vary depending on the construction chosen (for instance, $\mathcal{K}$ may some times have the exact amalgamation property and sometimes only the near-amalgamation property - this happens e.g whenever G is a subgroup of the permutation group of the integers whose topology does not admit a compatible complete left-invariant metric), however the theorem above implies that the approximate Ramsey property does not depend on the particular class $\mathcal{K}$, but only on whether G is extremely amenable or not. This is not surprising, as it reflects a similar phenomenon uncovered for discrete Fraïssé classes in [KPT].

An application. It is well-known that the orthogonal group of a separable, infinite-dimensional (real) Hilbert space is extremely amenable, from which we deduce that the isometry group of the unit sphere of the Hilbert space is extremely amenable (these groups are one and the same). Since this unit sphere is the Fraïssé limit of the class of finite spherical metric spaces of diameter at most 2, we obtain the approximate Ramsey property for this class. In turn, this easily implies the ARP for the class of all spherical metric spaces and, going back to the group side, we thus obtain that the isometry group of the unbounded universal spherical metric space is extremely amenable (see [P1], exercise 5.1.32 p112 for the definition of this space, and Blumenthal [B] for informations about spherical metric spaces).

4. Properties of a metric Fraïssé class that imply the approximate Ramsey property

We again denote by $\mathcal{K}$ a metric Fraïssé class, by $\mathbf{K}$ its Fraïssé limit, and by G the automorphism group of $\mathbf{K}$.

Definition 4.1. We say that $\mathcal{K}$ has the *extension property* if for any $\mathbf{A} \in \mathcal{K}$ there is $\mathbf{B} \in \mathcal{K}$ such that $\mathbf{A}$ embeds in $\mathbf{B}$ and any partial automorphism of $\mathbf{A}$ extends to an automorphism of $\mathbf{B}$.

Using a deep theorem due to Hervig and Lascar [HL], Solecki proved in [S2] that the class of finite metric spaces has the extension property.

In general, establishing that a class has this property seems very difficult. We will make use of a weaker form of this property to show that certain classes have the approximate Ramsey property.

Definition 4.2. We say that $\mathcal{K}$ has the *weak extension property* if for any $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ and any embedding $\alpha \in {}^{\mathbf{A}}\mathbf{B}$ there exists $B' \in \mathcal{K}$ and $\alpha' \in {}^{\mathbf{A}}\mathbf{B}'$ such that:

- (B', α') ε-approximates (B, α)
- For any partial automorphism g of $\mathbf{B}$ with domain $\alpha(A)$, there exists an automorphism g' of $\mathbf{B}'$ whose restriction to $\alpha'(A)$ ε-approximates g.

Intuitively, the meaning of this property is the following: up to perturbing $\mathbf{B}$ slightly, and possibly enlarge it a lot, we may approximate partial automorphisms of $\mathbf{B}$ with domain some fixed substructure of $\mathbf{B}$ by *global* automorphisms of $\mathbf{B}$. Of course, the weak extension property above is much weaker (at least formally) than the extension property.

Definition 4.3. Whenever (A, d) is a finite metric space, we let d_n denote the normalized ℓ_1-metric on A^n, defined by

$$d_n\left((a_1, \ldots, a_n), (a'_1, \ldots, a'_n)\right) = \frac{1}{n} \sum_{j=1}^{n} d(a_i, a'_i) \ .$$

Definition 4.4. We say that $\mathcal{K}$ is a ℓ_1 *metric Fraïssé class* (or that $\mathcal{K}$ *has the* ℓ_1 *property*) if for all $\mathbf{A} \in \mathcal{K}$ and all n one may turn (A^n, d_n) into an element $\mathbf{A}^n$ belonging to $\mathcal{K}$ in such a way that, whenever $(g_1, \ldots, g_n)$ are automorphisms of $\mathbf{A}$, the diagonal map

$$a \mapsto (g_1(a), \ldots, g_n(a))$$

is an embedding of $\mathbf{A}$ into $\mathbf{A}^n$.

For example, the class of all finite metric spaces is a metric Fraïssé class with both the ℓ_1 property and the extension property. Another example of such a class is given by the class $\mathcal{K}_X$ of finite metric spaces containing a common finite metric space X. A non-example would be, of course, the class of euclidean metric spaces.

The main result of this section is the following.

Theorem 4.5. *Let $\mathcal{K}$ be a ℓ_1 metric Fraïssé class with the weak extension property. Then $\mathcal{K}$ has the weak approximate Ramsey property (hence also the approximate Ramsey property), so $G = Aut(Flim(\mathcal{K}))$ is extremely amenable.*

Note that this enables one to recover the fact that the isometry group of the Urysohn space $\mathbf{U}$ is extremely amenable. Also, for X a finite metric space, the limit of $\mathcal{K}_X$ is $\mathbf{U}$ endowed with a distinguished copy of X, whose automorphism group is exactly the (pointwise) stabilizer of X. So we see that the group of isometries that fix X pointwise is extremely amenable for any finite subset X of $\mathbf{U}$. Actually, in both of these cases, Pestov's ideas, as reformulated in Theorem 4.6 below, imply that the groups are Lévy.

Proof. We consider a ℓ_1 metric Fraïssé class $\mathcal{K}$ with the weak extension property, and aim at proving that $\mathcal{K}$ has the WARP. Consider finite structures $\mathbf{A} \leq \mathbf{B} \in \mathcal{K}$ and $\varepsilon > 0$.

Let $\alpha_1, \ldots, \alpha_k$ enumerate the partial automorphisms of $\mathbf{B}$ with domain A and assume that $\alpha_1 = id_{|A}$. Applying the weak extension property, we know that we may find some $\mathbf{B}' \in \mathcal{K}$, an embedding $\alpha' \in {}^{\mathbf{A}}\mathbf{B}'$ and automorphisms $g_1, \dots, g_k$ of B' such that the restriction of each g_i to $\alpha'(A)$ ε-approximates α_i. Note that if $\varepsilon > 0$ is small enough, we may assume that $g_i \neq g_j$ for all $i \neq j$, which we do below.

We let p denote the cardinality of the subgroup H of $\mathrm{Aut}(B')$ generated by $g_1, \dots, g_k$. We extend the list $\{g_1, \dots, g_k\}$ to an enumeration $\{g_1, \dots, g_p\}$ of H.

For any n, we turn $((B')^n, d_n)$ into an element of $\mathcal{K}$ in a way that witnesses the ℓ_1-property, and call this structure $\mathbf{C}_n$. We want to show that for n big enough, for any coloring γ of ${}^{\mathbf{A}}\mathbf{C}_n$, there is $\beta \in {}^{\mathbf{B}'}\mathbf{C}_n$ such that the oscillation of γ on $\{\beta \circ g_{1|A}, \dots, \beta \circ g_{k|A}\}$ is less than ε. To see this, we will use concentration of measure - this is where the ℓ_1 property is useful.

For any integer n, we may consider the set $H_n = \{1, \dots, p\}^n$ and endow it with the normalized Hamming metric d_H, defined by

$$d_H(w, w') = \frac{1}{n} \left| \{i \colon w(i) \neq w'(i)\} \right| .$$

Given a coloring γ of ${}^{\mathbf{A}}\mathbf{C}_n$, we define a mapping $\gamma' \colon H_n \to [0,1]$ as follows: if $w = (i_1, \dots, i_n)$, we set

$$\gamma'(w) = \gamma((g_{i_1}, \dots, g_{i_p}) \circ \alpha') .$$

Let δ denote the diameter of B. Since γ is 1-Lipschitz, it is straightforward to check from the definition of ρ_A that γ' as defined above is a δ-Lipschitz map from (H_n, d_H) to $[0,1]$.

Now, endow H_n with the normalized counting measure μ_n, and recall that the concentration of measure phenomenon for products endowed with the ℓ_1-metric (see Theorem 4.3.19 in [P1], or Theorem 4.2 in [L]) ensures that for n big enough we have, for any δ-Lipschitz function $f \colon (H_n, d_H) \to [0,1]$, that

$$(*) \qquad \mu_n\left(\{w \colon\ |f(w) - E(f)| \leq \varepsilon\}\right) > 1 - \frac{1}{k} .$$

($E(f)$ denotes the expected value of f, i.e $\int f d\mu_n$)

Find such an n. We claim that C_n has the desired property. To see this, we first define for all $i \in \{1, \dots, k\}$ a bijection Θ_i from H_n to itself, as follows: given $w = (i_1, \dots, i_n)$, we look at the embedding $(g_{i_1}, \dots, g_{i_n}) \circ g_i \circ \alpha' = (g_{j_1}, \dots, g_{j_n}) \circ \alpha'$ and we set $\Theta_i(w) = (j_1, \dots, j_n)$.

Now, fix a coloring γ of ${}^{\mathbf{A}}\mathbf{C}_n$ and let γ' denote the corresponding mapping from H_n to $[0,1]$. Since each Θ_i preserves the measure μ_n, and γ' is δ-Lipschitz, we may apply $(*)$ to see that there exists some $w = (i_1, \dots, i_n) \in K_n$ such that

$$\forall i \in \{1, \dots, k\} \ \left|\gamma'(\Theta_i(w)) - E(\gamma')\right| \leq \varepsilon$$

From this, it follows that

$$\forall i, j \in \{1, \dots, k\} \left|\gamma'(\Theta_i(w)) - \gamma'(\Theta_j(w))\right| \leq 2\varepsilon .$$

Going back to the definition of γ', we have just obtained

$$\forall i, j \in \{1, \dots, k\} \ \left|\gamma((g_{i_1}, \dots, g_{i_n}) \circ g_i \circ \alpha') - \gamma((g_{i_1}, \dots, g_{i_n}) \circ g_j \circ \alpha')\right| \leq 2\varepsilon .$$

In other words, setting $\beta = (g_{i_1}, \dots, g_{i_n})$, we have shown that $\mathcal{K}$ has the WARP.

□

Let us conclude this section by stating a criterion for a Polish group to be Lévy - this consists simply in isolating the ideas used by Pestov in [P1] to show that the isometry group of Urysohn's universal metric space is a Lévy group. The proof is a straightforward adaptation of Pestov's proof as presented in [P1], so we content ourselves with stating the criterion below without proof.

Theorem 4.6. *Let $\mathcal{K}$ be a metric Fraïssé class with limit* $\mathbf{K}$*. Assume that:*

- *$\mathcal{K}$ has the extension property.*
- *$\mathcal{K}$ has the ℓ_1 property.*
- *For any finite substructure $\mathbf{A}$ of $\mathbf{K}$, and any finite group H acting on $\mathbf{A}$ by automorphisms, the action $H \curvearrowright \mathbf{A}$ extends to an action $H \curvearrowright \mathbf{K}$ by automorphisms.*

Then $G = Aut(\mathbf{K})$, endowed with its usual Polish topology, is a Lévy group.

Actually, there is a increasing chain of finite *subgroups which concentrates (for the normalized counting measure) and whose union is dense in G.*

5. An example

We work out an example in detail, and then present another possible example where we do not know whether the weak extension property holds or not.

5.1. **Metric spaces with a unary 1-Lipschitz predicate.** We consider the class $\mathcal{K}^{lip}$ obtained by considering all metric spaces (X, d) endowed with a 1-Lipschitz function $f: X \to \mathbf{R}$. Actually, everything we say below is also true if we consider metric spaces endowed with any (fixed) number of 1-Lipschitz unary predicates, but we're simply trying to describe an example so we'll stick with the simplest setup possible.

It is easy to check that $\mathcal{K}^{lip}$ is hereditary; the amalgamation property is also straightforward: take two structures (B, d, f_B) and (C, d, f_C) with a common substructure (A, d, f_A). Form the metric amalgam D of B, C over A; recall that this space is obtained by identifying the copies of A in B, C and setting

$$\forall b \in B \, \forall c \in C \; d(b, c) = \min\{d(b, a) + d(a, c) \colon a \in A\} \, .$$

Then extend f_B, f_C to a function f_D by setting $f_D(b) = f_B(b)$ for all $b \in B$ and $f_D(c) = f_C(c)$ for all $c \in C$ (note that, of course, both definitions coincide on $B \cap C = A$). It is clear that the restrictions of f_D to B and C are 1-Lipschitz; so we simply have to check that $|f_D(b) - f_D(c)| \leq d(b, c)$ for all $b \in B \setminus A$ and all $c \in C \setminus A$. Pick such a b and c, and find a such that

$$d(b, c) = d(b, a) + d(a, c) \, .$$

Then we have

$$|f_D(b) - f_D(c)| \leq |f_D(b) - f_D(a)| + |f_D(a) - f_D(c)| \leq d(b, a) + d(a, c) = d(b, c) \, .$$

The joint embedding property works similarly.

When it comes to separability and completeness, one may for instance identify $\mathcal{K}^{lip}_n$ with a closed subspace Z of $M_n(\mathbf{R}) \times \mathbf{R}^n$: any element of $\mathcal{K}^{lip}_n$ is just an enumerated metric space with at most n points (encoded by a symmetric $n \times n$ square matrix) along with a function (encoded by n reals); saying that the matrix encodes a metric space, and the vector in $\mathbf{R}^n$ encodes a Lipschitz function on that metric space, clearly are closed conditions.

Thus, the class $\mathcal{K}^{lip}$ is indeed a Fraïssé metric class, and its limit is Urysohn's universal metric space $\mathbf{U}$ endowed with a certain 1-Lipschitz real-valued function f. This function is *generic* in the following sense: denote by $\mathrm{Lip}_1(\mathbf{U})$ the set of Lipschitz-functions on $\mathbf{U}$, and endow it with the Polish topology of pointwise convergence. Then $\mathrm{Iso}(\mathbf{U})$ acts continuously on $\mathrm{Lip}_1(\mathbf{U})$ by pre-composition ($\varphi.f(x) = f(\varphi^{-1}(x))$), and the orbit of f under this action is a comeagre set. Note that any two such functions belong to the same orbit under the action of $\mathrm{Iso}(\mathbf{U})$.

One may check that a 1-Lipschitz function f on $\mathbf{U}$ is generic in the above sense if, and only if, f has the following property: for any finite subset A of $\mathbf{U}$ and any 1-point extension $(A \cup \{z\}, d, g)$ in $\mathcal{K}^{lip}$ of (A, d, f) there exists $z \in \mathbf{U}$ realizing this 1-point metric extension.

Let us now check that $\mathcal{K}^{lip}$ has the ℓ_1 property: fix some integer n, consider $\mathbf{A} = (A, d, f)$ in $\mathcal{K}^{lip}$, endow A^n with the normalized ℓ_1 metric d_n, and define f on A^n by setting

$$f(a_1, \ldots, a_n) = \frac{1}{n} \sum_{i=1}^{n} f(a_i) \ .$$

It is clear that diagonal embeddings of $\mathbf{A}$ into $\mathbf{A}^n$ will preserve f, so we simply need to check that f is 1-Lipschitz on A^n, and this is an immediate consequence of the triangle inequality:

$$\left|f(\bar{a}) - f(\bar{a}')\right| \leq \frac{1}{n} \sum_{i=1}^{n} \left|f(a_i) - f(a_i')\right| \leq \frac{1}{n} \sum_{i=1}^{n} d(a_i, a_i') = d_n(\bar{a}, \bar{a}') \ .$$

Our next task is to show that $\mathcal{K}^{lip}$ has the extension property; for this we'll apply Solecki's technique from [S2]: consider $\mathbf{A} = (A, d, f) \in \mathcal{K}^{lip}$. We may, and do, assume that A has at least two elements.

Using the back-and-forth technique, one may show that there exists a (generic) 1-Lipschitz function g on $\mathbf{U}$ such that $\mathbf{A}$ embeds into $(\mathbf{U}, d, g)$ and any partial automorphism of $\mathbf{A}$ extends to an automorphism of $(\mathbf{U}, d, g)$.

Let $\{r_0, \ldots, r_n\}$ denote the non-zero values taken by d on A, and $\{t_0, \ldots, t_m\}$ the values taken by f on A. We consider a (classical) relational language $\mathcal{L}$ with n binary symbols $R_1, \ldots, R_n$ and m unary symbols $F_1, \ldots, F_m$ and turn $\mathbf{A}$ into a $\mathcal{L}$-structure, which we denote by $\mathbf{A}_{\mathcal{L}}$, by setting

$$\forall i \in \{1, \ldots, n\} \ \left(\mathbf{A}_{\mathcal{L}} \models R_i(a, a')\right) \Leftrightarrow \left(d(a, a') = r_i\right)$$

and

$$\forall j \in \{1, \ldots, m\} \ \left(\mathbf{A}_{\mathcal{L}} \models F_j(a)\right) \Leftrightarrow \left(f(a) = t_j\right) \ .$$

We may also turn $(\mathbf{U}, d, g)$ into an $\mathcal{L}$-structure, which we denote by $\mathbf{U}_{\mathcal{L}}$, using the same procedure. Any partial automorphism of $\mathbf{A}_{\mathcal{L}}$ extends to an automorphism of $\mathbf{U}_{\mathcal{L}}$.

Following Solecki, we define:

- a *configuration of type I* to be any $\mathcal{L}$-structure $\mathbf{B}$ with universe $\{b_0, \dots, b_n\}$ such that for all i there is j_i such that $\mathbf{B} \models R_{j_i}(b_i, b_{i+1})$, there is j such that $\mathbf{B} \models R_j(b_0, b_n)$, and $\sum r_{j_i} < r_j$.
- a *configuration of type II* to be any $\mathcal{L}$-structure $\mathbf{B}$ with universe $\{b_0, \dots, b_n\}$ such that for all i there is j_i such that $\mathbf{B} \models R_{j_i}(b_i, b_{i+1})$, there is i such that $\mathbf{B} \models F_i(b_0)$ and j such that $\mathbf{B} \models F_j(b_n)$, and $\sum r_{j_i} < |t_i - t_j|$.

We let $\mathcal{T}$ denote the set of all configurations of type I or II, and note that $\mathcal{T}$ is finite (up to isomorphism, of course). The structure $\mathbf{A}_{\mathcal{L}}$ is $\mathcal{T}$-free, i.e there is no weak homomorphism from a configuration in $\mathcal{T}$ to $\mathbf{A}_{\mathcal{L}}$. Since $\mathbf{U}_{\mathcal{L}}$ is a $\mathcal{T}$-free $\mathcal{L}$-structure and $\mathbf{A}_{\mathcal{L}}$ embeds in $\mathbf{U}_{\mathcal{L}}$ in such a way that all partial automorphisms of $\mathbf{A}_{\mathcal{L}}$ extend to automorphisms of $\mathbf{U}_{\mathcal{L}}$, the main theorem of [HL] means that there exists a finite $\mathcal{L}$-structure $\mathbf{B}$ with the same property. For any partial automorphism p of $\mathbf{A}_{\mathcal{L}}$ we let $\bar{p}$ denote its extension to $\mathbf{B}$.

We are almost done: let C denote the subset of B made up of all b such that one has $\mathbf{B} \models F_i(b)$ for some i, and there is a *chain* from b to some (hence to all) $a \in A$, i.e a sequence $b_0, \dots, b_n$ such that $b_0 = b$, $b_n = a$ and for all i there exists some j_i such that $\mathbf{B} \models R_{j_i}(b_i, b_{i+1})$. Note that the fact that A has at least two elements means that A is contained in C.

For any partial automorphism p of $\mathbf{A}$ one has $\bar{p}(C) = C$. Note that for all $c, c' \in C$ there exists a chain from c to c'; we may then define a metric d_C on C by setting

$$d_C(c, c') = \min\{\sum_{i=0}^{n-1} r_i \colon c_0, \dots, c_n \text{ is a chain from } c \text{ to } c' \text{ with } \mathbf{B} \models R_i(c_i, c_{i+1}) \text{ for all } i\}$$

Clearly d_C is a metric, and the fact that there are no configurations of type I in $\mathbf{B}$ means that d_C extends the metric of A. We may then define a function f_C on C by setting $f_C(c) = t_i$ if $\mathbf{B} \models F_i(c)$. This function extends f, and the fact that there are no configurations of type II means that f_C is 1-Lipschitz.

It is then easy to check that any $\bar{p}$ preserves both d_C and f_C, and we are finally done proving that $\mathcal{K}^{lip}$ has the extension property.

Finally, we note that the characterization of the generic 1-Lipschitz function on the Urysohn space means (using a variation on Katětov's construction of the Urysohn space) that $\mathbf{K}^{lip}$ satisfies the third condition of Theorem 4.6. Let us state what our results on $\mathcal{K}^{lip}$ imply in terms of the Urysohn space.

Theorem 5.1. *There exists a generic* 1*-Lipschitz function on the Urysohn space* $\mathbf{U}$*. The group of isometries of* $\mathbf{U}$ *which preserve this function is a Lévy group.*

5.2. **Metric spaces with an isometry of order p.** One may consider, for some fixed integer p, the class $\mathcal{I}_p$ whose elements are of the form (X, d, σ), where (X, d) is a finite metric space, and σ is an isometry of X such that $\sigma^p = id_X$. Described in this way, it is not a relational metric class; however, it is easy to encode our structures using a (finite) relational language.

The class $\mathcal{I}_p$ is then a Fraïssé metric class, whose limit is naturally identified with the Urysohn space endowed with a *generic* isometry of order p. By generic, we mean the following: the subset $G_p = \{\varphi \colon \varphi^p = id_{\mathbf{U}}\}$ is a closed subset of

$\mathrm{Iso}(\mathbf{U})$, on which $\mathrm{Iso}(\mathbf{U})$ acts by conjugacy. The isometry σ is characterized (up to conjugacy) by the property that its conjugacy class is comeagre in G_p.

The class $\mathcal{I}_p$ has the ℓ_1 property, and if we were able to show that it has the weak extension property then we could deduce that the centralizer of a generic element of order p in $\mathrm{Iso}(\mathbf{U})$ is extremely amenable. However, we were not able to prove (or disprove!) this.

Question 5.2. Does the class $\mathcal{I}_p$ satisfy the weak extension property?

For our purposes, it would be enough to obtain a positive answer to the following question.

Question 5.3. Is the following assertion true?

If B is a finite rational metric space, σ is an isometry of B of order p and A is a σ-invariant subset of B, there exists a finite metric space C such that:

- σ extends to an isometry of C of order p.
- Any partial isometry p of B with domain A whose range is σ-invariant and which satisfies $\sigma \circ p = p \circ \sigma$ extends to an isometry of C which commutes with σ.

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